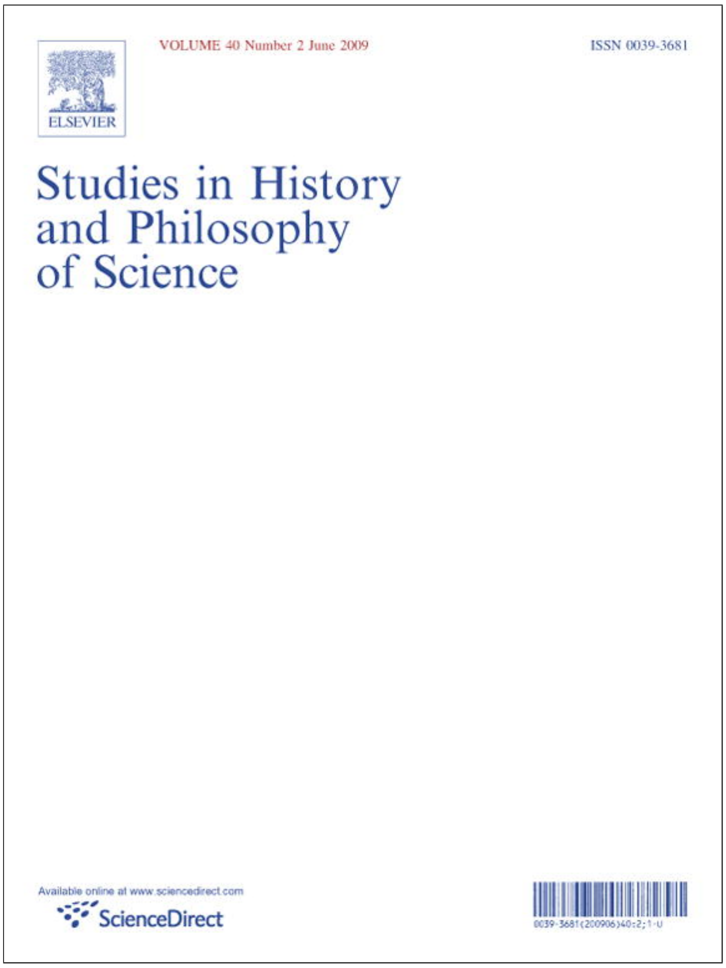
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Radical mathematical Thomism: beings of reason and divine decrees in Torricelli's philosophy of mathematics

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ABSTRACT

Evangelista Torricelli (1608–1647) is perhaps best known for being the most gifted of Galileo's pupils, and for his works based on indivisibles, especially his stunning cubature of an infinite hyperboloid. Scattered among Torricelli's writings, we find numerous traces of the philosophy of mathematics underlying his mathematical practice. Though virtually neglected by historians and philosophers alike, these traces reveal that Torricelli's mathematical practice was informed by an original philosophy of mathematics. The latter was dashed with strains of Thomistic metaphysics and theology. Torricelli's philosophy of mathematics emphasized mathematical constructs as human-made beings of reason, yet mathematical truths as divine decrees, which upon being discovered by the mathematician 'appropriate eternity'. In this paper, I reconstruct Torricelli's philosophy of mathematics—which I label *radical mathematical Thomism*—placing it in the context of Thomistic patterns of thought.

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1. Introduction: construction vs. discovery

Evangelista Torricelli (1608–1647) is perhaps best known for being the most gifted of Galileo's pupils, and for his works based on the geometry of indivisibles, which Torricelli developed especially in the wake of Bonaventura Cavalieri (1598?–1647).[1] In particular, in his 1644 *Opera geometrica* (the only work published by Torricelli himself), he presented a stunning result, the first cubature of an infinitely extended solid—a segment of hyperboloid. It had fascinating repercussions in seventeenth-century philosophcal literature.[2] Scattered among Torricelli's writings, we find numerous traces of the philosophy of mathematics underlying his mathematical practice. Though somewhat neglected by historians and philosophers alike, these traces reveal that Torricelli's mathematical practice was informed by an original philosophy of mathematics.

In this paper, I will reconstruct Torricelli's philosophy of mathematics, exposing the intriguing dynamics between his practice and philosophy of mathematics, in relation to the metaphysical and theological background of the early seventeenth century.

As we shall see, Torricelli's philosophy of mathematics was dashed with strains of Thomistic metaphysics and theology, which Torricelli might have absorbed from his uncle—a learned theologian from whom Torricelli received his first education—and/or later on when a student in the Jesuit schools.[3] Moreover,

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[1] On Torricelli's method of indivisibles, see De Gandt (1987, 1992) and Festa (1992). There are a number of old studies on Torricelli's mathematical works, which often tend to propose an anachronistic interpretation of Torricelli as the precursor of the infinitesimal method: that is, of the calculus. See, for example, Bortolotti (1927–1928, 1928, 1939).

[2] See Mancosu & Vailati (1991). The cubature of the infinitely extended solid is in Torricelli (1644), pp. 93–135—note, however, that the book, which includes different essays, has a different pagination for the different essays—and (1919–1944), Vol. 1, pp. 173–221.

[3] Little biographical information is known about Torricelli, especially the early period of his life. However, we know that the young Torricelli received his first instruction from his uncle, a priest and learned theologian who had earned his theology doctorate in Padua, in the early 1590s. See Torricelli (1919–1944), Vol. 4: a volume of biographical documents and essays on Torricelli. At the time that Torricelli's uncle was in Padua, the theology faculty there had been in place for more than a century. Indeed, with the exception of Rome, in the late sixteenth century Padua was the Italian university where the largest concentration of religious studies was offered. There were four chairs in total, two of Thomistic metaphysics and theology, and two of Scotist metaphysics and theology. Both theology and metaphysics were taught by members of religious orders. As is well known, Thomism and Scotism were the two most prominent currents of thought in theology and metaphysics during the time spanning the late middle ages to the seventeenth century. See Grendler (2002), pp. 353–392.

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as we learn from his correspondence, Torricelli was familiar with the intellectual milieu of the Collegium Romanum, the Jesuit university in Rome, especially with the professors of mathematics, successors of the then Europe-wide famous Jesuit mathematician, Cristoph Clavius (1538–1612).[4]

Torricelli's philosophy of mathematics emphasized mathematical constructs as human-made beings of reason, yet mathematical truths as divine decrees, which upon being discovered by the mathematician 'appropriate eternity', as Torricelli puts it. For reasons that hopefully will become clearer in due course, I label Torricelli's philosophy of mathematics *radical mathematical Thomism*—placing it in the context of early modern Thomistic patterns of thought.

I begin with two apparently contradictory statements by Torricelli about the nature of mathematics. The first statement concerns the nature of the constructs defined by the mathematician:

> But the things defined by geometry, that is, by the science of abstraction, have no existence in the universe of the world, other than that which is bestowed upon them by the definition in the universe of the intellect. Thus, no sooner are things defined in mathematics than they are born together with the definition.[5]

Torricelli regards the mathematical constructs defined by the geometer as existing only in the intellect, or mind, of the geometer, and as being born at the precise moment when they are defined. In the language of early modern metaphysical and theological thought, Torricelli's statement amounts to asserting that mathematical constructs, such as *circle*, for example, are beings of reason (*entia rationis*). In other words, they seem to be fictitious in the sense of not being real beings (*entia realia*), that is, beings existing outside the mind. However, as we will see, the distinction between *beings of reason* and *real beings* presupposes that a line can be drawn between the mind and the world, whereas in Torricelli's philosophy of mathematics this distinction is blurred when it comes to mathematical constructs.

The second statement by Torricelli concerns the nature of mathematical truths. The 'teachings of mathematics', says Torricelli, are 'divine decrees, and indubitable and eternal truths ... Geometrical truths, as soon as they are discovered, exclude contradictions, and appropriate eternity'.[6] Here we have a statement that at first glance seems to be in contradiction with the first statement.[7] If the constructs of mathematics are born at the precise moment when they are defined by the mathematician, and if they are no more than beings of reason, how can mathematical truths about them be eternal? Do the truths about the objects created by the mathematician exist before they are discovered by the mathematician? Further, how can mathematical truths appropriate eternity upon being discovered by the mathematician, and yet be divine, eternal decrees (*beneplaciti divini*, says Torricelli)?

Ettore Carruccio seems to have been the only Torricelli scholar who noticed the contradictions apparently raised by the two statements, and who offered a solution to the dilemma.[8] Carruccio argued that Torricelli was familiar with Augustine's writings—as indeed can be seen by the quotations made by Torricelli in the same text of the *Lezioni accademiche* from which the two above statements have been excerpted—and that Torricelli borrowed from Augustine the view that mathematical constructs exist both in the divine mind creating and contemplating reality, and in the human mind creating a purely intellectual reality.[9] However, I think that this suggestion, though fascinating, does not resolve the problem of the apparent conflict of the two statements. The reason for my perplexity with Carruccio's solution originates from what I call the *Augustinian prohibition*, which amounts to forbidding the claim that the human mind can beget eternal truths: it can only discover them.

The *Augustinian prohibition* can be stated as follows in Augustine's own words. Augustine states that:

> when we discover something about the liberal arts, either by reflecting within ourselves or when correctly questioned by someone else, we discover it in our soul; but to discover is neither to find nor to make. Otherwise, the soul would beget eternal things by means of a temporal discovery; for, eternal things are indeed sometimes discovered. What in fact is more eternal than the reason [*ratio*] of a circle?[10]

The human mind, says Augustine, can only discover eternal things; it cannot fashion them in any way whatever. Augustine, who was no mathematician, falls short of perceiving what Torricelli perceives well, that is, that the mathematician has a certain freedom in devising the constructs of mathematics. Augustine, I believe, would have been horrified by Torricelli's claim that geometrical truths, as soon as they are discovered, will exclude contradictions and appropriate eternity. This appropriation, Augustine would have objected, is more than theft of divine decrees. It is tantamount to a begetting of eternal things.

To penetrate deeper into this apparent conflict, we need to delve into metaphysical and theological issues concerning eternity as they were discussed in the late sixteenth and early seventeenth centuries. In the next section, I will sketchily review a debate about eternity within the tradition of Thomistic metaphysics and theology, which casts light on the problem at hand.

