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The Cognitive Development of Galileo's Theory of Buoyancy

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1. Introduction

Some thirty years ago, in a seminal book, William Shea argued that between 1610 and 1632 Galileo worked out "the methodology of his intellectual revolution", and that hydrostatics was one fundamental area of research Galileo concerned himself with at that time.[1] According to Shea, that methodology was deeply rooted in Archimedean mathematics and basically consisted in mathematically investigating classes of phenomena, such as floating bodies, under certain idealized conditions. I believe that Shea's view is fundamentally correct. I will develop it further, by reconsidering Galileo's methodology in finer detail, specifically in relation to the development of his theory of floating bodies. However, before proceeding I need to clarify my terminology. I will use the word "theory" to refer to linguistic and pictorial representations of classes of idealized phenomena, such as, for example, floating bodies; hence *theory of floating bodies*, or *theory of buoyancy*, for short. I will use the word "methodology" more broadly to refer to argumentative strategies, including forms of reasoning not necessarily mathematical.

The young Galileo took a profound interest in Archimedes. A few marginal postils to the latter's *On the sphere and the cylinder*, probably written in the late 1580s, strongly

[1] Shea 1972, pp. vii–viii.

suggest that Galileo scrupulously studied this work. At about the same time, Galileo furnished a solution to the problem of Hiero's crown different from that commonly related by the Archimedean tradition.[2] Though published many years later in *Two new sciences* (1638), Galileo's theorems on centres of gravity, whose Archimedean inspiration is all too evident, date from the mid-1580s.[3] Finally, we have the most Archimedean, and arguably one of the most important of his early writings, the so-called *De motu*, a set of drafts concerning questions of buoyancy, motion, and mechanics, which is generally attributed to the early 1590s.

De motu indicates that in the early 1590s Galileo developed a theory of buoyancy, which I will call "theory of Archimedean buoyancy", concerning the behaviour of bodies in the fluid elements, water, air, fire, under the action of gravity. As we shall see, Galileo's 1590s theory of buoyancy was grafted onto the Renaissance tradition of the text and figures of Archimedes' *On floating bodies*. This treatise, as is well known, concerns bodies floating on an idealized spherical shell of water whose centre coincides with the centre of the earth. In addition to floatation, the relevant sections of Galileo's *De motu* concern other forms of interaction between bodies and fluids, in a context gradually de-emphasizing their localization on the elemental globe. Though innovative, the theory of Archimedean buoyancy remained highly problematic and, as has long been recognized, was marred by a fatal error systematically repeated by Galileo.

Galileo never published *De motu*. He himself recognized and corrected that error about two decades later, in the early 1610s, at the height of his controversy on shape and buoyancy with some Aristotelian philosophers. At the same time Galileo developed his theory of buoyancy substantially. A set of drafts, brief notes, and diagrams, which have so far been little studied, bear witness to that extraordinary maturation in Galileo's methodology.[4] The immediate consequence of that maturation was dramatic. Galileo had to abandon an advanced version of the buoyancy treatise that he had been working on for some time. The payoff was immense, though. It propelled him beyond his theory of Archimedean buoyancy. Eventually it led to the publication of one of his masterpieces, the *Discourse on bodies that stay atop water, or move in it*. The *Discourse* indicates that now Galileo was in possession of a new theory of buoyancy, enriched by the tradition of sixteenth-century mechanics[5], and perhaps supported by the actual observation of phenomena with experimental apparatus. I will refer to Galileo's 1610s theory as "theory of mechanical buoyancy".

How did Galileo go about exploring the tenability of his initial commitment to the theory of Archimedean buoyancy? What methodology did he deploy in order to

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[2] In this paper I will quote passages from the so-called *National Edition* of Galileo's works in 20 volumes, in the abbreviated form of *Opere*, followed by the roman numeral of the volume and the Arabic numerals of the pages. All translations are mine, unless otherwise specified. Cf. *Opere*, I, pp. 215ff., 379 (on the problem of Hiero's crown), and ibid., pp. 233ff. (postils to *On the sphere and the cylinder*). Cf. Dollo 2003, pp. 63–86, in regard to the general influence of Archimedes on Galileo.

[3] Cf. *Opere*, I, pp. 187ff., and Di Girolamo 1999.

[4] Cf. *Opere*, IV, pp. 19–56.

[5] Cf. Drake 1999, II, the numerous essays in part IV are all relevant, pp. 121–376, Drake and Drabkin 1969, Laird 2000, pp. 36–44.

develop the theory of mechanical buoyancy? How did he fixate his belief in the theory of mechanical buoyancy? Eventually, to what extent did the methodology of his intellectual revolution shape the rhetoric he infused the *Discourse* with, in order to achieve the power of persuasion?

In this paper, I will answer these questions, focusing on Galileo's two theories of buoyancy. More specifically, I will discuss the creative interplay between representations of idealized phenomena and *proportional reasoning*[6]. It was that interplay that led Galileo to a new vista about buoyancy in the early 1610s. Further, I will suggest that Galileo gradually fixated his belief in the theory of mechanical buoyancy by pursuing one fundamental strategy, explanatory unification through paradox[7] building and resolution. The objective was to show that from the theory of mechanical buoyancy the explanatory mechanism could be derived underlying the early theory of Archimedean buoyancy. The strategy consisted in crafting a great paradox, as a form of cognitive self-challenge, which could then be resolved by the explanatory mechanism underlying the theory of mechanical buoyancy. Finally, I will argue that he populated the *Discourse* with paradoxical examples, precisely in order to bring about persuasion in the reader, in the image of his personal path to belief fixation.

To conclude this introduction, I wish to stress the importance of a close reading of all the relevant sources in their original languages. As for Archimedes' *On Floating bodies*, however, I will mostly make use of the Latin versions circulating in the Renaissance[8], since Galileo started out by assimilating the Latin tradition of Archimedes' *On Floating bodies* – the only tradition that he could have access to. This paper is a reflection of my broader interests in the history of forms of cognition, especially within the framework of a cognitive approach to language. I do not consider languages as accessorial means of expressing thought. I consider them as structuring cognitive processes that are insep-

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[6] Galileo's proportional reasoning is a form of reasoning based on the principled manipulation of ratios and proportions, according to the rules set forth in the fifth book of Euclid's *Elements*. As for Galileo's use of proportional reasoning in natural philosophy, a considerable body of literature is now available, which allows us to understand most of its technical aspects better. Cf. Armijo 2001, Drake 1973, 1974, 1987, Frajese 1964, Giusti 1981, 1986, 1990, 1992, 1993, 1994, and 1995, Maracchia 2001, Palladino 1991, Palmieri 2001 and 2002. For a general treatment of the various aspects of the Euclidean theory of proportions I have relied on: Grattan-Guinnes 1996, Sasaki 1985 and 1993, Saito 1986 and 1993. Rose 1975 is an extensive, immensely erudite survey of Renaissance mathematics in Italy from a non-technical point of view. Cf. also Sylla 1984, pp. 11–43.

[7] Occurrences of "*paradosso* [paradox]" in Galileo are rare. There is just one, for instance, in the *Discourse* (*Opere*, IV, p. 77), one in the *Assayer* (*Opere*, VI, p. 256), a few in the *Dialogue* (*Opere*, VII, pp. 65, 155, 452, for instance), one in the *Two New Sciences* (*Opere*, VIII, p. 68). In Galileo's writings concerning floating bodies, the meaning of this lemma is very close to that of a marvellous, incredible phenomenon [*accidente ammirando*, as we shall see], apparently contradictory. In June 1612, in a letter to Maffeo Barberini (the future pope, Urban VIII), Galileo wrote that some people held in high repute poked fun at the recently discovered solar spots, as if they were a paradox, an absurdity, "... mi viene scritto che huomini di molta stima di cotesta città se ne burlano come di paradosso et assurdo gravissimo..." (*Opere*, XI, p. 305).

