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# Superposition: on Cavalieri's practice of mathematics

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**Abstract** Bonaventura Cavalieri has been the subject of numerous scholarly publications. Recent students of Cavalieri have placed his geometry of indivisibles in the context of early modern mathematics, emphasizing the role of new geometrical objects, such as, for example, linear and plane indivisibles. In this paper, I will complement this recent trend by focusing on how Cavalieri manipulates geometrical objects. In particular, I will investigate one fundamental activity, namely, *superposition* of geometrical objects. In Cavalieri's practice, superposition is a means of both manipulating geometrical objects and drawing inferences. Finally, I will suggest that an integrated approach, namely, one which strives to understand both objects and activities, can illuminate the history of mathematics.

## 1 Introduction

Bonaventura Cavalieri (1598?–1647) has been the subject of numerous scholarly publications.[1] Most students of Cavalieri have placed his geometry of indivisibles

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[1] A number of studies have focused on specific aspects, often on the question of the epistemological status of indivisibles as new mathematical objects on the seventeenth-century scene. Cf., for example, Agostini (1940), Cellini (1966b), Vita (1972), Koyré (1973), Terregino (1980), Terregino (1987) and Beeley (1996, pp. 262–271). Other studies have tackled issues concerning Cavalieri's biography, for instance, Piola (1844), Favaro (1885); or the genesis of Cavalieri's *Geometry*, for example, Arrighi (1973), Giuntini (1985), and Sestini (1996/97). More rarely, historians have focused on historiography; cf., for example, Agazzi (1967), on L. Lombardo–Radice's translation into Italian, with commentary, of Cavalieri's *Geometry* (Cavalieri 1966).

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in the context of early modern mathematics, emphasizing the role of new geometrical objects, such as, for example, linear and plane indivisibles. Much light has been shed on Cavalieri's processes of demonstration and discovery by scholars who have investigated Cavalieri's method from the perspective of historical contextualization.[2] The form that this contextualization has tended to take, more recently, is an emphasis not only on the new objects which Cavalieri employed, namely, linear and plane indivisibles, but also on his infinitary techniques of proof and their Greek roots.[3] It is not my goal to repeat the work already done by others, notably, Enrico Giusti and K. Andersen, whose studies have illuminated the methods of indivisibles employed of Cavalieri.[4]

In this paper, I set myself the modest task of complementing Cavalieri scholarship by focusing on how Cavalieri manipulates geometrical objects. In particular, I will investigate Cavalieri's activity of *superposition* of geometrical objects. Cavalieri employed superposition flexibly as a tool for proof. However, superposition created some puzzles that Cavalieri was at pains to resolve, and was strongly objected to by the Jesuit mathematician, Paul Guldin (1577–1643). The dispute gave Cavalieri the opportunity to reflect, toward the end of his life, on the activities underlying the manipulation of geometrical objects, and to come to terms with the implications of superposition.

*Outline of the paper*. In Sect. 2, by way of preparing the reader for Cavalieri's uses of superposition, I will present a sketch of various views about superposition that can be found in the early modern period before Cavalieri, and which, as we shall see, would have been familiar to Cavalieri and Guldin. In Sect. 3, I will offer some analysis of superposition in Cavalieri, and present some of its diverse uses. In Sect. 4, I will discuss the controversy with Guldin, focusing on both Guldin's objections to superposition and Cavalieri's response. Finally, in Sect. 5, I will draw some conclusions.

## 2 Superposition in the early modern period before Cavalieri

In reference to Euclid's *Elements*, Book I, Proposition 4, Federico Commandino (1509–1575)—the celebrated Renaissance editor and translator of Euclidean and Archimedean works—noted that "[t]his mode of proof, which is carried out by superposing figures to one another, aside from being approved by Proclus,[5] a great expert in mathematics, is quite common among mathematicians, given that Archimedes used it not only with plane figures, as in the book on centers of gravity of plane figures, but also with solid figures, as in the book on conoids and spheroids".[6]

Consider Fig. 1. Euclid's Proposition I.4 states that triangles, having two equal sides and equal angles contained between the equal sides, are equal, and that they

[2] Giusti (1980), Andersen (1985), De Gandt (1992), Festa (1992), Malet (1996, pp. 11–20), and Muntersbjorn (2000)

[3] Mancosu (1996, pp. 34–24).

[4] Giusti (1980) and Andersen (1985).

[5] Proclus was a fifth-century CE philosopher and commentator on Book I of Euclid's *Elements*.

[6] Euclid (1575, 10 verso).

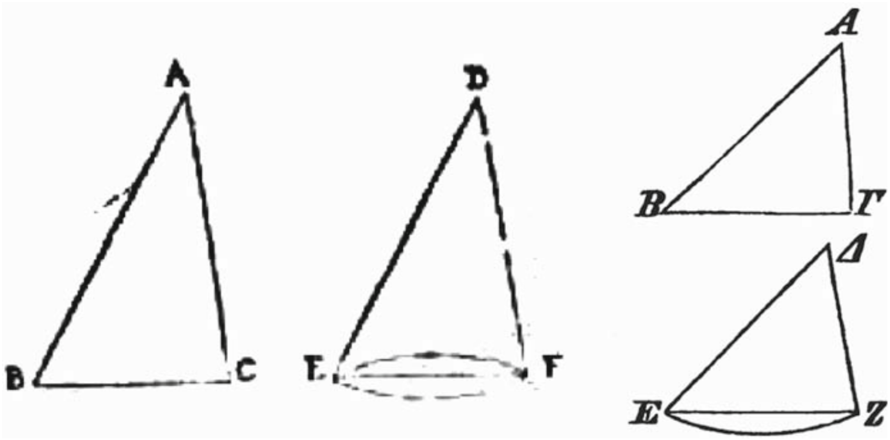



**Fig. 1** Euclid's Proposition I.4 states that triangles, having two equal sides and equal angles contained between the equal sides, are equal, and that they have the third side and the other angles equal (Euclid 1883, p. 17). On the left, the figure from the Commandino edition (Euclid 1575, 10 verso). On the right, the slightly different figure from the Heiberg edition

have the third side and the other angles equal.[7] Commandino has hinted at Proclus. Proclus' relevant remarks on this issue are as follows. I have excerpted them from the Renaissance Latin rendition by Francesco Barozzi (1537–1604), the edition which would have been accessible to Commandino and Cavalieri. "A base is said to be equal to another base, and a straight line to another straight line, if the conjunction of their ends makes the whole congruent[8] with the whole. For, any straight line is congruent to any straight line, but they are equal when their ends are congruent with each other. A rectilinear angle is said to be equal to another rectilinear angle if, one of the two sides including the angle being placed over one of the two sides including the other angle, the remaining side is congruent with the other remaining side".[9] Further on, Proclus comments that, as for the proof, from the equality of the sides, of the angles, and of the bases, the congruence of the sides, of the angles, and of the bases follows, and therefore the congruence of the triangles follows, and thus their equality. The cause of the proof, Proclus continues, seems to lie in equality considered in things which are of the same species. For, first, things that are congruent with one another are equal to one another (according to Euclid's *Elements*, common notion 7). The latter assertion, Proclus says, is true unconditionally and need not be limited in any way whatever. Second, things that are given equal to one another are congruent with one another, even though—Proclus warns us—this is not true unconditionally but only of things that are similar to one another, such as straight lines, arcs of circumferences of the

7 Cf. Thomas Heath's general comments on I.4, in Euclid (1956, I, 248–250), and Ian Mueller's analysis of the proof, in Mueller (2006, pp. 21–23), according to which Proposition I.4 "is certainly not deducible from Euclid's first principles as they stand (ibid., p. 23).

8 Barozzi's choice of the Latin "congruere" (Proclus 1560, p. 135) is also followed by Heiberg in his edition of Euclid's *Elements* (Euclid 1883, p. 19).