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[4] Torricelli's scientific correspondence is in Torricelli (1919–1944), Vol. 3, and Galluzzi & Torrini (1975–1984), Vol. I. On Torricelli's acquaintance with the work of Clavius on Euclidean proportion theory, and on Torricelli's own contribution to the foundations of proportion theory, see Giusti (1993) and Palmieri (2001).

[5] The prose is a bit convoluted but clear in its meaning. The original Ialian is as follows:

> Ma le cose definite dalla geometria, cioè dalla scienza dell'astrazione, non hanno altra esistenza nell'universo del mondo, fuor che quella che gli conferisce la definizione nell'universo dell'intelletto. Così quali saranno definite le cose della matematica, tali puntualmente nasceranno insieme con la definizione stessa.

The text is from the *Lezioni accademiche*, not published by Torricelli. See Torricelli (1975), pp. 553–651, and (1919–1944), Vol. 2, pp. 1–99. The passage quoted above is in Torricelli (1975), p. 585. Since Torricelli (1919–1944) is a rather rare edition, I will quote the *Lezioni accademiche* from Torricelli (1975).

[6] ... gl'insegnamenti della quale [i.e., mathematics] non sono opinioni di dottori, o fantasie di uomini, ma beneplaciti divini, e verità indubitabili, ed eterne ... le verità geometriche ritrovate una volta sola, subito che sono scoperte, escludono le contraddizioni, e s'impossessano dell'eternità. (See Torricelli, 1975, p. 616)

[7] There is no internal contradiction within this second statement (contrary to what one might suspect, since Torricelli first says that teachings of mathematics are divine decrees and indubitable and eternal truths, and immediately after this passage goes on to say that geometrical truths—as soon as they are discovered—exclude contradictions, and appropriate eternity). The teachings of mathematics are about truths *already* discovered, and therefore such that they have *already* appropriated eternity (in a sense that will become clearer as I develop the discussion in the remainder of the article). Hence mathematics can teach eternal truths.

[8] See Carruccio's comments in Torricelli (1955), pp. 17–18.

[9] I due punti di vista che si presentano nei due passi citati [i.e., the two apparently contradictory statements, here referred to by Carruccio in a reversed order] non sono in contraddizione. Secondo entrambi la matematica esiste in un mondo di oggetti del Pensiero, che nel primo di detti passi è il Pensiero divino che crea e contempla tutta la Realtà, nel secondo è il pensiero umano che costruisce un mondo intellettuale. (Torricelli, 1955, p. 18)

[10] Sed cum vel nos ipsi nobiscum ratiocinantes, vel ab alio bene interrogati de quibusdam liberalibus artibus ea quae invenimus, non alibi quam in animo nostro invenimus: neque id est invenire, quod facere aut gignere; alioquin aeterna gigneret animus inventione temporali (nam aeterna saepe invenit; quid enim tam aeternum quam circuli ratio ...?). (Augustine, n.d., *De immortalitate animae liber unus*, at 4.6)

2. Thomistic patterns of thought about eternity

What is eternity? I begin this exploration with Thomas Aquinas. In his *Summa contra gentiles*, Aquinas expounds his views about the eternity of God. What is the *ratio*[11] of God's eternity, asks Aquinas?

> Only the things that move are measured by time, for time is the number of motion, as is clear from [Aristotle's *Physics*], Book IV. But God is motionless, as has already been proved. He is therefore not measured by time. Thus God does not take the *before* and *after*. He has therefore no being after non-being, nor non-being after being, and thus no succession can be found in Him, since these things are unintelligible without time. God thus lacks beginning and end, while He has His being all at once. In this the *ratio* of eternity consists.[12]

Aquinas' passage can be illuminated by the comments made by a later Thomist commentator, Franciscus de Sylvestris (Francis Silvester, 1474?–1526). The crux of the matter, de Sylvestris seems to suggest, is that here Aquinas strives to interpret Boethius' definition of eternity (which reads 'perfect possession, all at once, of an interminable life').[13] De Sylvestris recalls that Aquinas expounds Boethius' definition at one point in his *Summa theologiae*. The Aquinas passage in question is as follows:

> Since in any motion there is succession, and a part after another, we apprehend time from this, that we number in motion the before and after, which is nothing else than the number of the before and after. But in that which lacks motion, and is always in the same way, there is no taking the before and after. Thus, as the *ratio* of time consists in numbering the before and after in motion, so the *ratio* of eternity consists in apprehending the uniformity of what is absolutely outside of motion.[14]

De Sylvestris pinpoints the difficulty with fine acumen. The question is how we are to read Aquinas' statement that the *ratio* of eternity is the apprehension of a uniform thing absolutely immutable. On this question, as we shall see, the metaphysicians and theologians of the early seventeenth century will hotly debate. Can the human intellectual act of apprehending a *ratio* of a being be constitutive of the *ratio* of that being (in this case the being *eternity*[15])? Here is a sketch of de Sylvestris' solution.

The key to de Sylvestris' solution is an analogy with time. The *ratio* of time consists in numbering the before and after in motion, because its complement—that is, the making discrete and thus separating the parts of time—inheres in time by virtue of the human soul's numbering operation. Hence, once the operation of the soul has been removed, there remains certainly in reality (*in re*) that thing which is time, but it does not have the complete *ratio* of time, since it does not have the separation and division in actuality, and therefore it is not number in actuality. By the same token, we can consider two things in eternity. One thing is the unity of the absolutely immutable divine being. A second thing is the human mensuration of the divine being. The first inheres (*convenit*) in the divine being in reality (*secundum quod est in re*), for the divine being is really one outside the human intellect. The second thing inheres in the divine being because of the operation of the human soul that distinguishes the unity of the divine being from His being, and which apprehends the divine unity as a certain measure and duration. Therefore, de Silvestris concludes, since the *ratio* of eternity is a unit measuring the absolutely immutable being—to which mensuration inheres only by virtue of the soul's distinguishing the immutable being's immutability from its being and apprehending its immutability with the *ratio* of measure (*sub ratione mensurae*)—then, the *ratio* of eternity can only be completed by the soul's operation. Hence, there certainly remains in reality (*in re*) that thing which is eternity, even if the soul does not consider it, but it does not have the complete *ratio* of eternity. Thus, the *ratio* of eternity consists 'completive' (*completive*) in apprehending the uniformity of that which is absolutely immutable, because the complement of the *ratio* of that which is absolutely immutable, namely, its being a measure, is only by virtue of the human soul's apprehending it.[16]

There is no question that de Sylvestris' reading of Aquinas' explanation of eternity places a great deal of emphasis on the completive contribution to the *ratio* of eternity that is brought about by the measuring act of the human intellect.

This interpretation of Aquinas' thought on eternity was vehemently impugned by one of the most influential of the early modern metaphysicians and theologians in the Thomist tradition, namely, the Jesuit Francisco Suárez (1548–1617).[17] Subsequently, another Thomist philosopher and theologian, John of St. Thomas (John Poinsot, 1589–1644), responded to Suárez' attack on Franciscus de Sylvestris, defending and further developing the latter's interpretation. The first complete edition of John of St. Thomas' *Cursus philosophicus Thomisticus* was published in Rome in 1637–1638.[18] I will examine Suárez' counter-arguments and John of St. Thomas' reply in order to piece together the sparse elements of the debate about eternity that constitutes the background to Torricelli's philosophy of mathematics.

First of all, Suárez recognizes that the difference posited by Aquinas between eternity and other durations, especially permanent and immutable durations, is true; but, Suárez claims, it is not derived from what formally and per se pertains to the *ratio* of eternity. Suárez argues that the main thrust of Aquinas' view

[11] The Latin *ratio* is perhaps the most challenging term in the philosophical lexicon of neo-Latin to render into a modern vernacular. It can typically mean 'reason', 'formal reason', 'cause', or 'essence'. The fact is that all of these semantic spheres overlap in philosophical neo-Latin. It is preferable, therefore, to keep to the original terminology.

[12] Illa sola tempore mensurantur quae moventur: eo quod tempus est numerus motus, ut patet in IV physicorum. Deus autem est omnino absque motu, ut iam probatum est. Tempore igitur non mensuratur. Igitur in ipso non est prius et posterius accipere. Non ergo habet esse post non esse, nec non esse post esse potest habere, nec aliqua successio in esse ipsius inveniri potest: quia haec sine tempore intelligi non possunt. Est igitur carens principio et fine, totum esse suum simul habens. In quo ratio aeternitatis consistit. (Aquinas, 1918, Book I, Ch. 15 n. 3)

See Leftow (1990) for an analysis of Aquinas' views on time and eternity. However, I think the author takes an excessively anachronistic approach, comparing Aquinas' views with today's relativity theory.

[13] Boethius' definition of eternity reads in Latin 'interminabilis vitae tota simul et perfecta possessio'. Cf. de Sylvestris' comment, in Aquinas (1918), p. 42.

[14] Cum enim in quolibet motu sit successio, et una pars post alteram, ex hoc quod numeramus prius et posterius in motu, apprehendimus tempus; quod nihil aliud est quam numerus prioris et posterioris in motu. In eo autem quod caret motu, et semper eodem modo se habet, non est accipere prius et posterius. Sicut igitur ratio temporis consistit in numeratione prioris et posterioris in motu, ita in apprehensione uniformitatis eius quod est omnino extra motum, consistit ratio aeternitatis.

Cf. Aquinas (n.d.), *Summa theologiae*, Part I, Question 10, Article 1, and Aquinas (1918), p. 42, where the text reported by de Sylvestris is slightly different.

[15] 'Eternity' is a being in that it is an attribute of God. Here, however, we must leave aside the broader theological question of whether in the early modern Thomist traditions God and eternity are beings separable by an intellectual act of the human mind.

[16] De Sylvestris' conclusion reads in its original as follows: 'Dicitur ergo ratio aeternitatis consistere completive *in apprehensione uniformitatis eius quod est omnino immutabile*, quia complementum rationis eius, scilicet esse mensuram, est tantum secundum animae apprehensionem'. See de Sylvestris' Latin text, in Aquinas (1918), p. 43.