[8] As is well known, only Latin versions of *On floating bodies* had circulated in Western Europe until Heiberg's discovery of the Constantinople palimpsest, in the early twentieth century.

arable from their phonetic substrata. Thus, though furnishing English translations for readability purposes, I will quote originals in footnotes, whenever I think it necessary.

2. Galileo's theory of Archimedean buoyancy

a. Water level

To understand Galileo's theory of Archimedean buoyancy we first of all need to understand its pictorial representations of the idealized floating bodies. They depict solid magnitudes floating in water, without encoding information concerning the chain of events immediately preceding the final state of floatation. These representations have antecedents in Archimedes. In Fig. 1 we have diagrams taken from two Renaissance editions of Archimedes' *On floating bodies*.

The diagrams on the left are from Niccolò Tartaglia's edition, those on the right from Federico Commandino's, two volumes which Galileo would have consulted. The diagrams of the first row represent bodies immersed in water (left portion of the hemisphere) and volumes of water (right portion of the hemisphere). They are located on the surface of a spherical shell of water covering the earth, whose centre coincides with the earth's centre. Those of the second represent a body within water and an equal volume of water. Those of the third row represent a body (*a*) wholly submerged but level with the surface of water, upon which another body is located (*d*), outside of water. Those of the fourth row represent magnitudes (the surfaces) and weights (the lines).

Like Archimedes Galileo depicts bodies floating on the surface of a mass of water at rest. However, in the first draft of *De motu* (cf. the diagrams in Fig. 2, left column), the water mass seems confined within a sort of circular sector.[9] Most importantly, Galileo always shows the raised level of water, level *g*, for instance, following immersion of the body *ab* (upper left diagram). In contrast to Archimedes' similar figures, the diagrams of *De motu* show bodies against the background of the raised level of water following immersion. This is an innovative feature. Two circular lines show the initial and final levels of water (*g, d,* for example, upper left diagram). On Archimedes' spherical shell of water covering the earth, the level of water does not change. On the other hand, no continuous process of immersion is hinted at in Galileo's diagrams, exactly like in Archimedes. The body is simply represented already immersed in water.

In subsequent drafts of *De motu*, Galileo abandons for good the spherical representation of water in favour of what (deceivingly) appear to be parallelepiped vessels (Fig. 2, right column). He never claims that this is the case, in fact. He keeps referring to water as if it had the mass-like nature of an undifferentiated element, regardless of its possibly being enclosed in a vessel. Although the lettering of the diagrams suggests that Galileo might have regarded them as a representation of bodies immersed in a vessel, he never specifies if the space enclosed by the vertical and horizontal lines is supposed to represent a vessel. Indeed he does not even mention what those lines are supposed

[9] See Fredette 1969, Camerota 1992, and Giusti 1998 for the problems concerning the date of the various parts of *De motu*.

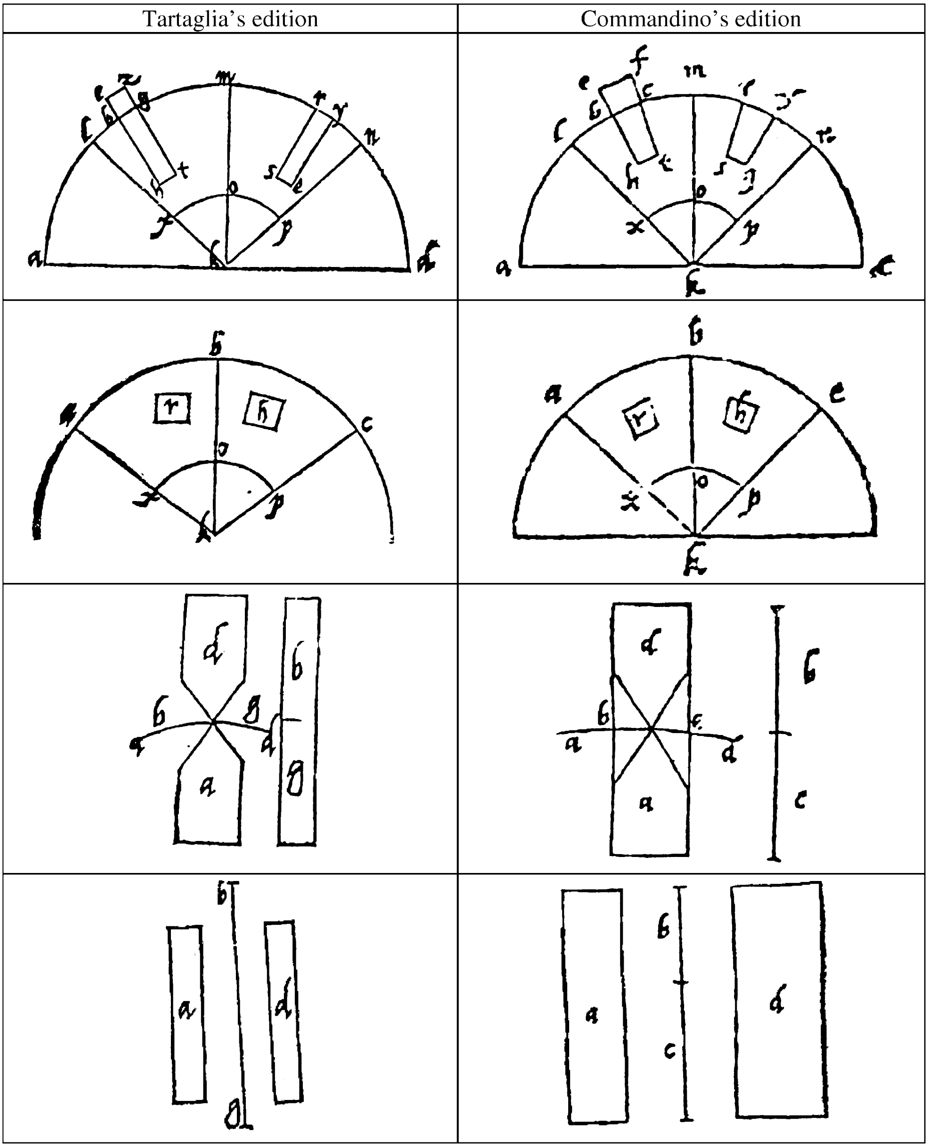

Fig. 1. On the left, diagrams from Niccolò Tartaglia's edition of Archimedes' *On floating bodies* (Archimedes 1543, pp. 32 verso – 34 verso); on the right, diagrams from Federico Commandino's edition of Archimedes' *On floating bodies* (Archimedes 1565, pp. 2 verso – 5 verso)