9 Proclus (1560, p. 135).

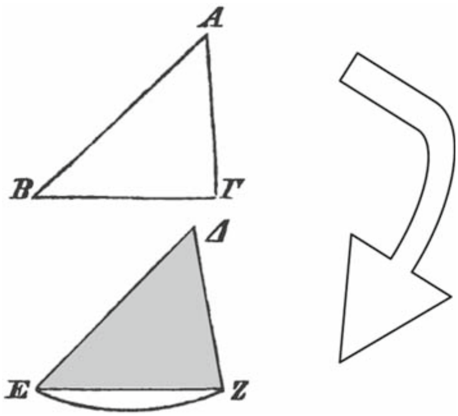



**Fig. 2** The congruence of wholes. Imagine transferring the above triangle over the one below. The congruence of the two triangles, the grey area, pops up in the diagram

same circles, and angles included by similar lines similarly placed.[10] Hence, Proclus concludes, the proof structure of *Elements* I.4 is the following. Two sides are equal to two sides and the angle included in them is equal to the angle included in the other two sides; therefore these things are all congruent (*conveniunt*) with one another. Thus, the two bases will be congruent with each other. But if all things are congruent with one another, they will also be equal to one another.

Proclus seems to suggest that, by partly assuming and partly deducing the congruence of all the objects that somehow include, or are included by, the triangles, i.e., the sides, and the angles, Euclid succeeds in showing the congruence of the triangles themselves, and therefore their equality. In other words, Proclus seems to suggest that once the congruence has been shown of all the objects including, or included by, two wholes, the geometer has the right to deduce the congruence of the wholes themselves, and hence their equality.[11]

Crucially, in my view, congruence of the wholes is not proven by Euclid as a conclusion following the deductive steps of the argument, but is being read straight off the diagram after showing the congruence of all the relevant objects (see Fig. 2).

Even though Euclid concludes the proof by saying "…therefore the whole triangle ABΓ is congruent with the whole triangle ΔEZ", the congruence of the two triangles—emphasized by the gray area in the figure—just pops up in the diagram by superposition of the relevant objects.

The French mathematician, Jacques Peletier (1517–1582), published an edition of the first six books of Euclid's *Elements*, in 1557, in which he vehemently attacked Euclid's mode of argumentation by superposition, exemplified by Proposition I.4.[12] "This [i.e., Euclid's proof of I.4] is the demonstration commonly taught by all inter-

10 Proclus (1560, p. 137).

11 I use the word "object" to avoid the word "part" since, according to Euclid, figures are comprehended by sides, an angle is an inclination (Greek, κλίσις, or *inclinatio*, in Heiberg's Latin rendition, or *inclinatione*, in Commandino's Italian), and therefore neither sides nor angles can be said to be parts of triangles in a Euclidean sense.

12 Cf. Thomas Heath's brief summary of Peletier's analysis of I.4, in Euclid (1956, I, 249).

preters, if one could call it a demonstration", exclaims Peletier.[13] Peletier argues that it is impossible to prove Proposition I.4, at that early stage in the *Elements*, and that ultimately Proposition I.4 can only be accepted as being per se evident, or rather as a sort of definition. For, Peletier says, "I cannot think of two equal angles unless I conceive what it means for two angles to be equal".[14] Peletier further elucidates his perplexity as follows. "Nobody could explain more perspicuously the equality of angles than by saying that two angles become equal to each other when the two sides containing an angle become equal to the two sides containing the other angle, while the bases which connect the sides also become equal. For, an angle is as much as the aperture, or expansion (*diductio*), of the two lines which contain it, but this [aperture] is as much as the base, that is, the line connecting the two sides".[15]

Peletier thinks that angles are as much as their "aperture, or expansion", namely, I believe, as much as the extended *figure* which pops up when one connects the two sides with a base, thus delimiting a finite *figure* included by boundaries. In the final analysis, Peletier asks, why did Euclid present Proposition I.4 as a theorem? The fact is, Peletier answers, that Proposition I.4 has a dual nature, partly principle, partly theorem. It is a principle insofar as it consists of a common judgment of mind (*quod in communi animi iudicio consisteret*), but it is a theorem insofar as it specifically compares triangles with triangles. Hence, Peletier concludes that Euclid placed Proposition I.4 among the theorems because, as a principle, it would not have been as simple and as naked as a principle should be.[16] To Peletier, the superposition of figures is somewhat mechanical, while only understanding (*intelligere*) is proper to the mathematician.[17] Thus, in his view, superposition has to be rejected. It is the act of mind's understanding that is required to grasp Proposition I.4, certainly not a mechanical proof.

The Jesuit mathematician, Christoph Clavius (1538–1612), responded to Peletier's attack on the validity of Euclid's proof of Proposition I.4. Clavius argued that Peletier did not understand the way in which mathematicians employed superposition. "Superposition is not a mechanical activity", says Clavius, "but a mental activity, an act of reason and of the intellect".[18] In theorems, Clavius notes, it is because one assumes the equality or inequality of magnitudes as known that the intellect can, without hesitation, recognize whether one magnitude will or will not exceed another magnitude when the two have been mentally superposed. Next Clavius goes on to castigate Peletier for proposing to assume Proposition I.4 as a principle instead of accepting Euclid's demonstration. For, Clavius concludes, "there is no doubt that Peletier's proposal is to be rejected on the following grounds. If Peletier is right in assuming as a principle that two angles become equal to each other when the two sides containing an angle become equal to the two sides containing the other angle, then, such figures as

13 Euclid (1557, p. 15).

14 Euclid (1557, p. 16).

15 Euclid (1557, p. 16).

16 Euclid (1557, p. 16).

17 Euclid (1557, p. 16).

18 Clavius (1999, p. 121).

squares and rhombs would all be equal if their sides are equal".[19] Ultimately, Clavius says, "if Proposition I.4 can be taken as a self-evident truth, in no need of demonstration, on the grounds of that shaky principle, it follows that all the propositions which depend on I.4 will fall apart, and together with I.4 the whole of geometry will fall apart".[20]

Let us note that Clavius seems to misquote Peletier's principle by forgetting that Peletier simply claims that two angles become equal to each other when the two sides containing an angle become equal to the two sides containing the other angle, *while the bases which connect the sides also become equal*.[21] Clavius is certainly right in arguing that mathematicians do not perform practically the mechanical act of superposition of figures. However, neither does Peletier think that this is the case. The crux of the matter seems rather to be that, while Peletier is uneasy with the fact that a crucial step of a demonstration, the superposition of figures, relies on the activity of reading off the diagram—that is, seeing in the diagram the popping up of something new—, Clavius finds it unproblematic because one assumes the equality or inequality of magnitudes as known, and then the intellect can recognize whether one magnitude will or will not exceed another magnitude when the two have been mentally superposed. As we shall see, Cavalieri will use superposition as an act of reading off the diagram, not by first assuming the equality or inequality of magnitudes as known.

Both Clavius and Peletier were quoted by Cavalieri in response to Guldin, who, against Cavalieri's use of superposition, had in turn quoted Christoph Grienberger (1564–1636), Clavius' successor in the chair of mathematics at the *Collegium Romanum*.[22] We shall see more about this fascinating controversy between Guldin and Cavalieri in the fourth section of this paper. For our present purposes, it suffices to say that Guldin also mentions Archimedes' use of superposition in the work *On conoids and spheroids*.[23] This leads us to a second crucial example of superposition, to be found this time in the Archimedean tradition, namely, in Proposition 20 of Archimedes' *On conoids and spheroids*.

Proposition 20 states that if any spheroid, namely, any solid obtained by rotating an ellipse around an axis, is cut by a plane through the center, then, both the spheroid and its surface will be divided into two equal parts. Archimedes' proof proceeds by means of two superpositions. Consider the diagram above (Fig. 3). In the case of the plane cutting the spheroid passing through an axis or being perpendicular to an axis the theorem is trivially true. In the general case, in which any plane whatever cuts the spheroid, consider two spheroids, one on the left, and one on the right equal and similar to the one on the left (according to Archimedes, two spheroids are similar to each other if their axes—defined by Archimedes as the fixed straight lines around which

[19] Clavius (1999, p. 121).

[20] Clavius (1999, p. 121).

[21] Further, one has a sense that, in rebutting Peletier, Clavius is emotionally responding to a challenge that strikes at the heart of the fundamental activity of superposing figures. This is signaled by Clavius' tone and Latin prose unusually becoming rather inordinate in this long and convoluted passage (Clavius 1999, p. 121).

[22] Cavalieri (1647, p. 215).

[23] A rare case, to be sure. Cf. Attilio Frajese's note, in Archimedes (1974, p. 279).