[17] The attack is in Suárez' *Metaphysical disputations*, published first in 1597, Disp. 50. See Suárez (1965), Vol. II, pp. 912–972.

[18] See the *Cursus philosophicus Thomisticus*' response to Suárez, in John of St. Thomas (1933–1937), Vol. II, pp. 369–385.

on eternity is eternity's lack of succession.[19] But it is unclear, Suárez says, how lack of succession is to be intended in created beings, such as, for instance, angels. For, in the being of angels there is no succession, but there is succession in their internal operations. Thus, Suárez concludes, the *ratio* of eternity can only be specified as follows. Eternity is duration per se and intrinsically necessary and dependent on nothing, and thus immutable.[20] In other words, firstly Suárez wishes to exclude the character of lacking succession from the *ratio* of eternity. Secondly, Suárez goes on to tackle the issue of whether de Sylvestris' analysis, in terms of the completive contribution to the *ratio* of eternity brought about by the human intellect, leads to a true or false conclusion. Suárez poses the question directly, asking whether 'eternity includes in its formal *ratio* a being of reason'.[21] Suárez recognizes that de Sylvestris' position is very common among Thomists and indeed seems to be in accord with Aquinas' own words. Further, Suárez says that Aquinas and other doctors often call eternity a measure of the divine being according to our apprehension, and that these authors claims that eternity measures the duration of God.[22] However, Suárez responds, the latter assertion is a false conclusion stemming from a false assumption. The false assumption is that the *ratio* of duration is constituted by a relation of measure, from which assumption, Suárez says, the Thomists conclude that, eternity itself being a duration, eternity's *ratio* must be a relation of measure.[23]

Suárez and the Thomists agree that a relation of measure is a being of reason (i.e., a relation of reason). Further, Suárez agrees that a relation of reason is a being that may have a foundation outside the human intellect, but to the *ratio* of which the human intellect always contributes its apprehension, completing it as a 'being to something else [*esse ad aliud*]'.[24] The disagreement is whether measure, insofar as it is a being of reason, enters the constitution of the *ratio* of eternity, hence of a divine attribute, hence *a fortiori* of God.

Suárez's argument against the Thomists who follow de Sylvestris is threefold. Suárez distinguishes between *active* and *passive* measure. Active measure, Suárez argues, can in turn be distinguished between *intrinsic* and *extrinsic*.

Firstly, eternity cannot be extrinsic active measure since neither we nor God can use eternity as a measure of duration. Secondly, eternity cannot be intrinsic active measure. For, if eternity is an intrinsic active measure of God, what is it supposed to measure? Not God, otherwise a measure would be a measure of itself. Thirdly, eternity cannot be a passive measure. For, if God were measurable because of His eternity, by which measure would God be measurable? Not by eternity, as has already been shown by Suárez with regard to active measure, nor, with greater reason, by any other created measure of duration. In the final analysis, eternity cannot be measurable because it is an infinite duration, and that which is infinite, Suárez concludes, is immeasurable, as such.[25]

Suárez wraps up his analysis by reiterating that no being of reason (*ens rationis*), insofar as it depends on the apprehension of our intellect, can be included in the formal *ratio* of eternity.[26] I now turn to John of St. Thomas.

In his *Cursus philosophicus Thomisticus*, John of St. Thomas presents a *Question on time*, in which he takes up some of the issues raised by Suárez.[27] John of St. Thomas thinks that time is real (i.e., a real being) according to its entity, in the same way as motion is real, of which time is the intrinsic duration. However, he points out that time requires a formality of reason (i.e., a being of reason) in order to be a measure. For, it cannot be a measure unless some of its parts are conjoined and distinguished by means of numbers. This, however, cannot happen but for the intellect that compares the past with the future. So, time has a dual nature, which is captured by John of St. Thomas with an analysis not too different from that of de Sylvestris. On the basis of his exposition of the nature of time, John of St. Thomas goes on to tackle the issue of eternity. The question is how to distinguish the *ratio* of time from the *ratio* of other permanent durations, such as eternity. Eternity is defined by John of St. Thomas following Boethius (see above). He comments that eternity, because of its infinity and immutability, not only in being but also in measuring, lacks succession, does not expect things successive, as if they flowed and thus coexisted with it, but rather contains them immutably, in the way of a superior measure from which they all derive. John of St. Thomas is at pains to illumine this admittedly obscure comment with a simile. He says that eternity is like an immense tree whose canopy covers all the flowing water of a river, so that it could be said that the tree coexisted with all the parts of the river's water, all at once.[28]

But what is measure? Remember that Suárez attacked the Thomists who follow de Sylvestris in reading Aquinas, by analyzing the *ratio* of measure, and concluding that eternity does not have the *ratio* of measure. Suárez' point was that it is a false assumption that the *ratio* of duration, hence of eternity, is constituted by a relation of measure. Suárez' attack on eternity as measure is countered by John of St. Thomas as follows. He acknowledges that a measure must be homogeneous with what it is supposed to measure. This requirement, though, is evidently necessary only when measure is taken *extrinsically*, that is, when measure is supposed to measure something else by application of itself to the something else it measures. For, when measure is taken *intrinsically*, that is, as *measure informing* something else, then the requirement of

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[19] Suárez claims that for Aquinas:

> Aliter exponit hoc discrimen D. Thom., I, q. 10, a. 5 ... aeternitas (inquit) est talis duratio quae nullam successionem admittit, neque in esse, neque in propriis et internis actibus seu operationibus rei aeternae; omnis autem alia duratio talis est ut vel in seipsa successionem habeat, vel saltem illam habeat adiunctam in aliquibus operationibus vel motibus eiusdem suppositi. (Suárez, 1965, Vol. II, p. 924)

However, I am unable to locate the passage quoted by Suárez in Aquinas' *Summa theologiae*. It looks as if Suárez is quoting from memory, perhaps mixing different elements that are not exactly to be found in Aquinas (n.d.), *Summa theologiae*, Part I, Question 10, Article 5. Aquinas' passage closest to Suárez' reading is:

> Sic ergo ex duobus notificatur aeternitas. Primo, ex hoc quod id quod est in aeternitate, est interminabile, idest principio et fine carens (ut terminus ad utrumque referatur). Secundo, per hoc quod ipsa aeternitas successione caret, tota simul existens. (Aquinas, n.d., *Summa theologiae*, Part I, Question 10, Article 1)

Article 5, referred to by Suárez, seems to be irrelevant to the point at issue.

[20] '... aeternitas est duratio per se et ab intrinseco necessaria et a nullo pendens, et consequenter omnino immutabilis' (Suárez, 1965, Vol. II, p. 925).

[21] Ibid., p. 926.

[22] Ibid., pp. 926–927.

[23] 'Verumtamen tota haec sententia procedit ex falso fundamento, nimirum, quod ratio durationis constituatur per relationem mensurae; inde enim intulerunt dicti auctores, etiam rationem talis durationis, nempe aeternitatis, sitam esse in tali ratione mensurae' (ibid., p. 927).

[24] Ibid., p. 1026. To the question of relation, Suárez devotes a whole disputation (Disp. 47). We need not dwell on the complex details of the discussion, especially concerning the partition between real relations and relations of reason. It will suffice to point out that, for Suárez, as for all Catholics, some relations are undoubtedly real relations, that is, real beings, since Catholic doctrine teaches that the persons of the trinity are constituted by relations, and that the persons of the trinity are undoubtedly real beings. See ibid., p. 784.

[25] Ibid., pp. 927–928.

[26] '... generaliter censeo nullum ens rationis, quatenus pendet ex apprehensione nostri intellectus, includi posse in formali ratione aeternitatis'. See ibid., p. 929.

[27] John of St. Thomas (1933–1937), Vol. II, pp. 369–385.

[28] Ibid., pp. 375–376.

homogeneity need no longer be met. In the latter case, measure will make a thing formally ordered and known to us. I call this second case of measure taken *intrinsically*, as discussed by John of St. Thomas, *mensura informans* (stressing its function of *informing*, i.e., of giving a form to the thing). To exemplify, consider the case of an accident supervening to a subject. The accident renders the subject known and manifest to us. Hence one finds in this case the same relation of *extrinsically* taken measure between subject and accident, in the sense that the accident renders the subject more known, or at least more knowable to us, even though subject and accident are not homogeneous. The accident is a *mensura informans* the subject. Time is a *mensura informans* motion. Eternity is a *mensura informans* the divine being.

Let us take stock. I discern two branches in the debate on eternity stemming from the same trunk. On the one hand, the Thomists who follows de Sylvestris emphasize the human intellect as actively involved in the constitution of the *ratio* of eternity. On the other hand, the strain of Thomism championed by Suárez tends to exclude the human intellect's activity from the constitution of the ratio of eternity. The Thomists who follow de Sylvestris include the products of the human intellect insofar as they are beings of reason in the *ratio* of a real being, indeed in the most real of all beings, namely, that of God. Suárez rejects such an inclusion. Armand Maurer, who made an exemplary study of the question of the status of mathematical objects and of abstraction in Aquinas' epistemology, noted that 'it seems clear that, in Aquinas's opinion, the objects of mathematics are not simply abstracted from the real world but owe their existence to a constructive or reconstructive activity of the mind'.[29] Maurer thus concludes that:

> abstraction is constructive or completive as well as selective. The mind must add to the real foundation of the mathematical notion and complete its formal character, as it does with the notions of species, universal, time, and truth. None of these, Aquinas has told us, enjoys a complete being in extramental reality. They have a foundation there, but their formal character comes from the mind. The same would seem to be true of mathematical notions.[30]

I agree fully with Maurer's conclusion.

To sum up, in the debate I sketched above, we have seen that in the late sixteenth and early seventeenth centuries it became possible to espouse the view that the completive activity of mind seems to enter into the very constitution of the formal nature, the *ratio*, of the divine attribute of eternity, hence of God himself. It is against this early modern theological and metaphysical background that I propose to investigate Torricelli's philosophy of mathematics, and its apparently contradictory stances on the eternal status of mathematical truths vs. the power of the mathematician to create beings of reason that will grasp eternal divine decrees. To which proposal I now turn.