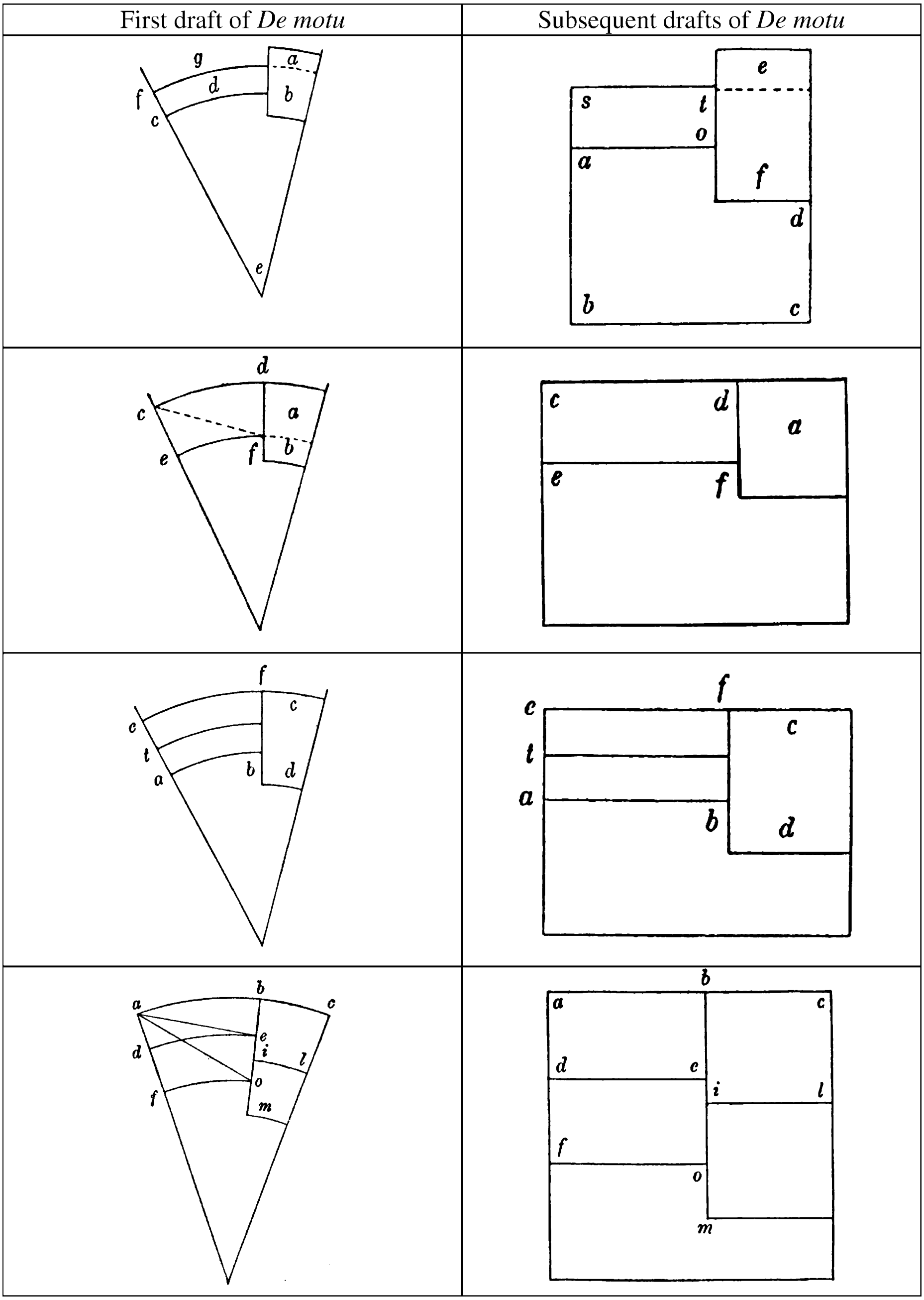

Fig. 2. On the left, diagrams from the first draft of *De motu* (*Opere*, I, pp. 381-384). On the right, diagrams from the subsequent drafts of *De motu* (*Opere*, I, pp. 255–272)

to represent. Galileo's language reflects this mass-like character of elemental water. For example, *abcd* (Fig. 2, upper right diagram) is simply called the "state of water".[10] So why is it that, rather puzzlingly, he represents water as if it were enclosed in a vessel? There is one subtle explanation for this perfectly studied ambiguity.

Attendant with the mass-like character of elemental water came the then hotly debated question of the weight of water in water. It was a natural-philosophical question. As we shall see, Galileo's explanation of buoyancy required the consideration of volumes of water counterbalancing the immersed bodies. In other words, it required volumes of *heavy* water. But in *De motu* Galileo convinced himself that water does not weigh in water! The only way out of this conundrum was to depict water as if it were enclosed in vessels. In this way the water volume needed to counterbalance the immersed body was not to be considered within water, but rather *outside* of it, and *heavy* with respect to the surrounding ambient. In the next two sections, we shall discuss Galileo's reconciliation of these competing views.

b. The weight of water in water

The question of the weight of water in water was an instance of the broader one of the weight of the elements within themselves, and more generally within their natural places. Galileo's mathematical explanation of the mechanism of buoyancy preliminarily required solving that thorny physical question. Let us see why.

The subject matter of the sections of *De motu* relevant for our purposes concerned the motion of heavy bodies in fluid media. According to Galileo, all upward and downward motions can be regarded as "violent" because they are the result of the thrusting out of the body determined by the relative *specific weights*[11] of body and medium. Thus, in *De motu* there no longer was a distinction between natural and violent motions, in sharp contrast to Aristotelian natural philosophy. In Galileo's words, "... when something moves upwards, it is raised by the gravity of the medium".[12] Thus, in Galileo's view, all things move

> [...] because of force and because of the extrusion of the medium. For water violently extrudes a wooden beam submerged forcefully, since by descending [water] returns to its proper region, and does not suffer that what is lighter remains below it. By the same token, a stone is extruded and pushed downwards because it is heavier than the medium.

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[10] "... sit aquae status, ante quam magnitudo in ipsam demittatur, *abcd* [...] Necessarium itaque est, ut, dum magnitudo f demergitur, aqua attollatur". Cf. *Opere*, I, pp. 255–256.

[11] The notion of "specific weight" [*gravitas in specie*] is paramount in *De motu*. Indeed *De motu* begins with the clarification of what it means to claim that something is "heavier than", "lighter than", or "equally heavy as" something else. Cf. *Opere*, I, pp. 251ff.

[12] *Opere*, I, p. 259.

> It is thus evident that this motion can be said violent, although in water, normally, a piece of wood is said to move naturally upwards, and a stone naturally downwards.[13]

Ultimately, Galileo claims, all motions are determined by one cause, gravity.[14] But can the elements actually be said to be heavy? In what sense, if any at all?

The context of the debate on the weight of the elements was Aristotelian natural philosophy, as the language of the elements and their natural places diffused throughout *De motu* clearly indicates. In *De Caelo*, IV, at 311a15, Aristotle argues that we distinguish the absolutely heavy, "as that which sinks to the bottom of all things, from the absolutely light, which is that which rises to the surface of all things".[15] According to Aristotle, whereas fire is absolutely light and earth is absolutely heavy, the intermediate elements, water and air, are neither absolutely light nor absolutely heavy. Furthermore, a few lines after the passage just quoted, Aristotle says that all the elements except fire have weight. Since for Aristotle the heaviness and lightness of all bodies have to be considered in accordance with their elemental composition, he then claims that water, for instance, which is a pure element, will always be heavy except in earth. Thus water in its own place has weight. Only fire does not weigh in its own place. In the late Middle Ages and in the Renaissance, Aristotle's ambiguity in regard to the issue of the weight of the elements gave rise to the question of whether the elements really weigh in their natural places.

It is beyond the scope of this paper, and beyond the competence of its author, even so much as to sketch the history of such a complex question until the late Renaissance, but a few general comments are relevant to my project. By the end of the sixteenth century no consensus was reached. Some Aristotelians, however, believed that the elements weigh in their proper places.[16] Others tried to build a halfway house between the two horns of the elemental dilemma.[17] In his juvenile compilation of natural philosophy, Galileo assembled material on the nature of the elements. We can catch a glimpse of the long-drawn-out debate concerning the nature of the elements in general by looking at Galileo's compilation.[18]

Further, we find two different positions as to the problem of the weight of the elements in Galileo's *De motu*, a fiercely anti-Aristotelian work. Their chronology signals a decisive development in favour of the second one.