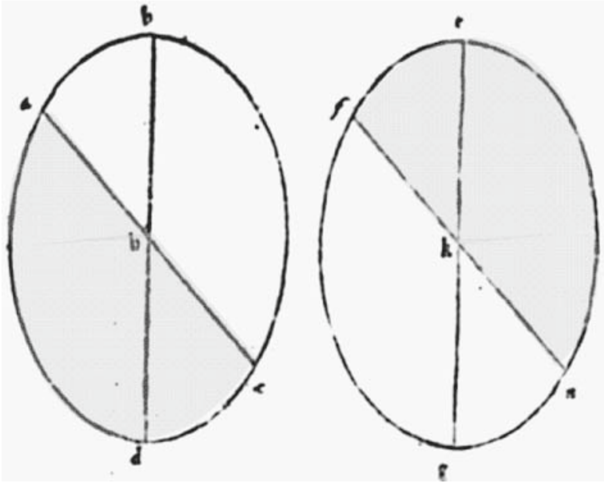



**Fig. 3** The diagram accompanying *Proposition 20* of Archimedes' *On conoids and spheroids*. This numbering of the propositions of *On conoids and spheroids* is according to Guldin's quotation, and to the 1615 Rivault edition (Archimedes 1615) used by Guldin, and also to the numbering of the 1544 Hervagius and 1558 Commandino editions (Archimedes 1544, and Archimedes 1558), but it is *Proposition 18*, according to the numbering of the 1974 Italian translation by A. Frajese (Archimedes 1974). The picture is from Archimedes 1544, 76. I have shaded two equal half portions of the spheroids, left and right, for clarity of exposition

the ellipse rotates—are in the same ratio of their diameters—defined by Archimedes as the straight lines perpendicular to the axes). The first superposition, then, shows the spheroid on the right to be congruent with the one on the left by translating the one on the right and placing the two vertical axes over each other, so that the two oblique straight lines, **fn**, **ec**—intersections of the planes cutting the spheroids with the spheroids themselves—will also be congruent. The second superposition shows the spheroid on the right to be congruent with the one on the left, by rotation and translation, that is, by making the vertical axes be congruent with each another, so that the two oblique straight lines, **fn**, **ec**- intersections of the planes cutting the spheroids with the spheroids themselves—will be congruent after rotation, and the two gray sections in the above picture will eventually overlap. Thus, Archimedes concludes, since the same segment of spheroid is congruent with both segments, it is clear that these two segments cut by the plane are equal to each other (and for the same reasons, their surfaces will equal to each other).[24]

We may notice that, whereas in I.4 Euclid proves by reductio ad absurdum that once the superposition of the two triangles has been made the bases of the triangles must also be congruent (a detail which I have not dwelled on in my discussion above), here Archimedes simply superposes the spheroids, and asks the reader to read off the diagram the congruence of the two portions in both cases. If we look at the section represented in the diagram, after the superposition has been made, we realize the congruence of the axes and of the straight lines resulting from cutting the spheroids

[24] Cf. the text of Proposition 20 according to Commandino's Latin edition, in Archimedes 1557, 37 verso–38 recto, and the Greek text and the Latin version by Heiberg, in Archimedes (1880, p. 370–375).

with the plane, because we know they are equal, but nothing is said by Archimedes concerning the curves joining the ends of the oblique straight lines, **fn**, **ec**.

Guldin marvels at the lack of comments by Commandino on Proposition 20 of *On conoids and spheroids*, given—says Guldin—the nature of the proof by superposition, and instead quotes an objection to the proof by David Rivault (1571–1616), editor of another edition of Archimedes appeared in 1615. Along not too dissimilar lines from Peletier's objection to *Elements*' I.4, Rivault argues that the force of the proof is wholly and solely due to the definition of spheroid by rotation.[25] He then goes on to reformulate the proof according to his own strategy by rotating only a portion of the ellipse determined by the oblique straight line (such as one of the gray areas in the above diagram). Guldin approves of Rivault's own proof strategy, saying that it is based on more intrinsic principles than superposition, and next goes on to give yet another slightly different version of it.[26] Cavalieri, in response to Guldin, counter-argues that proofs by superposition generate "true science", and that they rely on a most evident principle, namely, *things that are congruent to one another are equal to one another*.[27] Indeed, Cavalieri continues, if one eschews proofs by superposition of the type given by Euclid for I.4, then, one must also reject the most fundamental of Archimedean demonstrative techniques, namely, inscription and circumscription of figures, which, Cavalieri says, appears to be a certain kind of superposition (*quae videtur esse quaedam superpositio*).[28] In conclusion, Cavalieri exclaims, what could indicate equality to us more evidently than congruence (…*quid nobis apertius indicare potest aequalitatem quam congruentia*)? For, as Aristotle says, if what is one in substance is also equal in quantity, congruence itself will consequently be a most exact balance of equality, because, then, out of two quantities one will be made in a certain way.[29]

Euclid's proof of I.4—especially in the light of Proclus' comment—and Archimedes' proof of Proposition 20 of *On conoids and spheroids* exemplify the type of mathematical activity that I see important in later developments concerning Cavalieri's use of indivisibles. In the fourth section of this paper, I will discuss the intriguing objection raised by Guldin to Cavalieri's principles of indivisibles, and of course Cavalieri's response, but before that we need to consider how the activity of superposing geometrical objects shapes Cavalieri's practice of geometry, to which I now turn.

## 3 Cavalieri's activity of superposition

In this section, I will discuss a few examples of superposition taken from Cavalieri's geometrical works in order to show how the activity of superposing geometrical objects informs Cavalieri's practice of geometry.

25 Guldin (1640–1641, p. 315).

26 Guldin (1640–1641, pp. 316–317).

27 Cavalieri (1647, p. 215).

28 Cavalieri (1647, p. 215).

29 "...si enim, ut inquit Aristotiles, quod est unum in substantia, illud est aequale in quantitate, erit sane exactissima trutina aequalitatis ipsa congruentia, cum ex duabus quantitatibus tunc fiat quodamodo una" (Cavalieri 1647, p. 216).

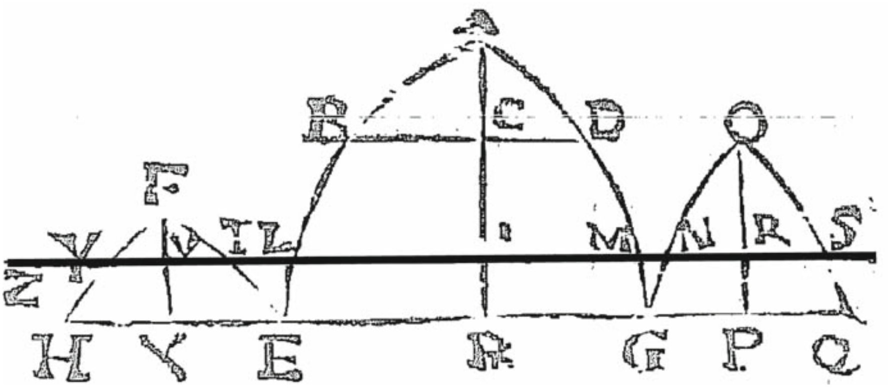



**Fig. 4** *All the lines* of plane figures have ratios to one another. The portion of figure AEG higher than OP, the height of figure OGQ, that is, portion BAD, is removed and translated into position FHE (Cavalieri (1635, Book II, 12)). I have emphasized line SZ

1. *'All the lines' of plane figures have ratios to one another*. Consider the diagram in Fig. 4.[30]