3. The foundations of Torricelli's philosophy of mathematics

The foundations of Torricelli's philosophy of mathematics come under three main headings, namely, *definition*, *relation*, *measure*. In this section, I will discuss them in turn, as they bear on the stance

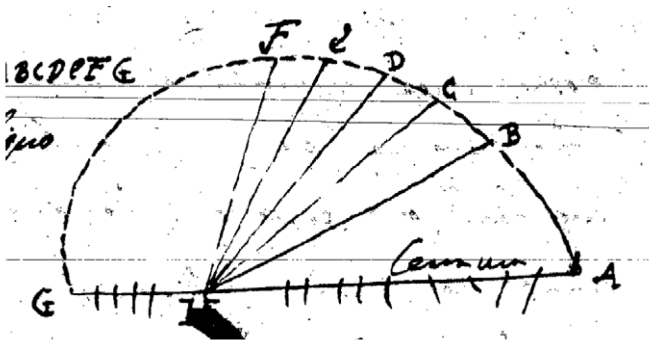

Fig. 1. The diagram accompanying the 'first, and true definition' of the geometric spiral in Torricelli's autograph manuscript. See Torricelli (2007), f. 40r (from a microfilm courtesy of the Biblioteca Nazionale, Florence, Italy).

Torricelli takes about eternal truths and the role of beings of reason in the mathematician's practice.

3.1. Definition

Perhaps the most dramatic moments in Torricelli's mathematical career, at least in relation to problems of definition, must have come when he set about constructing a wholly new geometrical object, a new being of reason, which today goes under the name of *logarithmic spiral*. Among many other relevant results, Torricelli brilliantly succeeded in rectifying the spiral. It is the first example of a rectification of a curve in the history of mathematics. However, Torricelli failed to publish his most original researches on the logarithmic spiral, which remained buried among his manuscripts until the twentieth century.[31] Amedeo Agostini, who reconstructed Torricelli's memoir, begins by presenting the definition that Torricelli called 'First and true definition'. It reads as follows (see Figure 1):

> *First and true definition*. Let us suppose a certain line ABCDEFG, the nature of which is such that, if from a point H any angles whatever are taken successively equal to one other, such as, for example, BHC, CHD, DHE, etc., these angles are contained by lines continuously proportional; that is, as HB is to HC so HC is to HD, [HD] to HE, etc., and so on, ad infinitum. Such a line ABCD etc. we call 'Geometric spiral'; but point H we call its center, and straight lines HB, HC, HD the radii of the Spiral.[32]

Torricelli calls his spiral 'geometric' in order to distinguish it from the Archimedean spiral, which, according to Torricelli, can be said to be truly 'arithmetic'.[33] Torricelli explains that the nature of the Archimedean spiral is *arithmetic*, in contrast to his own *geometric* spiral, for the following reason. The Archimedean spiral is generated by a point moving along a straight line, the radius (which revolves uniformly around a fixed point, the center of the spiral), in such a way that equal spaces are traversed by the point in equal intervals of time. On the other hand, in the geometric spiral, a point moving along the radius of the spiral traverses spaces that will be in a geometric proportion to one another.[34] If the point moves towards the center, for instance, and in the first interval of time it has traversed space S1, and in a second equal interval of time has traversed

[29] Maurer (1993), p. 52. See also Anderson (1969) on Aquinas and the question of the 'intelligible matter' of mathematical objects.

[30] Maurer (1993), p. 55.

[31] See Torricelli (1919–1944), Vol. 1, Part 2, pp. 350–399, 1949, 1955). The first edition simply reproduced most of the manuscript material still to be found in Florence at the National Library; the second and third editions tried to reconstruct both the logical order that Torricelli would have given the memoir if he had been able to polish it for publication, and some supposedly missing elements needed to assemble the manuscripts left by Torricelli into a more coherent whole.

[32] Prima definitio, et vera. Supponamus quandam lineam ABCDEFG eius natura ut si ab aliquo puncto H sumantur quocumque anguli deinceps, aequales inter se, puta BHC, CHD, DHE etc. huiusmodi anguli a lineis continue proportionalibus contineantur, nempe sit ut HB ad HC ita HC ad HD ad HE etc. usque ad infinitum. Talis linea ABCD etc. Spiralis geometrica a nobis appellabitur; punctum vero H ipsius centrum et rectas HB, HC, HD radios ipsius Spiralis vocabimus. (Torricelli, 1949, p. 17)

[33] Ibid., p. 17.

[34] Ibid., pp. 17–18.

space S2, and in a third equal interval of time a space S3, and so on, then S1 is to S2 as S2 is to S3, and so on. This property of traversed spaces, Torricelli shows, descends immediately from the first and true definition, since, the radii being in such a geometric proportion, it follows that their differences also (namely, the spaces traversed by the point) are in a geometric proportion.

This explanation hides a decisive clue to the problem of the definition of the geometric spiral. For, could the geometric spiral not be defined precisely by the kinematic property that distinguishes it from the Archimedean spiral? In a fascinating letter, written in 1645, Torricelli confesses to his correspondent that he has indeed three definitions of the geometric spiral, but he is unable to make up his mind as to which must be elected to define the curve.[35] Clearly, in 1645, Torricelli has not yet decided which, among the three candidates, will be the 'first and true definition' of the geometric spiral. The third candidate, we may easily speculate, is what Torricelli calls a 'description of the spiral by points', which can be found among his manuscripts.[36]

Correctly, in my view, Agostini emphasized the role that the definition played in Torricelli's investigative processes.[37] But what is a 'true' definition? In what sense can a definition be true? Torricelli is eerily silent on this point. We need to dig deeper into the historical philosophical background. Definition was a topic treated in logical treatises, and taught in the logic classes at various universities. At the Collegium Romanum, the prestigious Jesuit college in Rome where Torricelli might have been a student, we know that logic was taught in the first year of the philosophy curriculum.[38] Paulus Vallius (1561–1622), an eminent logician of the Jesuit order, had been a professor there.[39] Vallius published a massive logic, which may reflect something of the logic teachings that professors would have imparted in the Collegium at about that time.

Vallius devotes a whole disputation to the question of definition.[40] He explains the division between 'true definition [*definitio vera*]' and 'description [*descriptio*]' as follows. A *true definition* is a definition that is given by means of the essential parts of the defined thing. A *description* is a definition that is given either by means of many common accidents, which all together convene only to the *definiendum*, or by means of the true genus and a proper accident of the *definiendum*, or by means of something extrinsic, which however in some way can explicate the essence and quiddity of the *definiendum*.[41] Further, Vallius claims, the *true definition* is that which has all the conditions of a good and perfect definition. Vallius lists six conditions:

(1) The definition must be clearer than the definiendum.
(2) The definition must not contain anything superfluous.
(3) The definition must 'convene' with the *definiendum*, so that it does not convene to things other than the *definiendum*, nor to a part of it, but solely convenes to the *definiendum* as a whole.
(4) The definition must follow a good order, that is, it must first posit the genus and second the difference.
(5) The definition cannot omit any difference that implicitly or explicitly must be contained in it. So, for instance, in the so-called 'last difference [*ultima differentia*]' all superior differences are implicitly contained.
(6) The definition must only contain essential predicates, and all of them.[42]

More generally, Vallius says, a definition is an instrument of knowledge, and as such a definition pertains to the so-called *first operation of the intellect* (on which more in a moment); hence a definition yields a being of reason, which follows from that first operation.[43] The first operation of the intellect, Vallius explains, is an apprehension of the thing. Vallius' exposition is clearly modeled on Aquinas and the Thomistic tradition. We can find an illuminating text in Aquinas. Aquinas says that the first operation of the intellect is an imagination, which Aristotle names 'intelligence of indivisible things'. This first operation consists in the apprehension of a simple quiddity. The second operation of the intellect is called 'assent [*fides*]', and consists of a composition or division of a proposition. The first operation turns the attention to the quiddity of the thing. The second turns the attention to the very being of the thing. Now, Aquinas argues, since truth and falsity are founded in the being of a thing, and not in the quiddity of a thing, it follows that truth and falsity are properly to be found only in the second operation of the intellect and in its sign, which is *enunciation*. However, in a certain sense, truth and falsity can also be found in the first operation. For, the quiddity is a being of reason, and falsity can affect the first operation of the intellect, as follows. Firstly, a definition can be false in comparison to the *definiendum*; for example, the definition of circle is certainly false of triangle. Secondly, and more importantly, a definition can be false with respect to its parts' relation to one another, in which case a definition can imply an impossible consequence.[44]

In selecting a definition for the geometric spiral, Torricelli must have paid serious attention to the general conditions of a true definition. The kinematic property might have been equally attractive as a definition of the geometric spiral. But there is no doubt that the construction by points—a description, as Torricelli himself later called it—was eliminated rather quickly as a possible candidate, since it does not yield the curve but only an approximation to it by means of tangents. The real competition must have been between the geometric and the kinematic conditions. At first glance they seem to work equally well. However, Torricelli, who had assisted Galileo in Florence, and who published an elegant

[35] '... the definition, too, I have in three ways, but I have not decided which I must select [... anco la definizione ho in tre modi, ma non ho risoluto quale di essi io debba eleggere]' (Torricelli, 1919–1944, Vol. 3, p. 280).

[36] Torricelli (1949), pp. 19–20. This description by points comes very close to a similar mechanical construction of the geometric spiral that Torricelli describes in a letter to Marin Mersenne, where Torricelli talks about a mechanical implement that can be used to determine points of the spiral (more precisely, by means of tangents to the curve). See the passage from the letter to Mersenne, in Torricelli (1919–1944), Vol. 3, pp. 275–276. The diagram of the spiral that I showed above (see Figure 1) may have been drawn with the mechanical implement that Torricelli describes to Mersenne, since it is made by tracing a number of short rectilinear segments in succession, which determine tangents to the curve.

[37] Torricelli (1949), pp. 4 ff. Ettore Carruccio, who published a slightly different reconstruction of Torricelli's memoir, follows Agostini in placing the *first and true definition* at the beginning of the memoir. Gino Loria, who edited the 1919–1944 edition of the memoir, reproduced the material in the order in which it is still to be found today in the manuscript, where the *first and true definition* is given after a number of folios containing theorems about the spiral.