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[13] *Opere*, I, p. 259. Note that Galileo uses "*extrudere*" both for the upward motion of the piece of wood and for the downward motion of the stone.

[14] "... potest motuum omnium, tam sursum quam deorsum, causa reduci ad solam gravitatem" (*Opere*, I, p. 259).

[15] Aristotle 1984, I, pp. 508–509.

[16] Girolamo Borri (1512–1592), for instance, a professor of natural philosophy at Pisa University when Galileo was a student there, argued that air and water, as intermediate elements, are both heavy and light in their own places, and that this was Aristotle's opinion. Cf. Borri 1576, pp. 217ff., and, on Borri and Galileo, see De Pace 1990.

[17] Francesco Buonamici (†1603), a colleague of Galileo's at Pisa University in the 1590s, distinguished between "moment[*momentum*]", which cannot be attributed to the elements in their places, and "gravity[*gravitas*]", whose "force[*vis*]" remains in the elements within their places. Cf. the details, in Buonamici 1591, p. 469.

[18] *Opere*, I, pp. 122 ff.

First, in an earlier draft there is a somewhat neutral stance.[19] Since, however, Galileo polemically suggests that those who believe that the elements do not weigh in their proper place should prove their contention, we might tentatively deduce that he was initially skeptical about this claim. Furthermore, he gives the following definition of the meaning of *gravari*, which is repeated throughout *De motu*, and thus, I suggest, constantly held to be valid by him in the 1590s.

> We say that we feel burdened [*gravari*] when some weight is placed upon us which tends downwards because of its gravity, in which case we have to oppose a force so that the weight no longer descends; that *opposing* is what we call *to feel burdened*.[20]

Second, in a later draft Galileo reached a conclusion clearly in favour of the absence of either heaviness or lightness when the elements are located within their own places, so that water, for instance, does not weigh in water.[21] Galileo's argument, a reductio ad absurdum, runs as follows. Water in air is heavy and descends. Thus if a part of water in water is heavy it will descend. But when it reaches the bottom it is necessary that another part of water vacates the place being occupied. This part is forced to ascend, therefore. This part of water will consequently be light in water, a conclusion that seals the reductio.[22] In sum, in Galileo's words, "if a part of water weighed in water, it would descend; which it does not".[23]

Now that the physical ground had been cleared, i.e., the conclusion established that the elements are neither heavy nor light *within themselves*, the mathematical explanation of the mechanism of buoyancy could be reconciled with what we might term Galileo's "natural philosophy of indifference". But to this end water needed to be represented separate from water, outside of its natural place, as if being enclosed in vessels, so that it might be considered as heavy with respect to its surrounding ambient.

In Galileo's natural philosophy of indifference, the Archimedean explanation of the mechanism of buoyancy in terms of the action of water in water became untenable (Fig. 1, second row). Two magnitudes, r, a body lighter than water, and h, a portion of water, are represented located within the spherical shell of water. Both of them are supposed to press down, unequally, upon an underlying layer of water, xop. This pressing action turns out to be impossible for the water magnitude, h, which, according to Galileo, being located within water must be weightless. Furthermore, Galileo claims that his proofs are "less mathematical, and more physical", and his assumptions [*positiones*] "much clearer, and more manifest to the senses" than Archimedes's.[24] Why? We can tentatively answer

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[19] *Opere*, I, pp. 386 ff.

[20] "Tunc dicimur gravari, quando super nos incumbit aliquod pondus quod sua gravitate deorsum tendit, nobis autem opus est nostra vi resistere ne amplius descendat; illud autem resistere est quod gravari appellamus". *Opere*, I, pp. 288, 388.

[21] *Opere*, I, pp. 285ff.

[22] *Opere*, I, p. 286.

[23] *Opere*, I, p. 288. But what about the then commonly alleged experience of the divers' lack of pressure sensation? According to his definition of *gravitas*, Galileo explained the phenomenon claiming that since water does not weigh in water then divers do not experience the pressure of the above water.

[24] *Opere*, I, p. 379.

by looking at the diagrams in the third and fourth rows of Figs. 1 and 2. They accompany the proofs of the quantity of force impelling bodies in water, upward (third row) or downward (fourth row). In Archimedes's diagrams (Fig. 1), weights are represented by lines (Commandino's edition) or surfaces (Tartaglia's edition). In Galileo's diagrams (Fig. 2), no geometrical magnitude represents weight. Weights are "incorporated", so to say, into the solid magnitudes and in the volumes of water raised above the initial level. Galileo's following passage will clarify how (cf. Fig. 2, third row, right diagram). It introduces the proof that solid magnitudes specifically lighter than water, when forcefully pushed within water, move upward with as much force as the weight by which a water volume equal to the submersed portion of the magnitude exceeds the magnitude's weight.

> Let the first state of water, before the magnitude submerges, be according to surface *ab*; and let solid magnitude *cd* be forcefully immersed. Water will rise up to surface *ef*: and since the water which rises, *eb*, has the same volume as the entire immersed magnitude, and the magnitude is lighter than water, the weight of water *eb* will be greater than weight *cd*. Let a part of water, *tb*, be considered, the weight of which is equal to the weight of magnitude *cd*: it must be proven that magnitude *cd* moves upward with as much force as the weight of water *tf*. . . [25]

In Archimedes's version of the same theorem (cf. diagrams in Fig. 1, third row), two magnitudes, *a, d,* are represented upon one another. Their weights are represented by geometrical magnitudes adjacent to them, and magnitude *a* is immersed on the spherical shell of water, whose level does not change. Here is the concise beginning of the text of the proof in Commandino's version.

> Let magnitude *a* be lighter than water: and let the weight of magnitude *a* be *b*; let the weight be *bc* of a volume of water equal to *a*. It must be proven that. . . [26]

The structure of Archimedes's proof is significantly different from Galileo's, too. Archimedes begins by imagining (but not representing in the diagram) a volume of water whose weight is given by the sum of weights *b, g* (*b, c,* in Commandino's edition), which are represented in the diagram. Galileo begins by showing in the diagram the levels of water and the masses of water whose weights will be considered in the

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[25] "Sit itaque primus aquae status, antequam magnitudo in eam demittatur, secundum superficiem *ab*; et demittatur in eam, vi, solida magnitudo *cd*; et attollatur aqua usque ad superficiem *ef*: et quia aqua, quae attollitur, *eb* habet molem aequalem moli totius magnitudinis demersae, et magnitudo ponitur aqua levior, erit aquae *eb* gravitas maior gravitate *cd*. Intelligatur itaque pars aquae *tb*, cuius gravitas aequetur gravitati magnitudinis *cd*: demonstrandum itaque erit, magnitudinem *cd* sursum ferri tanta vi, quanta est gravitas aquae *tf* ...". *Opere*, I, pp. 269–270.

[26] "Sit enim magnitudo a levior *humido*: et sit magnitudinis quidem a gravitas b: humidi vero molem habentis aequalem ipsi a, gravitas sit bc" (Archimedes 1565, p. 4 verso, emphasis mine). Here a magnitude is considered in the "wet". Note that not the noun but the adjective, i.e., the term expressing the quality, is preferred by Archimedes to indicate the fluid. This is in accord with the original in Greek, "ἔστω τι μέγεθος τὸ A κουφότερον τοῦ ὑγροῦ ἔστω δὲ τοῦ μὲν μεγέθ εος τοῦ ἐν ᾧ A βάρος τὸ B, τοῦ δὲ ὑγροῦ τοῦ ἶσον ὄγκον ἔχοντος τῶ A τὸ βΓ" (Archimedes 1913, p. 330). On the other hand, Galileo consistently adopts the noun *aqua* to indicate the fluid masses.

proof. Galileo clinches the proof by showing that water *tb* presses magnitude *cd* upward with as much force as magnitude *cd* resists. Archimedes's proof hinges on showing that magnitudes *a, d* are at rest, and that consequently the forces they exchange cancel each other out.