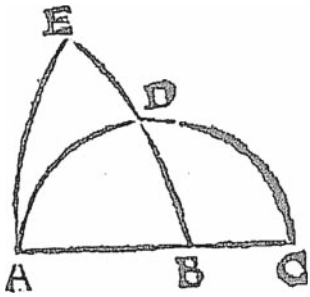
Two figures, AEG, OGQ, are given lying on the same straight line HQ. Cavalieri calls straight line HQ the 'rule' (*regula*), according to which *all the lines* of both figures can be generated by imagining a plane perpendicular to the plane of the sheet, passing through the rule, and traversing the figures while remaining parallel to its first position. All the lines generated in this way are called by Cavalieri *all the lines* of figures EAG, GOQ, and are the linear indivisibles of the two figures according to rule HQ. To establish that *all the lines* of figures have a ratio to one another, Cavalieri removes from the higher figure portion BAD—whose base is line BD parallel to the rule—relocating it into position FHE. If portion FHE is still higher than figure OGQ, he repeats the procedure until eventually a number of portions has been relocated along the rule that are all equal in height to, or less high than, figure OGQ. Consider now any straight line in figure OGQ—NS, say—parallel to the rule. Prolong straight line NS as far as necessary for the line to become longer than all the straight lines, ML, TY, of the figures placed along the rule. Let the straight line so prolonged be SZ. Then, SZ will embrace (*complectetur*) all the straight lines, ML, TY, of the figures placed along the rule, plus possibly some other straight lines, such as, for example, NM, LT (gaps in between the figures). Therefore, this being possible for *all the lines* of figure AEG, OGQ, it follows, Cavalieri concludes, that *all the lines* of figure AEG, OGQ have a ratio to each other, according to Euclid's *Elements*, Book V, Definition 4.[31] That straight line ZS can be made longer than straight lines SN, ML, TY is read off the diagram after prolonging it beyond point Y, and superposing it to straight lines ML, TY. This reading off the diagram is achieved by observing the diagram.[32]

30 Cavalieri (1635, Book II, pp. 13–17).

31 Euclid (1884, pp. 2–3).

32 It is interesting to note that Euclid never proves that straight lines are magnitudes that can have a ratio to one another. Cavalieri's procedure shows by superposition that this is the case.

**Fig. 5** 'All the lines' of equal plane figures are equal. The rules are line AB, BC. *All the lines* are generated in same manner described above




2. *'All the lines' of equal plane figures are equal.*[33] Consider Fig. 5.

Two equal plane figures, ADC, AEB, are given. The rules are line AB, BC. *All the lines* of the two figures are generated in the same manner as already described above. Imagine superposing figure ADC on figure AEB in such a way that the rules are superposed to each other. The portion of plane figure ADB pops up in the diagram. Portion ADB of figure AD is congruent with portion ADB of figure AEB. Then,*all the lines* of portion ADB of figure AD are congruent with *all the lines* of portion ADB of figure AEB. Repeat the procedure with the remaining portions BDC, ADE, an so on, until eventually all the residual portions have been superposed following the same strategy. Since figures ADC, AED are equal by hypothesis, then, all their portions—generated in the procedure—will eventually be congruent, and thus their successive *all the lines* will eventually become congruent (note, however, that nowhere does Euclid state that congruence follows from equality). But, Cavalieri concludes, magnitudes congruent to one another are equal to one another, according to Euclid's common notions.[34]

Cavalieri's argument that all the portions will eventually become congruent since the two figures are equal to each other rests on the possibility of indefinite repetition of the activity of superposing figures, and on the hope that it will terminate, thus 'exhausting' the superposition of the figures. Cavalieri does not comment on the possibility that the activity might never terminate. How can we make sure that the activity will terminate? This question nagged Cavalieri from the very beginning since, as we shall presently see, he subsequently devised another technique of superposition in order precisely to alleviate the fear that the activity of superposition might never terminate.

3. *Plane figures between parallel lines having the same lines at equal distances from the parallels are equal*. This proposition is generally referred to as Cavalieri's second method of indivisibles because he presented it in the seventh and final book of the *Geometria*, as an attempt to eliminate actual infinity from his geometry. Indeed, in the final book of the *Geometria*, Cavalieri re-obtained all the results of the previous books with this new method, which, according to him, is 'free of the concept of infinity' (…*alia ratione ab infinitatis exempta conceptu*).[35]

33 Cavalieri (1635, Book II, pp. 18–20).

34 Euclid (1883, p. 10–11).

35 Cavalieri (1635, Book VII, p. 3).

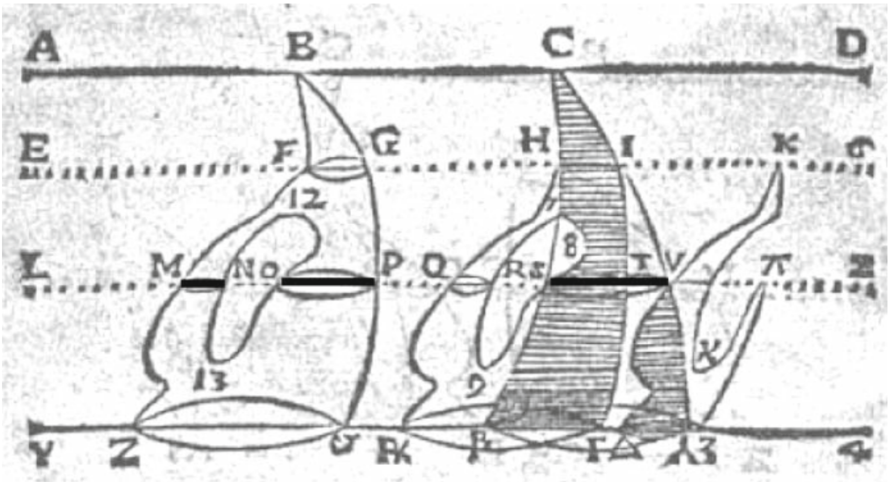



**Fig. 6** *Alia ratio ab infinitatis exempta conceptu*… another method free of the concept of infinity yet not free of superposition (Cavalieri (1635, Book VII, 7))

Consider Fig. 6.[36] Two figures are placed in between parallel lines, AD, Y4, that is, figure ZB$\varepsilon$, on the left, and figure CB$\lambda$, on the right. The two figures are such that the portions of straight lines cut by any line parallel to AD, Y4, and included in one figure are equal to the portions of straight lines, cut by the same line parallel to AD, Y4, and included in the other figure. Thus, for instance, MN, OP, in the figure on the left, will be equal to SV, in the figure on the right (thick, black lines added in the diagram). Cavalieri calls this type of figures 'equally analogous'. He then goes on to prove that equally analogous figures are equal.

The proof is again by superposition, as follows. Bring, for instance, the figure on the left over to the figure on the right in such a way that lines AB, Y$\varepsilon$, will overlap with lines CD, $\varepsilon$4 (i.e., by translating one figure parallel to itself). Since the figures are equally analogous there will be at least a portion of the figure on the left which is congruent with a portion of the figure on the right (shaded part, in the above diagram). Repeat the procedure for the residual portions of figures obtained by imagining subtracting the congruent part from both figures. The residual portions of figures have remained equally analogous given that an equal part has been subtracted by both. Repeat the procedure indefinitely until all the residual equally analogous parts have been superposed. Then, the figures will be congruent, and therefore equal.

Next, thanks to the limiting constraint which requires the figures to be placed in between parallel lines, Cavalieri succeeded in furnishing a second proof that equally analogous figures are equal, in the style of an Archimedean double redutio ad absurdum.[37] This success—a veritable Archimedean tour de force—must have alleviated the anxiety about the truth of the theorem, which, Cavalieri says in a scholium, is

[36] Cavalieri (1635, Book VII, pp. 4–9).

[37] Cavalieri (1635, Book VII, pp. 9–21).

of the utmost importance (…*maximi momenti*). But did it also answer the nagging question about superposition? I very much doubt this, since the double reductio argument demonstrates the conclusion in a different way, but does not sedate the worry that the activity of superposing equally analogous figures might have no end. In other words, the double reductio proof confirms a truth about a mathematical object—i.e., equally analogous figures are equal—but tells us nothing about the *activity* of superposing equally analogous figures.[38] Furthermore, we have evidence that Cavalieri himself very slowly came to terms with the possibility that the activity of superposing equally analogous figures might not terminate. Almost a decade after the publication of the *Geometria* (1635)—while informing his friend Evangelista Torricelli (1608–1647) of Guldin's attack on the geometry of indivisibles—Cavalieri is still convinced that the double reductio proof somehow confirms that the activity of superposing equally analogous figures will terminate.[39] About two months later, however, in a second letter, in which he asks Torricelli for his opinion about his doubts, Cavalieri has realized that the worry about the termination of the superposition process is more than legitimate. It is illuminating to read the relevant excerpt from the letter.