[38] Society of Jesus (1616), pp. 68 ff.

[39] Vallius is the very Jesuit professor whom William Wallace believes to be the author, or one of the authors, of the logic manuscripts from which Galileo culled the notes preserved in his MS n. 27. Wallace refers to Galileo's early notebooks, published by Wallace himself in Galilei (1988).

[40] Vallius (1622), Vol. II, pp. 337–409.

[41] Ibid., p. 387.

[42] Ibid., p. 390.

[43] Ibid., p. 381.

[44] Aquinas (n.d.), *Scriptum super sententiis*, Liber I, Distinctiones 19–21.

reworking of the third and fourth parts of Galileo's *Two new sciences* (in the 1644 *Opera geometrica*), was perfectly aware of a potential problem with the kinematic condition. In the third part of *Two new sciences*, Galileo had famously claimed that, if an accelerated motion were not to follow the proportionality speed–time, but the proportionality speed–distance traversed (an error that the young Galileo himself made, as his mouth-piece in the book, Salviati, admits), the motion would be impossible: that is, the motion would occur in an instant.[45] Now, the kinematic condition of the geometric spiral precisely describes such an impossible motion. Yet there is no question that the kinematic condition deductively follows from the first and true definition. So, I am convinced that Torricelli came to the conclusion that:

(1) the first and true definition really and truly defines a being of reason that is wholly possible: that is, free from impossible consequences, and that
(2) with the first operation of the intellect it is indeed possible to contemplate the motion of a point that follows the proportionality speed–time traversed (a motion which, when directed *towards* the center of the spiral may have started at any time, though not from rest, but will never reach the center; and, when directed *away* from the center, may either have started at any time but not from rest, or from rest but at a time placed infinitely back in the past (remember that the geometric spiral revolves indefinitely around the center without ever reaching it in finite number of revolutions)).

Indeed, it is quite remarkable that such a conclusion could be arrived at by Torricelli a priori, and that a priori Torricelli's spiral showed that Galileo's general conclusion was wrong.[46]

Thus, it is now clear why Torricelli eventually opted for the geometric condition in order to give the first and true definition of the geometric spiral. If he had opted for the kinematic condition he would have defined a being of reason that might have turned out to be a chimera. For, he would have assumed without proof the very possibility of the motion against which Galileo had issued his stern prohibition. Instead, Torricelli elegantly showed that the possibility of such a motion descends from a purely geometric condition that no doubt defines a being of reason wholly free from chimerical impossibilities. Torricelli, then, was entitled to call his definition of the geometric spiral 'first and true', following the tradition of Aristotelian–Thomistic logic in which a definition can be false with respect to its own parts' relation to one another—the most serious case in which a definition, insofar as it pertains to the first operation of the intellect, can go wrong, that is, imply an impossible consequence. Torricelli could contemplate his newly defined being of reason as spectacularly showing the type of accelerated motion that Galileo failed to grasp. The infinite gyrations of the point around the center of the spiral do not prevent the spiral from having a measurable, finite length. Torricelli rectified the curve, thus penetrating one divine decree about the nature of curves.[47] Remember that, for the Thomists following de Sylvestris, human mensuration of the divine being inheres in the divine being because of the operation of the human soul, which distinguishes the unity of the divine being from His being, and which apprehends the divine unity as a certain measure and duration; hence, in their view, the *ratio* of eternity can only be completed by the soul's operation. By the same token, the finite length of the geometric spiral is a divine decree constituting the *ratio* of eternity *completive*, insofar as it is apprehended by the mathematician's intellect contemplating the geometric spiral. In this sense, Torricelli can claim that mathematical truths take possession of eternity at the very instant they are discovered by the mathematician.

3.2. Relation

The language of Torricelli's geometrical works was that of Euclidean proportionality. Indeed, as Torricelli says, Euclid's theory of proportional magnitudes, as expounded in Books V and VI of the *Elements*, is the very foundation of geometry. Torricelli took a profound interest in questions concerning Euclid's definition of *sameness of ratios*, which is as follows:

> Magnitudes are said to be in the same ratio, the first to second, and the third to the fourth, when the equimultiples of the first and the third, both alike equal, alike exceed, or alike fall short of, the equimultiples of the second and the fourth—whatever this multiplication may be—and those equimultiples are being considered that correspond to each other.[48]

Indeed, Torricelli was deeply dissatisfied with Euclid's definition of sameness of ratios, and considered it to be obscure. Accordingly, he wrote a small tract in which he proposes to abandon Euclid's definition of sameness of ratios, in favor of a new approach based on Euclid's definition of proportionality as a *smilarity of ratios* (where a *ratio* between two magnitudes is in turn defined as a *habitude*[49] between the two magnitudes, so that, in effect, Torricelli's proposal amounts to defining a proportionality as a relation between two relations, a habitude between two habitudes).[50] Thus, Torricelli favored a definition of proportionality that invokes a relation of similarity between habitudes, rather than one that relies on the Euclidean computational technique of the equimultiples, as I will presently make clear.

Let us begin with an intriguing question: *Why was Torricelli so dissatisfied with Euclid's definition of sameness of ratios*?

> Yet, what could be conceived that is more obscure, and justly more worthy of doubt, than the sixth definition of Euclid's book V, where it is supposed that it is possible to find four magnitudes, the equimultiples of which, taken according to

[45] Galilei (1890–1909), Vol. VIII, pp. 203–204.

[46] Galileo's impossibility conclusion is generally wrong. It is correct only when taken to mean that the proportionality speed–distance traversed defines an accelerated motion that started from rest at any *finite* time in the past. However, Galileo is very ambiguous on this count, and it seems that he believed in the impossibility in an absolute sense. In the more restricted sense, Galileo's impossibility conclusion was proven to be correct by Pierre Fermat (1601–1665). Briefly, Fermat imagines a point uniformly accelerating along a line, and whose speed follows the ratio of the traversed distances. He then shows that if one considers a series of spaces in a continuous proportion along that line, then they will be traversed in the same interval of time. This part of the proof is the most complex because in its development Fermat follows an Archimedean method based on a double *reductio ad absurdum*. He then goes on to prove Galileo's assertion by constructing another *reductio ad absurdum* (Fermat, 1891–1912, Vol. II, pp. 267–276; Vol. III, pp. 302–309).

[47] Torricelli (1949), pp. 27 ff.

[48] In eadem ratione magnitudines dicuntur esse, prima ad secundam, et tertia ad quartam, cum primae et tertiae aeque multiplicia, a secundae et quartae aeque multiplicibus, qualiscunque sit haec multiplicatio, utrumque ab utroque vel una deficiunt, vel una equalia sunt, vel una excedunt; si ea sumantur quae inter se respondent. (Clavius, 1999, p. 209)

I take the Latin text of Euclid from Clavius' edition of the *Elements*, since Torricelli precisely owned an edition of Euclid, edited and commented on by Clavius. See the documents in Torricelli (1919–1944), Vol. IV, for a list of books possessed by Torricelli at the time of his death. See Giusti (1993) and Palmieri (2001) for discussions of problems to do with Euclid's definition of sameness of ratios in the Galilean school.

[49] 'Habitude' (Lat. *habitudo*) is standard terminology used to refer to a relation, both in the language of metaphysics and in that of theology in the period.

[50] The best edition of Torricelli's tract on proportions was published as an appendix in Giusti (1993), pp. 299–340.

any possible multiplication whatever among infinite possibilities, will always accord in excess, defect, or equalitzy?[51]

To find an answer, I will take a look at questions concerning relations, as they were discussed in the relevant literature of the time.

The whole edifice of geometry, Torricelli says, is erected on the doctrine of proportions, expounded by Euclid in the fifth and sixth books of the *Elements*.[52] Eternal geometrical truths are relations of similarity between ratios. Ratios are defined as habitudes of quantities. In the text of the *Elements* current in the sixteenth and seventeenth centuries, proportionalities are defined by Euclid as similarities of habitudes of quantities.[53] But proportionalities are not quantities, they are only relations. According to a traditional logical–metaphysical partition going back to Aristotle, relations of similarities are grouped together with relations of sameness and relations of equality, since they are all founded on unity. For, a relation of equality is founded on having one quantity, a relation of similarity is founded on having one quality, and a relation of sameness is founded on having one substance.[54] Since ratios (i.e., habitudes) are not quantities, there cannot exist a relation of equality among them, but at most a relation of similarity, which is often (linguistically, at least) conflated by both Euclid and Torricelli with a relation of sameness. Hence, ratios can only be similar to one another, never equal to one another. Torricelli's language is always very accurate. He never uses 'equality' or its cognates to refer to relations of ratios, or to proportionalities. He rigorously sticks to the similarity terminology. But what is similarity, then? Euclid does not tell us. Suárez proposes an illuminating thought experiment about similarity.[55] Imagine two white things. Now, either by the power of your imagination, or by assuming the absolute power of God, imagine taking away from the two things everything that is real except their being white. The two things will remain similar, and mind cannot conceive of anything else present in them except for their being white; that is, except for their having one form, one quality, whose *ratio* is the *ratio* of white (*ratio*, is here used in the sense explained above in the second section, not in the geometrical sense of ratio as habitude). Hence, Suárez concludes, to be similar is nothing more than to have qualities of the same *ratio*.