Thus Galileo abandoned the idealization of the Archimedean spherical shell of water covering the earth. Its underlying physics of the action of water in water was incompatible with the weightless condition of the elements within themselves. He went on to represent volumes of water under different idealized conditions, outside of water, in spaces delimited by an unspecified ambient, so that he could attribute weight to them (Fig. 2). The road was open to an innovative interpretation of buoyancy in terms of an equilibrium mechanism, still in the footsteps of Archimedes, yet such that at the same time it did not negate Galileo's natural philosophy of indifference.

c. *The equality of volumes*

Galileo's explanatory mechanism of buoyancy is basically a balance mechanism, the equality of the weights [*gravitates*] of the floating body and of a volume of water equal to the volume of the body's submerged portion. It was still inspired by Archimedes' analysis of buoyancy on the spherical shell of water. For it is by means of a reductio argument based on the equilibrium of portions of water that Archimedes proves that the surface of any water mass at *rest* is spherical, and that the centre of the spherical surface coincides with that of the earth. Let us consider the diagram in Fig. 3 (upper part). If the water surface were *abc*, instead of the spherical surface, *fbh*, with centre *k*, then different masses of water, *xabo* and *obcy*, would press differently the underlying layer of water, *xop* (*xoy*, in the figure, mistakenly). Thus the whole water could not be at rest. The diagram and the argument suggest that the equilibrium of the water masses could be modeled in analogy with a balance of equal arms. Let us now consider Galileo's representation of a balance of equal arms in Fig. 3 (part below).

According to Galileo, three events might occur in relation to weight *e*. It might remain at rest, move upwards, or move downwards. If it is heavier than weight *o*, it will move downwards, if it is less heavy than *o* it will move upwards, not because it does not have

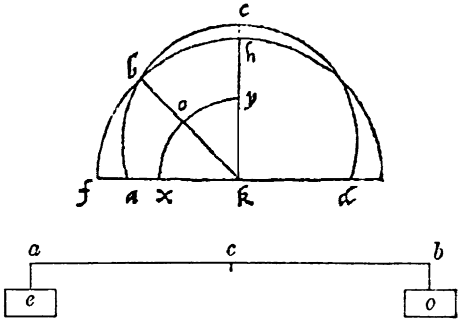

Fig. 3. Above: the equilibrium condition of the spherical shell of water in *On floating bodies* (Archimedes 1565, p. 2 recto). Below: the balance of equal arms in *De motu* (*Opere*, I, p. 257)

gravity, but because *o* is heavier. From which, Galileo argues, it is evident that in the balance both upward and downward motions originate from gravity, but in a different way. For *e*'s upward motion occurs because of *o*'s gravity, whereas *e*'s downward motion occurs because of its own gravity.[27]

The balance of equal arms perfectly models the motions of a body within fluid media. The body is represented by one weight. The other weight will represent a portion of the medium having the same volume as the body's volume. The explanatory mechanism of buoyancy is the same as the more general explanatory mechanism of motion and rest. The body's motion or rest will follow according as the body is heavier than, lighter than, or as heavy as a volume of the medium *equal* to the body's own volume.[28]

In accordance with the balance analogy, in all cases of Fig. 2, Galileo considers volumes of medium *equal* to the volume of the submerged portion of a body. And since he realized that the level of the medium varies as the body submerges, he intuited that a volume of medium equal to the volume of the body's submerging portion must vacate the place gradually being occupied by the descending body. For example, in Fig. 2, upper left diagram, the body's portion *b* is assumed to be equal to the displaced volume of water, i.e., the volume rising from level *d* to level *g*. In the following passage from the first draft of *De motu* Galileo explains his intuition.

> It is thus evident that the volume of water delimited by surfaces *fg, cd,* which is raised, must be equal to the volume of that part of the magnitude that is submerged, that is *b*. For it is most clear that the said volume can neither be smaller, otherwise interpenetration of bodies would occur, nor greater, otherwise a void place would be left.[29]

So (deceivingly) obvious was the intuition of the equality of volumes of the displacing body and the displaced medium that in subsequent drafts of *De motu* Galileo did not even feel the need to repeat it explicitly.[30] Indeed, it was sustained by the even more (deceivingly) obvious axiom of the impossibility of the interpenetration of bodies. Unfortunately this intuition undermined Galileo's theory of Archimedean buoyancy. It led him to commit a fatal error systematically. In the next section, we shall discuss its nature and reasons, and see how in the 1610s Galileo's discovery of the fallacy lurking behind his intuition steered him towards the mechanical theory of buoyancy.

3. Galileo's theory of mechanical buoyancy

a. A new vista

To answer the question of how Galileo explored the tenability of the theory of Archimedean buoyancy, and finally reached a new vista, we first of all need to consider the

[27] *Opere*, I, p. 258.

[28] "... ita ut, nempe, mobile naturale unius ponderis in lancem vicem gerat; tanta autem moles medii, quanta est mobilis moles, alterum in lance pondus repraesentet". *Opere*, I, p. 259.

[29] *Opere*, I, p. 381.

[30] *Opere*, I, pp. 254ff.

manifesto statement, especially in reference to Archimedes, which Galileo announced in the opening sections of the *Discourse on buoyancy*.

> I thus claim that the reason why certain solids descend to the bottom of water is the excess of their gravity on the gravity of water. By the same token, I claim that the excess of the gravity of water on their gravity is the reason why other [solids] do not descend, but rather ascend from the bottom, rising to the surface. This was subtly proven by Archimedes, in the treatise *On floating bodies*; [. . .]. With a different method and with other strategies, I will arrive at the same conclusions, while reducing the causes of these effects to principles both more intrinsic and more immediate, which could also allow us to discern the causes of marvellous phenomena, apparently unbelievable, as that a smallest quantity of water might with its own modest weight raise a body, a thousand times heavier than it.[31]

Now let us consider the correspondent manifesto statement opening the first draft of the *Discourse*, let us call it *Discourse*$_0$, that is preserved among his manuscripts, and which Galileo eventually rejected.

> I thus claim that the reason why certain solids descend in water is the excess of the bodies' gravity on the gravity of water. By the same token, I claim that the excess of the gravity of water on their gravity is the reason why other [solids] do not descend, but rather ascend from the bottom, rising to the surface. This was subtly proven by Archimedes, in the tre atise *On floating bodies*; [. . .]. However, I will try to explain this more clearly, for the benefit of everybody.[32]

Thus Galileo started out with the modest intention of better clarifying what Archimedes had already 'subtly' proven. No mention here of the "principles both more intrinsic and more immediate" [*principii più intrinsechi e immediati*], and of the "marvellous phenomena, apparently unbelievable" [*accidente ammirando e quasi incredibile*], which must have caught Galileo's imagination some time later, as we shall see in the next section.