In reference to both strategies of superposition (that is, the one employed to show that *all the lines* of equal plane figures are equal, and the other employed to show that equally analogous figures are equal), Cavalieri admits

> …which superposition— it being possible to suspect that it might never terminate, since, because of the mutual superposition, it is unknown whether such parts will become congruent that are one-half or more than one-half themselves (for, in this way, it would be certain that one could arrive at a residue less than any given quantity)— might therefore give reason to doubt. Thus, I wanted to add the second proof [i.e., the double reductio] of the first proposition of book VII, in order to eliminate doubts at least as regards plane figures, even though it would have been necessary to add another similar proof for solid figures. However, the multiple, nay, infinite variety of solids frightened me to the point that I abandoned the enterprise, while assuming that the [double reductio] proof given for plane figures could somehow guarantee that a similar theorem is true for solids; even more so, since this [sequence of] superpositions need not be made in actuality, but by one [act of] superposition one can intend all the remaining acts which can be performed in the same way.[40]

Cavalieri refers implicitly to Euclid's *Elements* X.1, by saying "…whether such parts will become congruent that are one-half or more than one-half themselves (for, in this way, it would be certain that one could arrive at a residue less than any given quantity)". Indeed, if it were possible to prove that this is always the case, one might

[38] Cavalieri was eventually able to extend the reductio argument, which in the *Geometria* is limited to plane figures, to a large class of equally analogous solids, in the posthumous *Exercitationes* (Cavalieri 1647, pp. 120–173).

[39] "E sebbene per levare ogni ombra a chi avesse dubbio per questo non finir mai, soggiungo un' altra dimostrazione dell' istesso quanto alle figure piane, conforme allo stile d' Archimede, finalmente [Guldin] cavilla pure anco contro di questa…" (Torricelli, 1919–1944, III, p. 93).

[40] Torricelli (1919–1944, III, pp. 114–115).

conclude that the process of superposition will converge. In section four, I will come back to Guldin's doubts about superposition, and will articulate a more detailed discussion of Cavalieri's elaborate answer to Guldin.

4. *Quadrature of the parabola*. Cavalieri says that the parabola has always been his favorite curve, and presents its quadrature in the *Exercitationes* as one example, perhaps the most spectacular example, of the power of the method of indivisibles.[41] Indeed, Cavalieri presents two quadratures, the first according to the method based on 'all the lines'; the second according to method based on equally analogous figures. We need not dwell on the second quadrature, since its superposition technique, based on equally analogous figures, has already been discussed.[42] The first quadrature, on the other hand, shows another intriguing example of superposition of indivisible objects.[43]

The general strategy of the proof is based on 'all the squares' of plane figures, and depends on a preliminary result that Cavalieri proves in the *Geometria*. Imagine erecting a square perpendicularly on every line generated by the process we have called 'all the lines' of a plane figure.

Now consider the diagram in Fig. 7. A parallelogram is cut by a diagonal into two equal triangles. Generate all the lines of one of the triangles. Then imagine elevating planes perpendicularly on every line of the triangle.

A solid is thus created that Cavalieri calls *all the square* of the triangle. On any figure whatever in which one can consider *all the lines* one can also imagine *all the squares*. Cavalieri proves that for any parallelogram whatever all the squares of the parallelogram, according to one of the sides taken as rule, are three times *all the squares* of the triangle cut by the diagonal, taken according to the same rule.[44] We are now in a better position to consider quadrature of the parabola (see Fig. 8).

Given the parabola OAC, Cavalieri sets out to prove that parallelogram TDCO is to parabola OAC as 3 is to 2. Consider any line GK in between lines AB and DC. By definition, since curve AHC is a semi-parabola, then, the square on BC, that is, on SQ, is to the square on SH as BA is to AS, that is, as GK is to KH. Now, construct the square ADEF on line AD, draw its diagonal AE, and prolong line GK up to the point R. Thus, SH is equal to AK, which in turn will be equal KI. This means that the square on SH will be equal to the square on KI. Therefore, by definition and by construction of the diagram, the square on ED, that is, on RK, is to the square on IK as GH is to HK. Next, one has to imagine all the squares generated on the semi-parabola by lines parallel to SH, and all the corresponding squares generated by lines parallel to IK in the triangle. Equally, one has to imagine all the squares generated on the parallelogram ABCD by lines parallel to BC, and all the corresponding squares generated by lines parallel to ED in parallelogram AFEF. By the first method of indivisibles, then, *all the squares* on parallelogram ABCD, rule BC, that is, *all the squares* on parallelogram AFED, rule ED, are to *all the squares* on the semi-parabola, rule BC, that is, to *all the squares* on triangle AED, rule ED, as *all the lines* of parallelogram ABCD, rule

41 Cavalieri (1647, p. 81).

42 Cavalieri (1647, pp. 113–115).

43 Cavalieri (1647, pp. 81–83).

44 Cavalieri (1635, Book II, pp. 78–80).

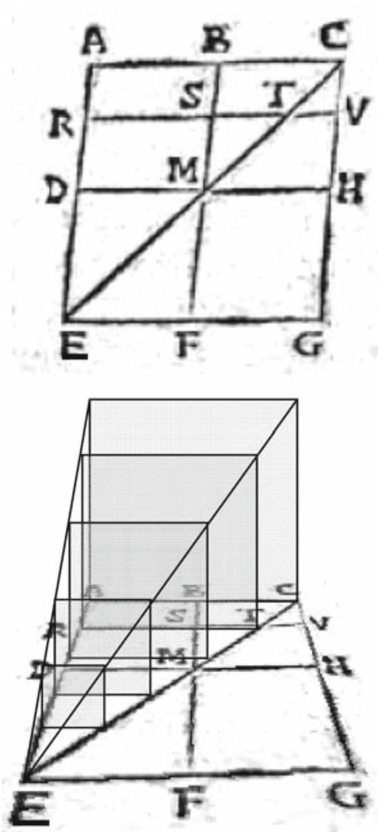



**Fig. 7** *All the squares* of a triangle formed by sides and diameter of a parallelogram. Above: the original diagram (Cavalieri 1635, Book II, p. 79). Below: I have deformed the original diagram into a 3-D perspective so as to visualize better *all the squares* of the triangle cut by the diagonal

AB, are to *all the lines* of mixed-triangle ADC, rule AB. Now look at the recreated diagram in Fig. 9.

- As we have seen, Cavalieri has proved that for any parallelogram whatever *all the squares* of the parallelogram, according to one of the sides taken as rule, are three times *all the squares* of the triangle cut by the diagonal, taken according to the same rule.
- Since he has now shown that *all the lines* of parallelogram ABCD, rule AB, are to *all the lines* of mixed-triangle ADC, rule AB, as *all the squares* on parallelogram AFED are to *all the squares* on triangle AED, Cavalieri clinches the proof by concluding that *all the lines* of parallelogram ABCD, rule AB, are to *all the lines* of mixed-triangle ADC, rule AB, as three is to one. The rest is obvious and need not concern us.

The argumentative strategy hinges on the assumption that for each square on triangle AED there is one square on the semi-parabola, and that for each square on parallelogram AFED there is one square on parallelogram ABCD. However, whereas the squares elevated on triangle AED and on parallelogram AFED, with rule ED, are evenly spread across the figures, the squares elevated on thesemi-parabola

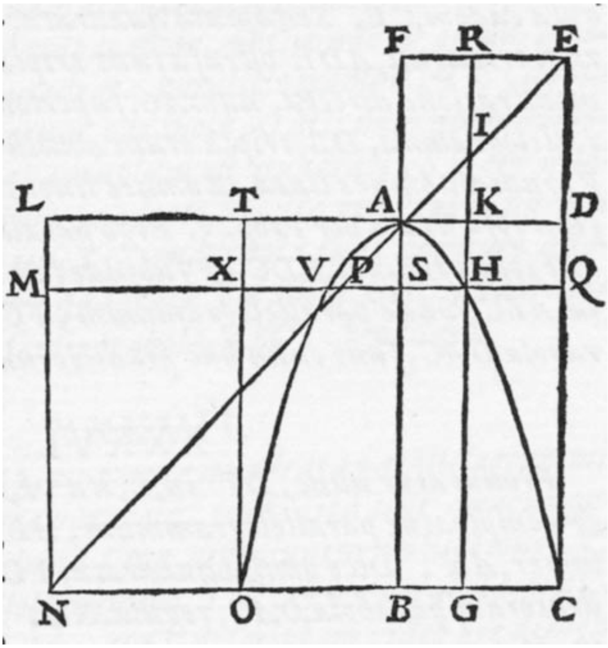



**Fig. 8** The diagram of the first quadrature of the parabola (Cavalieri 1647, p. 81)

and on parallelogram ABCD, according to rule BC, are not evenly spread across the figures (as the above diagram suggests, Fig. 9). In fact, the squares elevated on the semi-parabola and on parallelogram ABCD (both according to rule BC) are generated at each intersection between the semi-parabola and straight line KG, while the latter moves parallel to rule AB (or DC).[45]

In this proof, superposition works as a means of controlling the direction of the sequence of deductive steps at a bifurcation where two options are possible. It operates as follows. First of all, let us note that, once Cavalieri has shown that, for any such line as GH, GK is to KH as the square on BC is to the square on SH, that is, as the square on KR is to the square on KI, he has two options to move forward.