Does Euclid's equimultiple definition furnish a criterion to evaluate the similarity, or sameness, of two ratios, their having one quality? By no means, according to Torricelli. For, Torricelli argues, in order to define a subject in mathematics we cannot make use of any affection of the subject whatever, especially when such an affection is either difficult to grasp, or we have reason to doubt its very possibility. To clarify his position Torricelli puts forward the following simile. Suppose we define 'circle' as the plane figure such that, if any two straight lines within it intersect in any way, the rectangles formed by the four parts cut by the two intersecting straight lines will always be equal. This is certainly true of circles, as we know from Euclid. However, if we were to start our investigations about these sorts of figures from scratch, this definition of circle would not be a good one, since we could not be sure whether such a figure exists at all; that is, if there is a subject to which such an affection inheres. Even though a geometer could prove that such a subject can indeed exist, Torricelli reiterates, yet we could still have good reason to doubt whether the subject to which such an affection inheres is the same circle whose nature we already know.[56]

The moral is clear, according to Torricelli. In putting forward the equimultiple definition of proportionality, Euclid makes the same error. He defines a proportionality by an affection of the subject whose existence in a proportionality we have reason to doubt, that is, in a similarity of ratios. It is not at all clear whether the accord among the first and third equimultiples, in always alike being equal to, or alike exceeding, or alike falling short of, the second and the fourth equimultiples—whatever this multiplication may be—is at all possible among four proportional magnitudes. Finally, Torricelli argues, one cannot take refuge in saying that a geometrical definition is no more than an imposition of names, so that geometers are free to give the thing being defined the name they please. For, in this case, Torricelli says, if the equimultiple definition is no more than name imposition, then the whole doctrine of proportions will fall apart. If, by proportional magnitudes, I only need to form in my mind the concept of that eternal accord among the equimultiples, I will always doubt whether the ratio of the first quantity to the second is similar to the ratio of the third to the fourth.[57]

We now have the building blocks that allow us to assemble a tentative answer to our initial question, namely, what the motivations are behind Torricelli's refusal of Euclid's equimultiple definition.

The fact is that Euclid's equimultiple definition interferes with the categorical separation between quantity and quality. Quantities can be equal to one another. A relation of equality can be predicated only on quantities. Ratios are not quantities, but habitudes of quantities. Proportionalities are relations of similarity between habitudes. Habitudes can be similar to one another, not equal to one another. In effect Euclid's equimultiple definition is a criterion to quantify similarity. In other words, it is a criterion to transform the qualitative relation of similarity among habitudes (the having one quality) into a criterion to quantify similarity. It takes geometry and its qualitative similarity relations—namely, its truths—into the domain of quantification. This does not mean of course that geometry does not have commerce with quantities. The fact is that the eternal truths of geometry cannot be subjected to quantification. The eternal truths of geometry are relations of similarity among habitudes of quantities.

Thus, Torricelli goes on to rewrite the fifth and sixth books of the *Elements*. He does away with the equimultiple definition and needs to reconstruct the whole machinery of proportion theory

[51] Sed quidnam obscurius, et iustissima dubitatione dignius concipi unquam potest quam definitio sexte libri V Euclidis, ubi supponitur dari posse quatuor magnitudines, quarum aequemultiplicia iuxta quamlibet ex infinitis possibilibus multiplicationibus sumpta, ut ibi iubetur, semper in excessu, sive defectu, vel aequalitate conveniunt?

Torricelli refers to the equimultiple definition of proportionality as the sixth definition of Book V, which is today normally presented as the fifth, since at that time small differences in the order of presentation of Book V were current. See Giusti (1993), pp. 303.

[52] Ibid., pp. 301.

[53] Aside from the equimultiple definition of proportionality Euclid furnishes a second definition of proportionality as 'similarity of ratios [*similitudo rationum*]'. This second definition is nowadays considered to be spurious. It was expunged from the 'genuine' Euclidean text by Heiberg in his influential nineteenth-century edition of the *Elements*. See ibid., p. 10, and Euclid (1884), pp. 2–4.

[54] Suárez (1965), Vol. II, pp. 820.

[55] Ibid., pp. 782.

[56] Giusti (1993), pp. 302–303.

[57] ... si dicamus sextam illam definitionem nihil aliud esse praeter nominis quandam impositionem, quis non videat omnem opera perdi, omnemque huius doctrinae fructum interire? Quotiesqumque enim imposterum magnitudines proportionales sive proponi, sive demonstrari, sive quoqumque alio modo nominari audiam, numquam alium conceptum de illis in intellectu constituam praeter illum in definitione imperatum ... Caeterum ignorabo penitus quod magis ad rem attinet, nimirum an prima ad secundam sit ut est tertia ad quartam, hoc est an ratio quae inter primam et secundam est, vere et sine ulla dubitatione similis sit rationi quae inter tertiam et quartam intercedit. (Ibid., pp. 305–306)

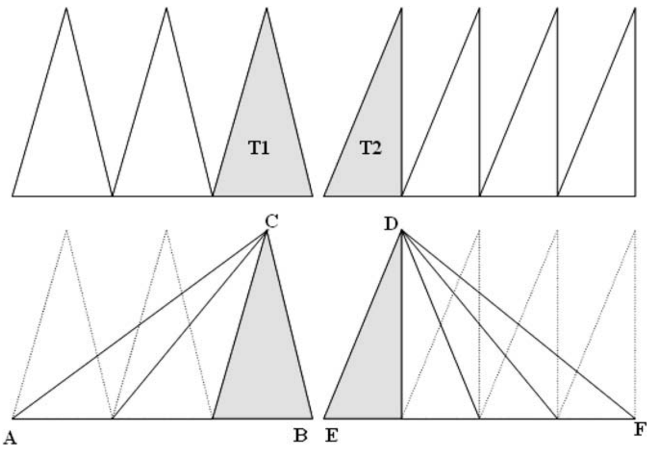

Fig. 2. Euclid's *Elements*, Proposition VI.1. Above: triangles T1 and T2 are to each other as their bases. Below: the assembled triangles. If base AB is greater than, less than, or equal to base EF, then triangle ABC is greater than, less than, or equal to triangle DEF.

by demonstrating the relevant propositions independently of the equimultiples, typically by means of *reductio ad absurdum* proofs. It will suffice to compare and contrast, by way of example, how Euclid and Torricelli go about demonstrating the first proposition of Book VI of the *Elements*. Proposition VI.1 states that triangles having the same height are to one another as their bases.

With reference to Figure 2, let us consider two triangles having the same height, T1 and T2. Proposition VI.1 asserts that T1 and T2 are to each other as their bases, that is, the ratio of T1 to T2 is the same as that of their bases. The first step of Euclid's proof is the construction of equimultiples of T1 and T2 and of their bases. We consider, for example, two triangles equal to T1 and three triangles equal to T2 (the choice of the number of multiples is totally arbitrary). We need to show that conditions given in the equimultiple definition hold true for the chosen equimultiples of triangles T1 and T2, and for those of their bases.

Since Euclid has proven in Book I, Proposition 38 that triangles having the same height and the same base are equal to each other, the multiples of T1 and T2 can be replaced with multiples having the same base and height but inclined in such a way as to have also the same vertices, C and D. Thus far we have constructed and cleverly assembled the equimultiples. We now need to compare them. This is easily done by looking at triangles ABC and DEF. Both of them have the same height. Thus, if their bases are equal to, greater than, or less than, each other, then the triangles themselves will be equal to, greater than, or less than each other (note that Book I, Proposition 38 guarantees the equality case, while the inequality cases are left unproven by Euclid). Therefore, triangles T1 and T2 are to each other as their bases.

Torricelli begins by showing that if, in a given triangle, the base is cut into any number of equal parts, then sums of triangles, constructed on corresponding sums of parts of the base, will be to one another as the sums of the bases (Figure 3, top). This is proven by simply recalling Euclid's Book I, Proposition 38, and by postulating that multiples of a quantity are to one another as the numbers of multiplicity (so, for instance, if Q is a quantity, then, nQ is to mQ as n is to m, with n, m integers).[58]

The core of Torricelli's proof of VI.1 is as follows (see Figure 3, bottom). If the ratio of triangle ABC to triangle CBD is not similar to that of their bases, AC, CD, let it be such that the ratio of the

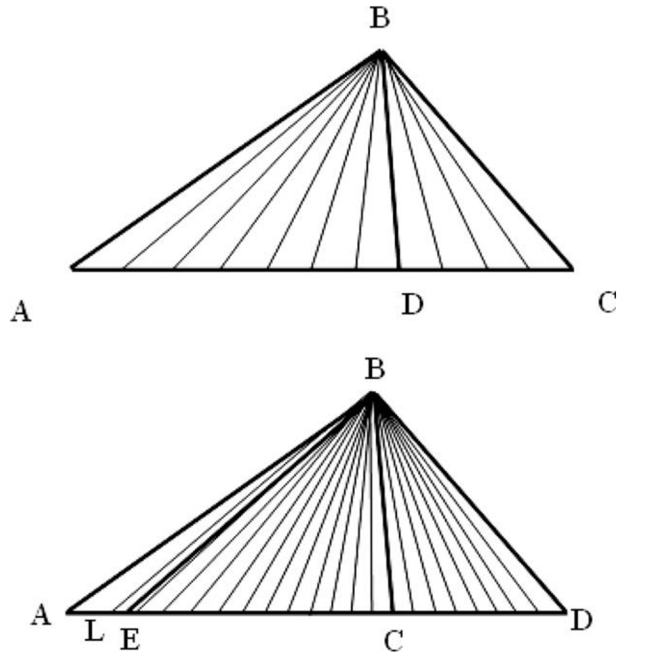

Fig. 3. Torricelli's diagrams for proposition VI.1. Above: a triangle whose base is divided into any number of equal parts. Below: two triangles on the same height, ABC, CBD.

triangles to each other is similar to that of another line, EC, smaller than AC, to line CD (we will not repeat Torricelli's reasoning for the second case of the double reductio argument, that is, if line EC is greater than line AC). Note that Torricelli carefully refrains from saying: 'if the ratio of the triangles is not that of their bases, then let it be *greater* or *smaller* than ... etc.'. For, this would imply that ratios, that is, habitudes, can be said to be smaller or greater than one another. But Torricelli has no criterion for comparing ratios with one another directly. Next, Torricelli bisects line CD, then he bisects half of it, and so forth, until he eventually gets a line, CI, that is smaller than AE.[59] Subsequently, Torricelli divides lines CD, AC into parts equal to line CI. Of course, there is in CD an integer number of parts equal to CI. On the other hand, on line AC, by starting the division from point C, it is certain that one of the points of division will fall within AE. Let such a point be called L, and let lines be drawn from point B to all points of division. Thus, line LC is greater than it should be in order for it to have the same ratio to line CD as line EC. Now, triangle LBC is to triangle CBD as line LC is to line CD (for the preliminary result, shown above). Therefore, triangle LBC is greater than it should be for it to have the same ratio to triangle CBD as the ratio of line EC to line CD. Therefore, *a fortiori*, triangle ABC is greater than it should be for it to have the same ratio to triangle CBD as the ratio of line EC to line CD. Which is absurd.