For the time being, Galileo, closely following the sequence of Archimedes's theorems on floating bodies, goes on to state that a body specifically lighter than wa-

[31] "Dico, dunque, la cagione per la quale alcuni corpi solidi discendono al fondo nell' acqua, esser l' eccesso della gravità loro sopra la gravità dell' acqua, e, all' incontro, l'eccesso della gravità dell' acqua sopra la gravità di quelli esser cagione che altri non discendano, anzi che dal fondo si elevino e sormontino alla superficie. Ciò fu sottilmente dimostrato da Archimede, ne' libri Delle cose che stanno sopra l' acqua; [. . .]. Io con metodo differente e con altri mezzi procurerò di concluder lo stesso, riducendo le cagioni di tali effetti a principii più intrinsechi e immediati, ne' quali anco si scorgano le cause di qualche accidente ammirando e quasi incredibile, quale sarebbe che una piccolissima quantità d' acqua potesse col suo lieve peso sollevare e sostenere un corpo solido, cento e mille volte più grave di lei". *Opere*, IV, p. 67.

[32] "Dico, dunque, la causa per la quale alcuni corpi solidi discendono nell' acqua, esser l' eccesso della gravità di essi corpi sopra la gravità dell' acqua; ed, all' incontro, l'eccesso della gravità dell' acqua sopra la gravità di altri solidi esser cagione che quelli non discendino, anzi dal fondo si elevino e sormontino alla superficie. Ciò fu sottilmente dimostrato da Archimede, ne' libri Delle cose che stanno sopra l' acqua; [. . .]. Ma io, per facile intelligenza di ognuno, tenterò di spiegarlo più chiaramente". *Opere*, IV, p. 36.

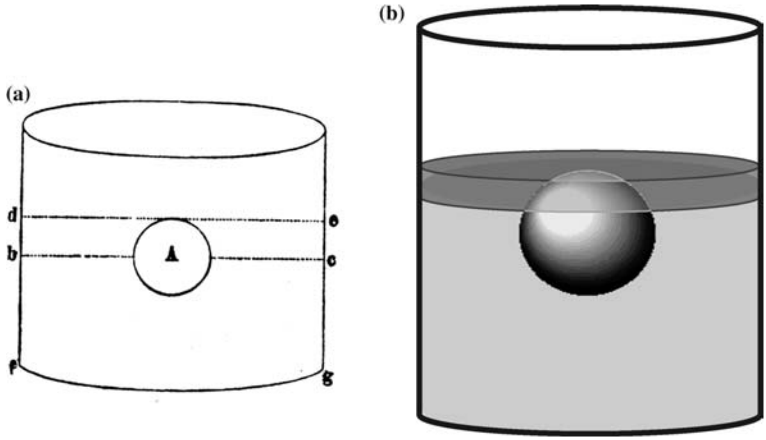

Fig. 4. **(a)** Galileo's original Fig.. **(b)** A reconstruction of Galileo's figure. The darker area represents a volume of water displaced equal to the volume of the sphere

ter cannot submerge wholly under water in a vessel.[33] Galileo furnishes two proofs of this theorem, the first is complete, the second abruptly breaks off in mid-sentence toward the end of the text. We need not analyze both, the second is in fact based on virtually the same strategy as the first. But we need analyze the first since the error it contains reveals the crisis point that Galileo's theory of Archimedean buoyancy reached in the 1610s. The broad argumentative strategy is as follows (cf. Fig. 4).

First of all, Galileo defines what it means to be *equally heavy* ("[t]hose bodies or those matters are called equally heavy, equal volumes of which weigh equally"), and *heavier than* ("[a] matter will be said to be heavier than another one, if a volume of the former weighs more than an equal volume of the latter").[34] Secondly, he posits the axioms that a heavier body cannot be raised by a lighter one, and that according to nature heavy bodies must remain below light bodies. Finally, he draws the conclusion that if the sphere could entirely submerge, then an absurd consequence would follow (i.e., the sphere would raise a weight greater than its own). Therefore the floating of the sphere wholly under water is impossible. Let us now consider the argumentative structure of the proof in finer detail.

If a sphere (specifically) lighter than water could possibly stay wholly under water, since the volume of the water displaced (the darker grey area, in the above figure) must be equal to that of the sphere, then, being lighter than an equal volume of water the sphere would raise a weight greater than its own. This is an absurd consequence according to the axiom posited by Galileo that a heavier body cannot be raised by a lighter one. The whole argument hinges upon the following assumption, which Galileo makes explicit in the text of the proof: *a volume of water equal to that of the submerged solid body must*

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[33] *Opere*, IV, p. 37.

[34] "Chiamonsi egualmente gravi quei corpi o quelle materie, delle quali moli uguali pesano egualmente" (*Opere*, IV, p. 36). "Più grave si dirà una materia di un' altra, se una mole di quella perserà più che un' altra egual mole di questa" (ibid.).

vacate the place that the solid body is to occupy. From this assumption Galileo derives the erroneous consequence that the volume of water raised above the initial level by the submerging body is equal to the volume of the submerged part of the body. Because of the changing level of water during submersion the amount of water vertically displaced above the initial level is actually smaller than the volume of the sphere, and depends on the geometry of the vessel (being indeed equal to the volume of the sphere in a vessel containing an infinite quantity of water). Galileo's fatal error was noted long ago by William Shea.[35] Further, observe that the configuration of bodies in Fig. 4 presents the (impossible) equilibrium of the sphere with respect to the level of water *after* immersion. In other words, we do not know the temporal sequence of events immediately preceding the sphere's floating. The sphere just sits there, as if in an atemporal state of affairs.

What is not hinted at in any of Galileo's diagrams depicting floating bodies from the 1590s to the 1610s is information on the process leading to equilibrium (cf. Figs. 2, 4). The sequence of the changing configurations of the body's positions in relation to the water mass during submersion is never depicted in the diagrams. Hence came the systematic error when Galileo brought that information to bear on his reasoning. He derived a faulty conclusion because the assumption that an equal volume of water must vacate the place being occupied by the body does not apply to fluids trivially. To clinch the proof Galileo relied on deductive reasoning, but his diagrams only capture what I call the "imperfective configuration" of the floating sphere. This fundamental characteristic of imperfectivity, on the other hand, we find throughout Archimedes' treatise on floating bodies. In Archimedes, hydrostatics primarily concerns imperfective configurations of bodies in water. So, I would argue that $\textit{Discourse}_0$ was still based on the unquestioned expectation that hydrostatic principles could be applied independently of the dynamic process leading to floatation.

Gradually, however, Galileo came to terms with two new surprising regularities in the behaviour of floating bodies. First, in small vessels the relation between the volume of the displacing body and that of the displaced water somehow depends on the change of water level during submersion. In any event, it cannot be the relation of equality. If this is the case, Galileo must have wondered, what will become of the apparently obvious principle of the equality of volumes? Second, buoyancy may well be possible in quantities of water smaller than the submerged volumes. But if bodies specifically lighter than water can float in small vessels, even though nowhere near enough water is poured in the vessel to equal the volume of their submerged part, what will the new principles governing this bizarre behaviour be like?

Figure 5 presents diagrams preserved among Galileo's drafts for the *Discourse*. In my view, they are the most extraordinary traces surviving of the function of diagrams in Galileo's methodology. They were developed by Galileo in the 1610s. The first three diagrams (Fig. 5a) depict instances of an idealized phenomenon that I call *mersion*, in order to emphasize Galileo's realization of the geometrical identity of the processes of

[35] Shea 1972, pp. 18–19. Cf. also Galluzzi 1979, pp. 234–235.