1. An illegitimate conclusion (we shall presently see why *illegitimate*), i.e., that *all the lines* of parallelogram ABCD, rule AB, are to *all the lines* of mixed-triangle ADC, rule AB, as *all the squares* on parallelogram ABCD, rule BC, are to *all the squares* on the semi-parabola, rule BC, and therefore that *all the squares* on the semi-parabola are three times those on parallelogram ABCD.[46]
2. The correct conclusion that *all the lines* of parallelogram ABCD, rule AB, are to *all the lines* of mixed-triangle ADC, rule AB, as *all the squares* on parallelogram

[45] Cf. Struik (1969, pp. 209–219), for a brief overview of the entire proof in English.

[46] This illegitimate conclusion would be in conflict with the result proven by Cavalieri in the very next theorem, namely, that *all the squares* of parallelogram ABCD, if taken with rule BC, are just two times *all the squares* on the semi-parabola taken with the same rule (Cavalieri 1647, pp. 82–83). But this result does not reveal the reason of the illegitimacy of the conclusion.

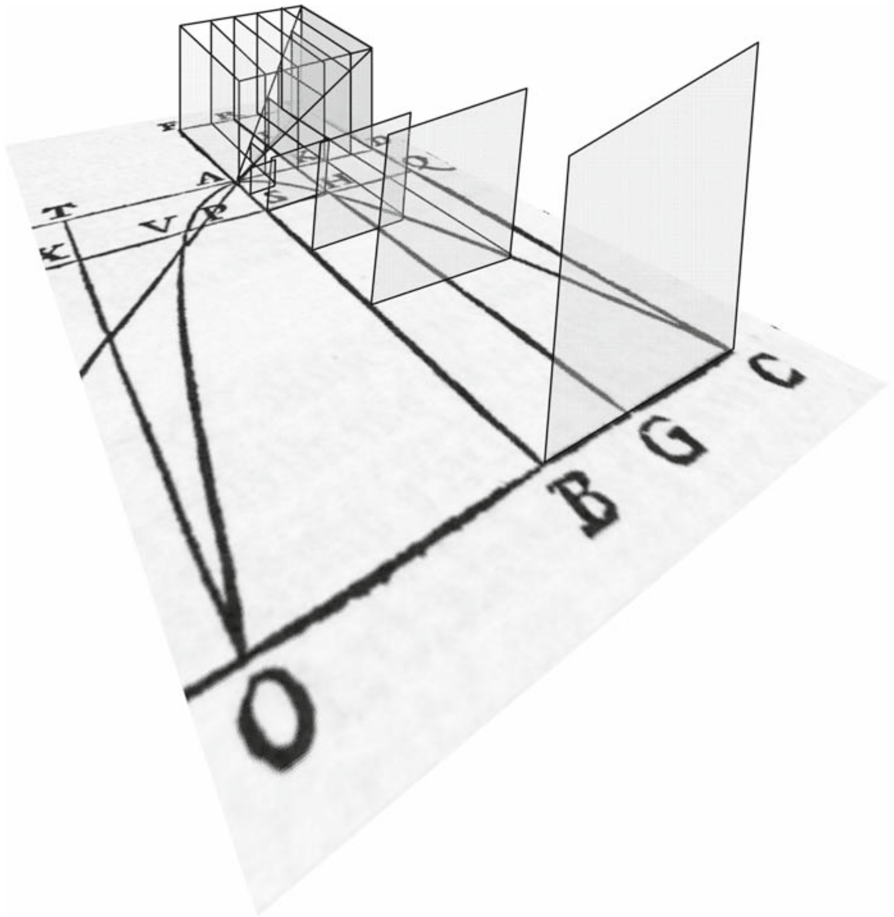



**Fig. 9** A 3-D recreation of the diagram of the first quadrature of the parabola, showing *all the squares* involved in the proof, except, for clarity, *all the squares* of parallelogram ABCD

AFED, rule ED, are to *all the squares* on triangle AED, rule BC, and therefore that *all the lines* on of parallelogram ABCD are three times those of mixed-triangle ADC.

How does Cavalieri steer clear of the illegitimate conclusion at this bifurcation? Correlatively, what warrants the legitimate conclusion?

Cavalieri avoids the illegitimate conclusion by having recourse to superposition of indivisibles. Let us see how. At the beginning of the *Exercitationes*, Cavalieri warns us that, if indivisibles are to be compared correctly, they must be taken according to the same degree of thickness, or constipation (*gradus spissitudinis, aut constipationis*). Cavalieri states that if the indivisibles are generated by the motion (*transitus*) of a plane cutting a plane-figure perpendicularly to it, then the indivisibles are called of "straight transit" (*transitus recti*). If the indivisibles are generated by the motion of a plane cutting the plane-figure obliquely, the indivisibles are called of "oblique transit"

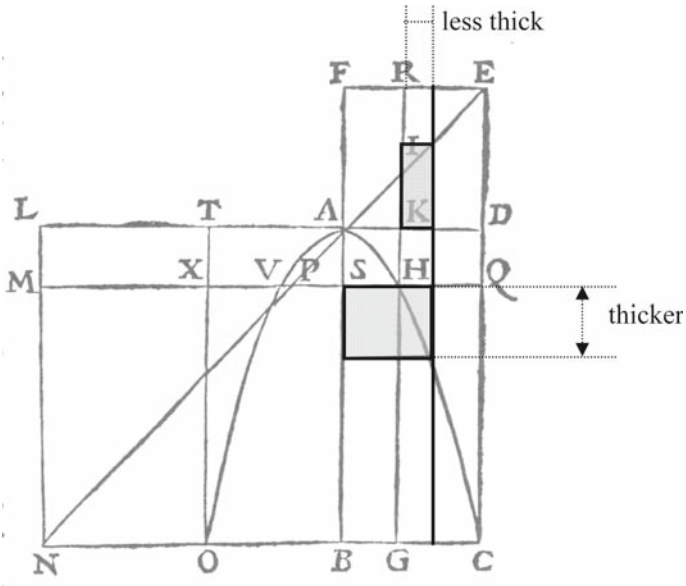



**Fig. 10** The thickness of indivisibles. I have added the gray portions and black lines to mark off indivisibles of the same height but different thickness in the semi-parabola and in the triangle. Of course the diagram is only a suggestion since the thickness of indivisibles cannot be finitely small

(*transitus obliqui*). The degree of thickness of indivisibles of straight transit is different from the degree of thickness of indivisibles of oblique transit. Therefore, Cavalieri argues, if indivisibles are to be compared correctly, they must be taken according to the same transit.[47]

In the quadrature of the parabola, there is one feature of the indivisibles that is unknown, but can be read off the diagram. It is their thickness. However, here the criterion of transit fails to discriminate between the legitimate and the illegitimate conclusion since all the indivisibles are of straight transit (Cavalieri does not say this explicitly but it can be evinced from the fact that, at the beginning of the *Exercitationes*, in Definition I, he calls *all the lines of straight transit* simply *all the lines* without further qualification. At any rate, the argument I am proposing here would be the same even though all indivisibles were of oblique transit).[48]

How, then, does Cavalieri discriminate between the degrees of thickness of the indivisibles? An answer can be found in the diagram that supports the proof. Cavalieri notices at the beginning of the *Exercitationes* that, when the indivisible lines of a figure appear not to have the same distance from one another, this is a "sign" (*signum*) that the indivisibles do not have the same thickness.[49] Now consider Fig. 10. Cavalieri seems to

[47] Cavalieri (1647, pp. 15–16), where Cavalieri introduces the *gradus spissitudinis*, and a few examples, and 238–241, where Cavalieri elucidates his argument with the analogy of the density of threads in a canvass explaining how to solve a related paradox (incurred by different degrees of thickness).