Euclid constructs the equimultiples of the magnitudes involved in VI.1 according to the equimultiple definition of proportional magnitudes. The latter definition quantifies the relation of similarity by subjecting the *having one quality* to the 'obscure' constraint of the eternal accord, as Torricelli says, of the equimultiples. Torricelli, having rejected the equimultiple definition, shows that the ratio of the triangles is similar to that of their bases by a double reductio. The proof ultimately stands on the postulate on which the preliminary result is based: namely, that multiples of a quantity are to one another as the numbers of multiplicity. This

58 Ibid., pp. 311–313.

59 That this is possible is proven by Torricelli at the beginning of the tract on porportions (ibid., pp. 311–312).

is the *having one quality* that defines similarity, and the model of which Torricelli finds in Euclid's *Elements*, Book VII, where Euclid expounds a second definition of proportional magnitudes valid only for integers.[60]

To sum up, eternal geometrical truths are expressed in the language of proportionality: that is, as relations of similarity. In Torricelli's philosophy of mathematics, relations of similarity, as all geometrical constructs, are beings of reason. The fabric of geometry, then, is a framework of beings of reason, such as triangles, circles, and habitudes of quantities, related to one another by other beings of reason, such as relations of equality between quantities, and relations of similarity between habitudes of quantities. Yet geometrical truths are eternal divine decrees expressed in the language of relations of similarity among habitudes of quantities. They are not accessible via the obscurity of the equimultiple definition of proportionality. Above all they cannot be subjected to quantification. To avoid fabricating chimeras the geometer's mind must steer clear of the equimultiple definition of proportionality.

3.3. Measure

Our last concern will be Torricelli's stunning discovery, by means of indivisibles, that an infinitely extended solid can occupy a finite volume (the cubature of a segment of an infinite hyperboloid). Torricelli's discovery has some interesting implications for his philosophy of mathematics. For, as we shall see, although it introduces a tension in the view that geometry is the science of abstracted quantity—the *science of abstraction*, as Torricelli puts it—it also hints at how Torricelli's philosophy of mathematics could meaningfully violate the Augustinian prohibition against the mind's begetting of eternal things.

The discovery impinges on the philosophical problem of the nature of continuous quantity. A Euclidean measure of continuous quantities was established with a proportionality, that is, a relation of similarity between two habitudes, the habitude of a finite continuous quantity to another finite continuous quantity, and the habitude of an integer to another integer. Torricelli's discovery alters the meaning of habitude and hence that of measure of quantity. It amounts to establishing a relation of similarity between two habitudes, the habitude of two non-homogeneous volumes, one finite, the other *infinitely* extended along one dimension, and the habitude of any two equal integers. It thus amounts to asserting a habitude between a finite volume and an infinitely extended volume. In the final analysis, it amounts to asserting that measure of continuous quantity is not habitude of *extensions*—regardless of whether they be finite or infinite extensions—but rather habitude of *locations*.

I will now illustrate my point about Torricelli's use of indivisbles in more detail.[61] Let us consider Figure 4.

Let EN and LD be two branches of a hyperbola whose asymptotes are AB, AC. Let FENBLDC be an infinitely extended solid generated by the rotation of the hyperbola around axis AB. Within the infinitely extended solid, Torricelli imagines infinite cylindrical surfaces, such as, for example, surface NLOI. Torricelli easily proves that, if we consider a cylinder, AHGC, whose base's radius is equal to diagonal AD (which Torricelli calls the semi-axis of the hyperbola: that is, the bisector of angle BAC), then any cylindrical surface whatever within the infinitely extended solid, such as, for example, NLOI, will be equal to its corresponding circle within the finite cylinder, such as, in this case, circle IM.[62] Hence, Torricelli argues, all the cylindrical surfaces taken together will be equal to all the circles taken together. Thus, the infinitely extended solid will be equal to the finite cylinder AHGC.[63]

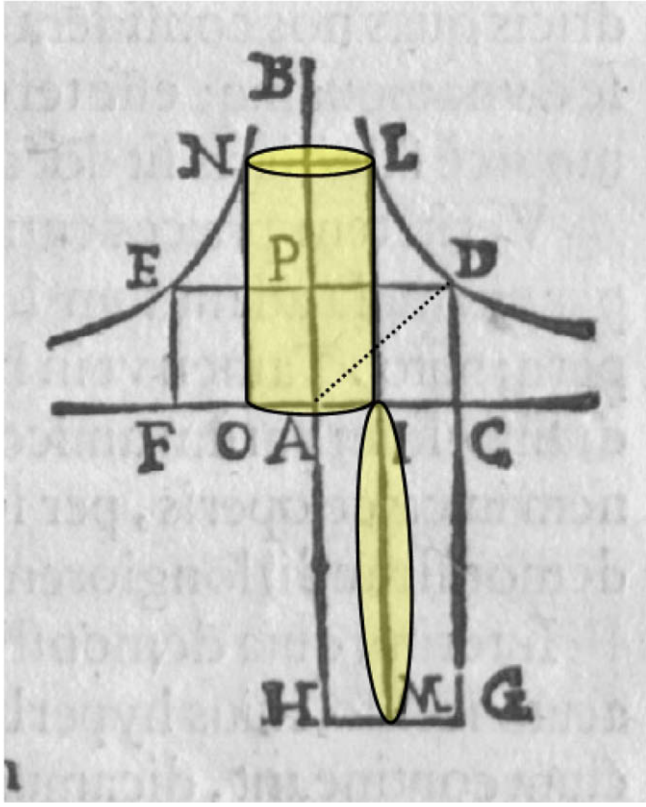

Fig. 4. The diagram accompanying Torricelli's proof by indivisibles that the volume of the infinitely extended solid is finite. FENBLDC is the infinitely extended solid generated by the rotation of the hyperbola around axis AB. I have added the bisector, line AD, cylindrical surface NLOI, and circle IM in order to visualize the basic elements that constitute the proof (from Torricelli, 1644, *De solido hyperbolico acuto*, p. 115).

The device that makes the proof work is the recognition that within the infinitely extended solid there is an infinitude of cylindrical surfaces (an example of Torricelli's so-called *curved indivisibles*), and that these cylindrical surfaces are all equal to indivisible slices of the finite cylinder AHGC. The important principle that emerges with this technique of proof is that the infinitely extended solid can be unpacked, so to speak, into countless surfaces according to the order of their locations imposed by the infinitely extended solid generated by the hyperbola, and that these surfaces can be re-stacked according to another order of

[60] . . . four magnitudes are proportional when the first is the same multiple of the second, or the same part, or the same parts that the third is of the fourth; or certainly, as we [i.e., Clavius] have added to the definition, when the first contains the second, or the second plus a part of it, or parts of it, in a way exactly equal to that in which the third contains the fourth, or the fourth plus a part of it, or parts of it.

I have taken the text again from Clavius' edition of the *Elements*. Cf. Clavius' extension of Euclid's Book VII, Definition 20 to ratios of rational quantities (not necessarily numbers), in Clavius (1999), p. 211.

[61] In my view, substantially the same result was independently reached by Torricelli's friend and fellow indivisibilist, Bonaventura Cavalieri—albeit in less spectacular manner, in that Cavalieri shows this result while demonstrating the quadrature of a finite figure, a parabolic segment—in his posthumously published *Exercitationes geometricae sex* (which Torricelli did not see, since both Cavalieri and Torricelli died within a very short time of each other, towards the end of 1647). See Cavalieri (1647), pp. 81–84.

[62] See Torricelli (1644), pp. 113–115: *De solido hyperbolico acuto*, for the details of the preliminary results that are needed before the cubature can be carried out.

[63] Ibid., pp. 115–116.

locations, the order of the horizontal finite cylinder. In other words, every cylindrical surface within the infinitely extended solid is located in a place that is determined by the infinitely extended solid generated by the hyperbola, whereas the same surface, once it has been recognized that it is equal to a circle within the finite cylinder, can be relocated in a place that is determined by the finite cylinder. There are interesting implications.

The volume of the infinitely extended solid and the volume of the finite cylinder are the same. Thus, a habitude, in this case a relation of equality, between a finite volume and an infinitely extended volume can be established. Volume is three-dimensional extension, but the proof by indivisibles suggests that three-dimensional extension is ultimately reducible to a relation between locations of infinite indivisible surfaces. A measure of continuous quantity, a volume, is not a habitude of *extensions*—regardless of whether they be finite or infinite extensions—but rather a habitude of *locations* of infinite indivisible surfaces.

A tension looms large in Torricelli's view that geometry is the science of abstracted quantity—the *science of abstraction*. For, according to traditional logical and metaphysical conceptions, current in the intellectual milieu in which Torricelli was educated, abstracted quantity is the subject matter of geometry. However, the species of continuous quantity are only three: namely body, line, and surface. Location is not a species of quantity.[64] Thus, the question arises whether geometry can embrace a more universal science of relations between new objects, new beings of reason, such as, for example, abstracted locations. This question must be left open for the time being. We have reached a threshold, so to speak, within Torricelli's philosophy of mathematics, a limit determined in part by the status of our present knowledge, which could be improved only by an extensive survey of the manuscript heritage not fully exploited until now.