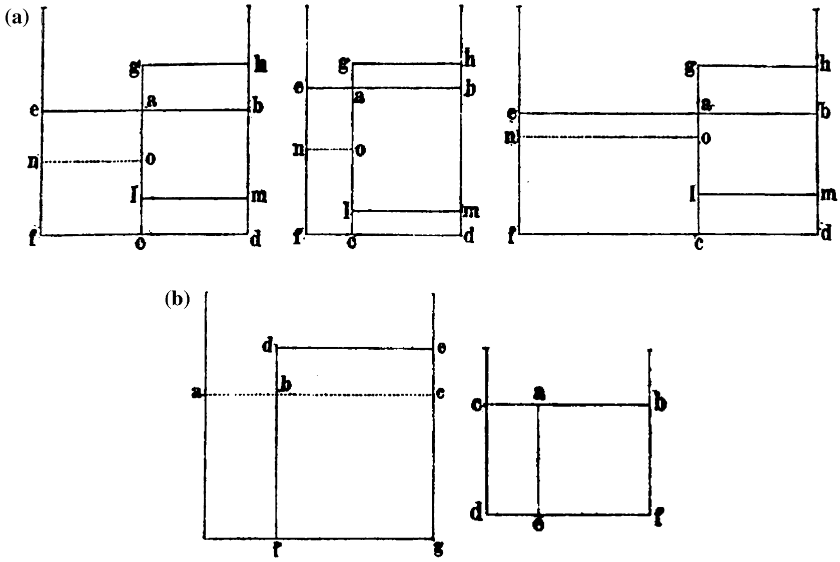

Fig. 5. The most extraordinary remains of the process of maturation in Galileo's understanding of buoyancy in the 1610s (*Opere*, IV, p. 53). **(a)** Pictorial representations of the idealized phenomenon of mersion. **(b)** Pictorial representations of the idealized phenomenon of mechanical buoyancy

im-*mersion* and e-*mersion*. This realization challenged the Archimedean principle of the equality of volumes. Let's see how.

> If a solid cylinder or prism is located in a hollow cylinder or prism, surrounded by water, I say that when the prism is extracted, perpendicularly and with the base parallel to the level of water, the surface of water will descend, and the descent of water will have the same ratio to the ascent of the solid as the solid's surface to the water's surface.[36]

A vessel contains a solid (Fig. 5a). As Galileo's text clarifies, the diagrams are meant purely to illustrate the geometric interaction of body and water level during emersion, in three configurations of quantity of water and solid's volume. Most importantly, the diagrams display the emerging body and the water level in two different positional states, thus hinting at the new information that *mersion* must encode concerning the sequence of events (possibly) leading to floatation.

More generally, the phenomenon of *mersion* profiles:

a) the initial level of water at a certain time, t_0 (either before im-*mersion*, when the body is just above the surface of water, or before e-*mersion*, when the body is totally submerged, but just under the surface of water);
b) a body's subsequent movement of gradual im-*mersion*, or e-*mersion*, and
c) the concomitant change in the level of water.

[36] *Opere*, IV, pp. 52–53. The proof is based on proportional reasoning.

In addition, mersion is open to different final outcomes, namely, the floating or sinking of the body. In other words, it does not profile the eventual positional state of the body with respect to the level of water at a subsequent time. Mersion is a processual phenomenon, it is not static. Moreover, it does not profile the size of the vessel, in fact it profiles the prototypical interaction between body and water level, quite independently of the geometrical characteristics of body and vessel. Mersion, in other words, profiles a body's actions of gradually becoming wet, i.e., being gradually surrounded by water, or gradually becoming dry, i.e., being gradually surrounded by air. And, it does so in relation to a mass of water whose geometry varies according as the body moves in it.

Eventually Galileo was able to discover the ratio of the volume of displaced water to the volume of the portion undergoing mersion of a prism or cylinder in a vessel (a theorem which, however, he only published as an addition to the second edition of the *Discourse*).[37]

The diagrams must be mentally animated, so to speak, to generate the frames of a moving picture. One has to visualize the motion of emersion, vertically, and the effects of the vessel's varying width, horizontally. The case in the centre of Fig. 5a is crucial. A vessel (*efdb*) can actually contain much less water (*efca*) than the submerged portion of the solid (*acdb*), represented both in its initial position, and in a subsequent position, *glmh*, after the beginning of emersion. The diagram reveals that the principle of the equality of the displacing body and the displaced water is totally inadequate to explain the geometric interaction of solid and fluid bodies. Geometrical considerations apart, Galileo must have asked himself, will the prism's floatation still be possible? Under which circumstances?

The answer is immediately found by Galileo, in two steps, with the help of the two diagrams shown in Fig. 5b. Within the framework of the theory of Archimedean buoyancy, both diagrams *might* be interpreted as anomalous instances of buoyancy phenomena. The quantity of water (*abf*, on the left, and *caed*, on the right) is smaller than the submerged portion of the solid (*cbfg*, on the left, and *abfe*, on the right). But in fact they need not be interpreted as anomalous instances, at all. Here Galileo begins to challenge the explanatory mechanism of his early theory of Archimedean buoyancy. He introduces the idealized phenomenon of mechanical buoyancy, on the analogy of mechanical equilibrium in the balance of different arms. First (Fig. 5b, left diagram), Galileo proves that a certain ratio is required of the water's level to the solid's height for the equilibrium of water and solid (specifically lighter than water). Second (Fig. 5b, right diagram), he proves that a solid specifically lighter than water will start rising if the level of water reaches the same height as the solid.

The texts associated with the diagrams are extremely interesting, because in the first Galileo, almost groping for a proof, adopts an "analytic" style, in conjunction with Euclidean proportional reasoning. In the second, he makes explicit the idealization of

[37] *Opere*, IV, pp. 71–72.

buoyancy as an instance of the mechanical equilibrium in the balance of different arms. It is worth reading the passages in their entirety, in turn. Here is the first.

> Let water *af* be heavier than *dg* as *df* is to *fb*: water *af* will be at rest. It will be proved if one shows that descent *ab* is to ascent *de* as weight *dg* is to weight *af*: but descent *ab* to ascent *bc* is as *bc* to *ba*, or, as *bg* to *af*: one has to show, therefore, that as *bg* is to *af*, so is weight *dg* to weight *af*. But weight *dg* to weight *af* has the ratio compounded[38] of [the ratio] of volume *dg* to volume *af* and of [the ratio] of the specific weight of *dg* to the [specific] weight of *af*: one has thus to show that volume *bg* has to volume *af* the ratio compounded of [the ratio] of volume *dg* to volume *af* and of [the ratio] of the specific weight of *dg* to the specific weight of *af*: and this will be the case if the specific weight of *dg*, or *bg*, to the specific weight of *af* is as the volume *bg* to the volume *gd*; which is true.[39]

Note that in accord with the analytic style, when Galileo eventually hits upon a true proposition, the fragment ends with the sentence "which is true [*quod verum est*]", instead of the canonical *quod erat demonstrandum* which would seal a synthetic proof. Here is the second fragment.

> Let the solid *af* be [specifically] lighter than water, and let *ce* be water: I say that if freed [the solid] will rise. For if *af* were as heavy as water, then as the weight of volume *ce* is to the weight of *af* so volume *ce* would be to volume *af*: but, being water [specifically] heavier, gravity *ce* will have a greater ratio to gravity *af*, than volume *ce* to volume *af*, that is, than surface *ca* to surface *ab*, that is, than the ascent of the solid to the descent of water: therefore water *ce* will descend, while solid *af* will rise.[40]

In a subsequent draft, Galileo reworked the two texts associated with the two diagrams. Now that he had found a demonstrative sequence he reversed the procedure. He

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[38] *Compounded* ratio is a difficult concept that cannot be translated into the concept of multiplication of ratios, which is based on algebraic symbolism. It has a long history of interpretations, rooted in the 23^{rd} proposition, Book VI, of Euclid's *Elements*, which scholars have just begun to explore. For compounding ratio in Euclid, cf. Drake 1973, 1974, and 1987, Saito 1993 and Rusnock and Thagard 1995. For Galileo's use of compounded ratio, I have mostly relied on Palmieri 2002, pp. 130–172, and Giusti 1993, pp. 44–56. Sylla 1984 studies the passage from an ancient and medieval tradition of compounding ratios (whose best expositions are to be found, according to her, in Thomas Bradwardine and Nicole Oresme), in which compounding was regarded as addition, to the modern tradition in which compounding is regarded as multiplication (purportedly sanctioned by Newton). See also Sylla 1986. Jesseph 1999, pp. 153–59, expands on the significance of compounded ratio, in the seventeenth century, especially in the context of the 'mathematical war' between J. Wallis and T. Hobbes.