[48] Cavalieri (1647, pp. 6–7).

[49] Cavalieri (1647, p. 17).

imagine that indivisible squares on the semi-parabola, rule BC, and on triangle AED, rule ED, would not have the same thickness if they were superposed, that is, they would not have the same distance from one another (and analogously that indivisible squares on parallelogram ABCD, rule BC, and on parallelogram AFED, rule ED, would not have the same thickness if they were superposed, that is, they would not have the same distance from one another). On the other hand, he also seems to imagine that indivisible squares, such as those on the triangle and on parallelogram AFED, rule ED, and indivisible lines, such as, GH, HK, rule DC, do have the same thickness, that is, they do have the same distance from one another (in other words, the indivisibles are evenly spread).[50]

In the first case, therefore, the diagram affords a *signum* that the indivisibles do not have the same thickness, whereas in the second case it affords a *signum* that the indivisibles do have the same thickness. The first *signum* bars the illegitimate conclusion. The second *signum* warrants the legitimate conclusion.

To sum up, Cavalieri is able to compare the degree of thickness of indivisibles by reading *signs* off the diagram. On this basis, even though here the criterion of transit fails since all indivisibles are of the same transit, the proof can still proceed to its correct conclusion by comparing indivisibles having the same degree of thickness. One sign suggests that certain indivisibles do have the same thickness, thus warranting the correct deductive step. Superposition affords another sign that makes it possible to rule out the illegitimate deductive step.

## 4 Cavalieri's controversy with Guldin about superposition

In this section, I will focus on one particular objection raised by Guldin, which questions the use of superposition in geometry. This objection, as we shall see, gave Cavalieri the opportunity to expand on superposition, and ultimately reflect on its implications. Eventually, when Cavalieri wrote the *Exercitationes*, he expounded his final meditations on superposition, propounding a fascinating analysis of the activity that had innervated his practice of geometry.

In relation to superposition, specifically as used by Cavalieri to prove that equally analogous figures are equal (cf. previous section), Guldin asks how we can tell whether one figure is congruent with the other, so that it really overlaps with the other.[51] Who, Guldin asks, is going to be the judge of superposition? The hand, the eye, or the intellect? It is obvious, says Guldin, that only the intellect can judge; hence, he says,

[50] In the second case, the *signum* of same thickness, i.e., same distance of the indivisibles from one another, need not be afforded by superposition. That the indivisibles have the same distance from one another could have been deduced tacitly by Cavalieri by falling back on Euclid's *Elements*, Proposition I.33, which states that "The straight lines joining equal and parallel straight lines (at the extremities which are) in the same directions (respectively) are themselves also equal and parallel" (Euclid 1956, I, p. 322). However, it is worth noting that the proof of I.33 still depends on I.4. In the first case, I think that we can confidently rule out that the *signum* of different thickness could have been deduced tacitly by Cavalieri since this deduction—if at all possible within the constraints of Cavalieri's geometrical means—would have been rather complex (the thickness being variable all along the semi-parabola), and Cavalieri would no doubt have incorporated it in the proof.

[51] Cavalieri (1647, 208ff).

**Fig. 11** Guldin's objection. Consider small portions of equally analogous figures around the line which is being superposed (dotted horizontal line), according to the procedure set forth by Cavalieri for equally analogous figures (you can identify the two figures by the contour lines of same thickness)

the necessary condition to decide congruence of figures is the following, that is, that equal, corresponding straight lines can be superposed to one another. This, however, he insists, is already a possibility granted by the hypothesis that corresponding lines are equal to one another. So, in effect, Guldin concludes, nothing can be said about the figures included between the parallel lines. For, he notices, the intellect can say nothing about the other two lines which include the figures between the two parallels, whether these could become congruent or not, since these two other lines are generally supposed to be irregular curves, and indeed totally arbitrary.[52]

In the final analysis, according to Guldin, nothing can be concluded about the surfaces themselves, and what is said about equal lines is already contained in the hypothesis. I have tried to visualize the substance of Guldin's objection with a sketch of two simple equally analogous figures (see Fig. 11). Consider a limited portion of the two figures, around the dotted line which is being superposed, according to the procedure set forth by Cavalieri.

The crucial point of Cavalieri's answer is as follows. "When Guldin asks who is the judge of the superposition, I respond the intellect, to which it is evident by hypothesis that a certain superposition has been made, in any way whatever, according to the law prescribed by the procedure. And when Guldin adds, 'Let them be superposed, and be congruent, what are we going to conclude? That the lines are equal?', I say that this is not what is being concluded, but that it is concluded that a part of the superposed figure is congruent to a part of the other figure, whatever that part may be… for, it is enough for the intellect to suppose that some superposition has occurred".[53]

We can break down the controversy in this way. Guldin claims that once the two lines—assumed equal by hypothesis of equally analogous figures—have been super-

[52] Cf. Lombardo-Radice's comment on this question (Cavalieri 1966, p. 822), which, I think, is somewhat misleading, in that it reframes Guldin's objection in terms of modern set theory, attributing to Guldin a 'finitist mentality', which does not do justice to the serious point raised by Guldin's objection. Cf. also Cellini (1966).

[53] Cavalieri (1647, p. 210).

posed, nothing more can be said about the area around them, which might have been superposed. In effect, we can say that Guldin does not concede that from congruence of lines congruence of some extended area follows. Cavalieri replies that the intellect, knowing that a superposition of equally analogous figures has occurred, concludes that an extended part of the superimposed figure is congruent to an extended part of the other figure.

Essentially, Guldin insists that nothing can be read off the diagram after superposing the figures, while Cavalieri—notwithstanding his insistence on the act of the intellect which, according to him, deduces the congruence of some extended area around the figures—de facto claims that it is the very reading off the diagram that justifies the deduction.

In my view, Cavalieri remains steadfast in defending the fundamental activity that he performs on the object, even though he cannot further justify this activity. This activity has two components, or rather two salient actions. First, the figures are superposed according to a certain procedure (in this case, the lines which determine the property of equally analogous figures must remain parallel to the two parallel lines in between which the figures are initially placed). Second, the mathematician reads of the diagram the portion of figures which turns out to be congruent after superposition has occurred. Nothing can be concluded about the actual extension of the congruent portion of the two figures. This obviously depends on the particular figures. However, it can be observed by the mathematician that some extended portion of the two figures will be congruent, and this is all the mathematician is allowed to read off the diagram.

Was Cavalieri satisfied with his answer to Guldin? I think not. I have just suggested that Cavalieri cannot further justify the activity he performs on the object. I believe that this is why he eventually re-focused his analysis of the objection, by offering a theory of superposition in the subsequent sections of the third *Exercitatio*. I will now briefly expound his theory.[54]

Cavalieri begins his analysis with an original interpretation of *Elements* I.4—the Euclidean proposition from which we started. According to Cavalieri, never does the intellect assent to the congruence of two magnitudes, unless the intellect presupposes the equality of certain things, and thus a correct superposition of these things, and therefore unless the intellect [observes, or deduces] their congruence, according to the following principle. *Things which are equal are congruent to one another if they are correctly superposed*, as most evidently appears from *Elements* I.4.[55] Here Cavalieri moves beyond Euclid, since Euclid refrains from defining equality of figures, only admitting, among the common notions, that congruent figures are equal. Thus, Cavalieri reverses the strategy, that is, he claims that equal figures are indeed congruent if they are correctly superposed. The fine balance capable of assaying equality of figures, says Cavalieri, is their congruence.

54 Cavalieri (1647, pp. 216–219).

55 A difficult passage in which the elliptic tendency of Cavalieri's prose does not allow a resolution of the action verb that refers to the subject of the sentence (namely, the intellect). Perhaps this signals Cavalieri's uneasiness with the problem. "Numquam enim ille [i.e., intellectus] assentitur duas magnitudines invicem congruere, nisi ex praesupposita quorundam aequalitate, ac mutua, debitaque superpositione, et subinde ipsorum necessaria congruentia, iuxta aliud hoc principium…" (Cavalieri 1647, p. 216).