Finally, I conclude on a note that may clarify Torricelli's stance on the mind's contribution to the creation of eternal truths. Torricelli's philosophy of mathematics could violate the Augustinian prohibition against the mind's begetting of eternal things. Extension is an accident of created being; it is not an accident of uncreated being. The proof by indivisibles of the equality of the infinitely extended solid and the finite cylinder opens up a new vista. It is much more, I think, than an astounding geometrical feat, showing the possibility of a habitude between an infinite continuous quantity and a finite continuous quantity. It brings the finite and the infinite closer to each other in a subtle way. Extension seems to evaporate as a feature of created being. If extension is habitude of locations, that is, relation of locations, then the distance between created and uncreated being seems to be annihilated. For, every educated Catholic in Torricelli's counter-Reformation Italy would have known that God is three real relations that constitute and distinguish three persons. The tricky point was that, for Torricelli, geometrical relations are not real beings but beings of reason. However, the distinction between *beings of reason* and *real beings* presupposes that a sharp line can be drawn between the mind and the external world. It presupposes a dualistic stance that seems to be alien to Torricelli's radical take on his Thomistic background. In Torricelli's philosophy of mathematics, the distinction between *beings of reason* and *real beings* is blurred when it comes to mathematical constructs.

4. Conclusion: Torricelli's radical mathematical Thomism

To sum up, I will furnish a suggestion as to how, in the light of the debate sketched in the second section, it is possible to reconcile the apparent contradictions in the statements by Torricelli from which I started.

I once again begin with a crucial passage from Aquinas. In the wake of Augustine, Aquinas broaches the question of the status of the ideas present in the mind of God. There certainly are ideas in the mind of God since God created single beings according to their own natures: that is, their own reasons (*rationes*). The latter are the ideas present in the mind of God. Therefore, in the mind of God there is a plurality of ideas.[65] However, Aquinas raises the possible objection that, since all nouns referring to divine things must either be essential, such as *God*, or personal, such as *father*, or notional, such as *generating*, it follows that *idea* must refer to an essence, that is, be an essential noun. For, it cannot be either personal or notional, in that it does not refer to all of the three persons of the trinity. Therefore, multiple ideas cannot be present in the mind of God.

Aquinas' position favors the first option (a plurality of ideas). It is as follows. There can be two types of plurality. In the sense of a plurality of things, there cannot be a plurality of ideas in God. However, idea means exemplar: that is, a thing that is imitated. There is indeed one thing that is imitated by everything else, and this one thing is God's essence. Hence, there can be a plurality in another sense, that is, according to human intelligence. For, though every thing imitates the divine essence, not every thing does so in the same way, but each thing imitates in a different grade. Thus, the divine essence, insofar as it is imitated by a particular creature, is the *ratio* and the idea of this creature. In this way, there is a plurality of ideas in God.[66]

Back To Torricelli. Geometrical truths are eternal divine decrees. The geometer's mind forms beings of reason that are manipulated in order to discover geometrical truths. But the truths being discovered are not present in the mind of God as a plurality of ideas. In this sense, they are not eternal until the mathematician completes them with a creative act of the mind, which explicates the unity of God's essence. In the language of John of St. Thomas, the mathematician's creative act becomes a *mensura informans* the being of God.

I have labeled Torricelli's philosophy of mathematics *radical mathematical Thomism*—placing it in the context of Thomistic patterns of thought. Torricelli's radicalism is opposed to Suárez' arguments against Aquinas and the Thomists who follow de Sylvestris, such as John of St. Thomas. It is radical in the sense of throwing into sharp relief the responsibility of the mathematician, hence of a created being, for participating *completive* in God's act of promulgating eternal decrees.

References

Anderson, T. (1969). Mathematics in Aquinas. *The New Scholasticism*, *43*, 555–576.

Aquinas, T. (n.d.). *Corpus Thomisticum*. http://www.corpusthomisticum.org/. (Accessed 20 September 2007)

[64] Suárez (1965), Vol. II, pp. 529 ff., discusses continuous quantity at length.

[65] Aquinas (n.d.), *Quaestiones de quodlibet*, Quodlibet IV, Question 1, Argument 1:

> Dicit enim Augustinus, in libro LXXXIII quaestionum, quod Deus singula propriis rationibus creavit, et alia ratione hominem, et alia ratione equum. Sed rationes rerum in mente divina dicuntur ideae, ut patet per Augustinum, ibidem. Ergo sunt plures ideae.

[66] Dicendum, quod duplex est pluralitas. Una quidem est pluralitas rerum; et secundum hoc non sunt plures ideae in Deo. Nominat enim idea formam exemplarem. Est autem una res quae est omnium exemplar: scilicet divina essentia, quam omnia imitantur, in quantum sunt et bona sunt. Alia vero pluralitas est secundum intelligentiae rationem; et secundum hoc sunt plures ideae. Licet enim omnes res, in quantum sunt, divinam essentiam imitentur, non tamen uno et eodem modo omnia imitantur ipsam, sed diversimode, et secundum diversos gradus. Sic ergo divina essentia, secundum quod est imitabilis hoc modo ab hac creatura, est propria ratio et idea huiusmodi creaturae; et similiter de aliis: unde secundum hoc sunt plures ideae. (Ibid., *Quodlibet IV*, Question 1)

Aquinas, T. (1918). *Summa contra gentiles, Tome I* (Leonine ed.). Sancti Thomas Aquinatis Opera Omnia, 13. Rome: Garroni.

Augustine (n.d.). *S. Aurelii Augustini opera omnia*. http://www.augustinus.it/latino/index.htm. (Accessed 20 September 2007)

Bortolotti, E. (1927–1928). Le prime rettificazioni di un arco di curva nella memoria 'De infinitis spiralibus' di Torricelli. *Rendiconto delle sessioni della R. Accademia delle Scienze dell'Istituto di Bologna, 32*, 127–139.

Bortolotti, E. (1928). *I progressi del metodo infinitesimale nell'opera geometrica di Torricelli*. Bologna: Zanichelli. (Offprint from *Periodico di matematiche*, *8*, 19–59)

Bortolotti, E. (1939). L'opera geometrica di Torricelli. *Monatshefte für Mathematik und Physik, 48*, 457–486.

Cavalieri, B. (1647). *Exercitationes geometricae sex*. Bologna: Monti.

Clavius, C. (1611–1612). *Christophori Clauii Bambergensis e Societate Iesu Opera mathematica V tomis distributa. Ab auctore nunc denuo correcta et plurimis locis aucta* (5 vols.). Mainz: Eltz.

Clavius, C. (1999). *Commentaria in Euclidis Elementa geometrica*. Hildesheim: Olms-Weidmann. (Facsimile ed. of Clavius, 1611–1612, Vol. I)

De Gandt, F. (1987). Les indivisibles de Torricelli. In idem (Ed.), *L'œuvre de Torricelli: Science Galilèenne et nouvelle géométrie* (pp. 147–206). Nice: Publications de la Faculté des lettres et sciences humaines de Nice/Paris: Les Belles Lettres.

De Gandt, F. (1992). L'évolution de la théorie des indivisibles et l'apport de Torricelli. In M. Bucciantini, & M. Torrini (Eds.), *Geometria e atomismo nella scuola Galileiana* (pp. 103–118). Florence: Olschki.

Euclid. (1884). *Euclidis opera omnia* (I. L. Heiberg, & H. Menge, Eds.) (8 vols.) (Vol. II). Leipzig: Teubner.

Fermat, P. (1891–1912). *Œuvres* (P. Tannery, & C. Henry, Eds.) (5 vols.). Paris: Gauthiers-Villars.

Festa, E. (1992). La notion d''agrégat d'indivisibles' dans la constitution de la cinématique Galiléenne: Cavalieri, Galilée, Torricelli. *Revue d'histoire des sciences, 45*, 307–336.

Galilei, G. (1890–1909). *Le opere di Galileo Galilei. Edizione nazionale* (Antonio Favaro, Ed.) (20 vols.). Florence: Barbèra.

Galilei, G. (1988). *'Tractatio de praecognitionibus et praecognitis' and 'Tractatio de demonstratione'* (W. F. Edwards, & W. A. Wallace, Eds.). Padua: Antenore.

Galluzzi, P., & Torrini, M. (1975–1984). *Le opere dei discepoli di Galileo Galilei. Carteggio* (2 vols.). Florence: Barbèra.

Giusti, E. (1993). *Euclides reformatus: La teoria delle proporzioni nella scuola Galileiana*. Torino: Bollati-Boringhieri.

Grendler, P. F. (2002). *The universities of the Italian Renaissance*. Baltimore & London: Johns Hopkins University Press.

John of St. Thomas. (1933–1937). *Cursus philosophicus Thomisticus* (B. Reiser, Ed.) (3 vols.). Turin: Marietti.

Leftow, B. (1990). Aquinas on time and eternity. *The New Scholasticism, 64*, 387–399.

Mancosu, P., & Vailati, E. (1991). Torricelli's infinitely long solid and its philosophical reception in the seventeenth century. *Isis, 82*, 50–70.

Maurer, A. (1993). Thomists and Thomas Aquinas on the foundation of Mathematics. *Review of Metaphysics, 47*, 43–61.

Palmieri, P. (2001). The obscurity of the equimultiples: Clavius' and Galileo's foundational studies of Euclid's theory of proportions. *Archive for History of Exact Sciences, 55*, 555–597.

Society of Jesus (1616). *Ratio studiorum*. Rome: Collegium Romanum.

Suárez, F. (1965). *Disputationes metaphysicae*. Hildesheim: G. Olms. (Facsimile ed. of *R. P. F. Suárez e Societate Jesu opera omnia* [C. Berton, Ed.] [28 vols.]. Paris: Vivès, 1856–1878)

Torricelli, E. (1644). *Opera geometrica*. Florence: Landi e Massa.

Torricelli, E. (1919–1944). *Opere di Evangelista Torricelli* (G. Loria, & G. Vassura, Eds.) (4 vols. in 5 tomes). Faenza: G. Montanari.

Torricelli, E. (1949). *La memoria di Evangelista Torricelli sopra la spirale logaritmica riordinata e completata* (A. Agostini, Ed.). Livorno: Accademia Navale.

Torricelli, E. (1955). *De infinitis spiralibus* (Ettore Carruccio, Ed.). Pisa: Domus Galilaeana.

Torricelli, E. (1975). *Opere scelte* (L. Belloni, Ed.). Turin: UTET.

Torricelli, E. (2007). *Manoscritto 138*. Discepoli di Galileo, XXVIII. Firenze: Biblioteca Nazionale Centrale, Collezione Galileiana.

Vallius, P. (1622). *Logica* (2 vols.). Lyon: Rouille.