Cavalieri's theory of superposition aims at exploring the possibility that the axioms (*axiomata*) of equality and congruence are convertible; in other words, if two geometrical objects are congruent then they will be equal, and vice versa, if they are equal they will also be congruent. Cavalieri suggests that there are three modes of superposition, namely, *simple motion*, *inflexion*, and *fluxion*. According to these three modes congruence can be represented to the intellect.

By a *simple motion*, for example, a straight line can be made congruent to a circumference, that is, by making the circle roll over the straight line. While the circumference rolls over the straight line the intellect recognizes that the congruence is being realized, though not in static manner, but in a dynamic manner.

By *inflexion*, we imagine that one of the given equal quantities can be divided in parts, or components, which move one by one, that is, inflect, so that they adapt to the other quantity. An example would be a straight line which can be made congruent to a polygon, such as, for instance, an equilateral pentagon by inflecting its five parts around the pentagon. However, if by inflection we want to make a straight line congruent with the circumference of a circle, then, the circumference must be divided into a number of parts greater than any number that can be assigned. For, if the number were finite, it would follow that if the minimal particle of the straight line adhered to the circumference there would be something rectilinear in the circumference, which is clearly absurd.

We have *fluxion* when the parts not only move, as in inflection, but they also change their place reciprocally, though without any space being interjected among them. If a space were interjected then we would have a case of rarefaction. Here again, the number of parts which have to be permuted can be finite, or their number can be greater than any given number. Consider as an example of a finite number of parts which have to be permuted the following. An curvilinear angle is equal to a rectilinear angle. We want to make them congruent. In reference to the diagram in Fig. 12, we have three rectilinear angles, ABC, a right angle (case 1), an obtuse angle (case 2), and an acute angle (case 3). Further, consider semicircles constructed on the dotted lines which delimit the rectilinear angles, and a third semicircle, in the third case, constructed on BD, perpendicular to AB. By construction, then, the curvilinear angles FBG are equal to the rectilinear angles ABC. How do we make them congruent?

In cases 1 and 2, since the mixed angles ABG are common, it will suffice to superpose curvilinear angles FBA to curvilinear angles CBG. For, in this way, the two parts that compound each curvilinear angle become apt to the congruence, so that the whole becomes congruent with the whole.

Case 3 is more complex, but more intriguing. Semicircle FBG is divided by straight line BE into the mixed angle FBE and the contact angle EBG. But also ABK is a contact angle, so that contact angle EBG can be superposed to contact angle ABK. Now, the residual figure, FIBE, is equal to residual figure HKBC and therefore they can be made congruent. Eventually, the whole curvilinear angle FBG can thus be made congruent with the whole rectilinear angle ABC, but in order to achieve this congruence the two parts of curvilinear angle FBG—the residual figure, FIBE, and contact angle EBG—will have to change their place reciprocally, that is, the parts will have to flow, or slide over each other (see the diagram).

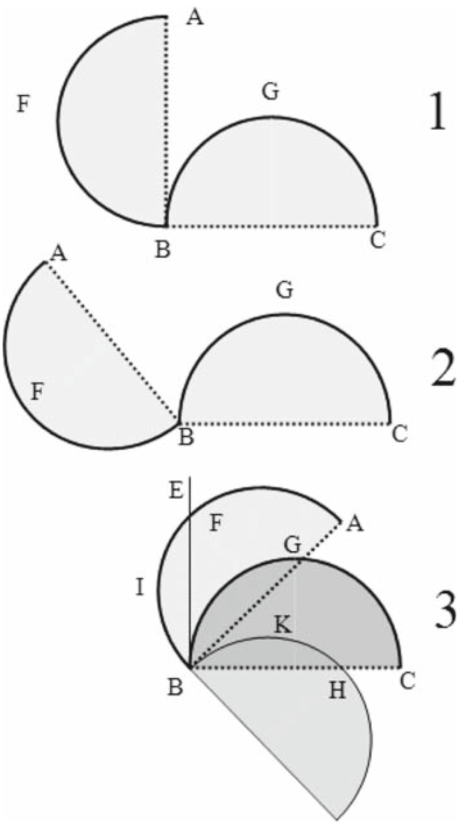



**Fig. 12** Cavalieri's theory of superposition. The three modes of superposition: (1) *simple motion*; (2) *inflection*; (3) *fluxion*. I have used dotted lines to mark off the rectilinear angles better. See the original diagram in (Cavalieri 1647, p. 218)

Further, Cavalieri argues, an example of this third mode of superposition involving a number of parts greater than any given number can be seen in the case of a circle equal to a square. For, in order for the two to be congruent, it might be necessary that an infinite number of parts can flow. That this is the case strongly suggests the following fact. Let us imagine superposing a square on the circle only by means of rectilinear pieces. It will never be possible to make something that is compounded by a finite number of rectilinear objects congruent with a curvilinear object. To achieve this congruence between a square and a circle, an infinite number of rectilinear objects in the square will have to flow.

Thus, Cavalieri concludes, that according to these three modes of superposition, it is possible for the intellect to grasp the convertibility of the axioms of equality and congruence, and to recognize congruence as the exact balance of equality.

Let us take stock. Under the pressure of Guldin's attack, Cavalieri has pushed his analysis of the activity of superposing figures to the limit of his practice of geometry. He has confronted the fundamental activity that underlies the new geometry of indivisibles. The geometry of indivisibles ultimately rests on the human subject's willingness to observe the effects of superposing certain mathematical objects. Cavalieri's final musings on the theory of superposition hint at the difficulties encountered by the human subject trying to visualize the effects of infinite superpositions of straight and curvilinear objects. There is an irreducibly empirical element in this practice.

## 5 Conclusion

In this paper, I set myself the task of focusing on activities. In particular, I have investigated one activity, namely, *superposition* of geometrical objects.Cavalieri's geometry of indivisibles was always concerned with superposition. It gradually dawned on Cavalieri that the activity of superposition was of the essence. He realized that both Euclid and Archimedes had used superposition as a viable method of proof under certain circumstances. Cavalieri used superposition by reading off the diagram the equality or inequality of magnitudes, not by first assuming the equality or inequality of magnitudes as known. Indeed, as we have seen, Clavius argued that mathematicians use superposition by first assuming the equality or inequality of magnitudes as known, and then recognizing the congruence of the superposed magnitudes with the intellect. But Cavalieri's activity of superposing geometrical objects does not conform to Clavius' analysis of the practice of geometers.

When Cavalieri came under attack by Guldin, he re-focused his efforts in order to gain further insights into the nature of superposition. His dissatisfaction with his own infinite demonstrative process of the equality of equally analogous figures—as evidenced by the epistolary exchange with his fellow indivisibilist Torricelli—suggests that Cavalieri sensed the deep tension underlying his new geometry. To Cavalieri, progress in geometry must have appeared to be staked on a fundamental activity of the human subject that could not be reduced to more evident principles.

In the final part of the response to Guldin (which we could examine here), Cavalieri's rebuttal culminates, so to speak, by furnishing a proof by indivisibles of the so-called *Guldin theorem*, the fundamental theorem of Guldin's *De centro gravitatis*, which Guldin himself—an irreducible opponent of the indivisibles, unable or unwilling to accept geometry as an irreducibly human activity- could never prove. Why did Guldin fail to prove his theorem? The fact is that Guldin was a geometer steeped in philosophical norms rooted in his Aristotelian theory of science, as one can easily see by glancing at the opening sections of the *De centro gravitatis*.[56] On the other hand, Cavalieri, who in his youth started out as a brilliant theologian and metaphysician—presumably imbued with the philosophy of Aquinas, and much admired in Milan for his disputation skills[57]—, never let the fetters of philosophical prejudice stand in the way of his grappling with the geometrical objects that fascinated him.

The present case study suggests an integrated historiographic approach. It is rewarding to place mathematical objects in the context of activities, even when, to the protagonists, such activities escape (or seem to escape) mathematical justification. Focusing on mathematical objects and the activities employed to manipulate the objects and to draw inferences about them can illuminate the history of mathematical thought.

56 Guldin (1635, pp. 1–20).

57 Piola (1844).

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