[39] *Opere*, IV, p. 53. The language of this proof raises an interesting question. It is in Latin, whereas virtually all of the remaining preparatory material for the *Discourse*, as well as the *Discourse* itself, is in Italian. Perhaps at the beginning Galileo was undecided as to what language to choose for the *Discourse*.

[40] *Opere*, IV, p. 53.

abandoned the analytic style of the first text, offering a synthetic proof of the following theorem (still cast in the language of proportional reasoning, though).

> If a solid prism is less heavy than water, [when it is] placed in a vessel with parallel walls vertically with respect to the horizon, and water is poured in, it will not be raised until its entire height has the same ratio to the height of the immersed part as the ratio of the specific weight of water to the [specific] weight of the solid; if more water is poured in the solid will be raised.[41]

According to the synthetic style, the final version of this theorem's proof, which Galileo published in the *Discourse*, ends with the Italian equivalent of the canonical *quod erat demonstrandum*.[42]

In both drafts, it was the explanatory mechanism of the balance of different arms and weights that allowed Galileo to clinch the arguments. "It will be proved if one shows that descent *ab* is to ascent *de* as weight *dg* is to weight *af*. . . ". Again, ". . . gravity *ce* will have a greater ratio to gravity *af* [. . .] than the ascent of the solid to the descent of water". These sentences implicitly make appeal to the principle of equilibrium in a balance of different arms and weights, according to the Archimedean inverse proportionality of weights and distances from the fulcrum. Galileo will make the proportionality explicit in the published version of the *Discourse*, with an example, as follows.

> It happens, thus, [. . .] the same that happens in the Roman balance, in which a weight of two pounds will counterbalance another of two hundred, providing that in the same time the former moves through a space one hundred times greater than the latter; which will happen when one arm of the balance is one hundred times longer than the other.[43]

A decade earlier, around the 1600s, Galileo had worked on the theory of mechanics, possibly compiling notes for a treatise which he never published, and which are nowadays collectively referred to by scholars as *Le mecaniche*.[44] In one of those notes he explained his analysis of the principle governing the functioning of the balance of different arms in detail.

> Now, let us consider the motion of weight B, while it descends in E, and that of [weight] A, while it ascends in D. Doubtless we will find space BE to be as greater than space AD as distance BC is longer than [distance] CA, since the two angles formed at centre C, DCA and ECB, are equal, being at the apex, and since consequently the two circumferences, BE, AD, are similar, having to each other the same ratio of the radii BC, CA, which describe them. Thus the speed of weight B's motion, which descends, is as greater than the speed of the other mobile, A, which ascends, as the gravity of the latter exceeds the gravity of the former. And since weight A cannot but be raised in D slowly, as long as the other weight descends in E swiftly, it will neither be astonishing, nor a violation of the natural order, that the speed of weight B's motion compensates the greater resistance of

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[41] *Opere*, IV, p. 55. In this case, Galileo reverted to Italian.
[42] "...che è quello che bisognava dimostrare". *Opere*, IV, p. 76.
[43] *Opere*, IV, p. 78.
[44] *Opere*, II, pp. 149–191.

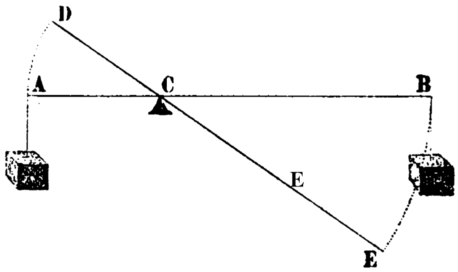

Fig. 6. Galileo's diagram illustrating the principle governing the functioning of the balance of different arms (*Opere*, II, p. 163)

> weight A, so long as A moves toward B with laziness, while the other [weight, B,] swiftly descends in E.[45]

To sum up, mersion's processual nature suggested to Galileo that he needed to explain buoyancy not in terms of weights of equal volumes, but in terms of different weights, different displacements, and different speeds. The pictorial representations became the basis of a new idealization of buoyancy phenomena. A theory of mechanical buoyancy was rapidly fledging. One cognitive pursuit, the unification of the two theories of buoyancy through paradox building and resolution, now became crucial to the transformation of the mind, and eventually to belief fixation. Might the same mechanical principles be capable of explaining buoyancy phenomena under all idealized conditions? The next two sections are devoted to this final stage.

b. Explanatory unification

There are two stages in Galileo's approach to explanatory unification. The first concerns a proof that solids specifically lighter than water submerge up to such a depth that a volume of water equal to the submerged portion of the solid weighs as much as the entire solid. The second concerns a proof that the explanatory principle of the theory of Archimedean buoyancy could be derived from that of the theory of mechanical buoyancy. The whole strategy turned around the construction and resolution of a paradoxical case of buoyancy, which only the mechanical theory could resolve. I shall discuss the two stages in turn.

The draft accompanying the reworking into a general theorem of the two fragments discussed above (cf. the diagrams in Fig. 5b) ends with the unproven statement that "... equilibrium will only follow when a part, *bg*, of solid *dg*, is submerged equal to a volume of water which would weigh as much as the entire solid, *dg*".[46] It was almost like a promissory remark that Galileo made to himself. He duly honoured the promise in the published *Discourse*, where he furnished the proof (Fig. 7).

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[45] *Opere*, II, p. 164.
[46] *Opere*, IV, p. 55.

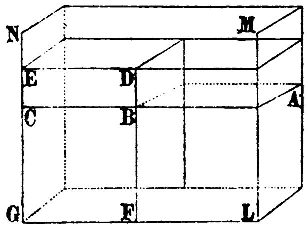

Fig. 7. The diagram associated with the proof that that the explanatory principle of the theory of Archimedean buoyancy could be derived from that of mechanical buoyancy (*Opere*, IV, p. 75)

Galileo considers a solid prism, DFGE, specifically lighter than water, within a vessel, MLGN. The ratio of the prism's height, DF, to height FB (level of water) is chosen equal to the ratio of the specific weigh of water to that of the prism (this is the level for the water to remain at rest, as already known to Galileo, cf. the first fragment associated with Fig. 5b). The proof is as follows.

> According to the preceding lemma[47], the absolute weight of a volume of water equal to volume BG to the absolute weight of prism DG has the ratio compounded of the ratio of volume BG to volume GD, and of the specific weight of water to the specific weight of the prism: but the specific weight of water to the specific weight of the prism has been chosen as the volume GD to volume BG: therefore the absolute weight of a volume of water equal to volume BG to the absolute weight of solid GD has the ratio compounded of the ratio of volume BG to volume GD, and of [the ratio] of volume DG to volume GB, which is a ratio of equality. Thus, the absolute weight of a volume of water equal to the portion BG of the prism is equal to the absolute weight of the entire solid DG.[48]

Thus, in the first step, Galileo has indirectly shown the validity of an assumption that, as we shall see in a moment, he had already called into question in *De motu*, i.e., the legitimacy of comparing equal volumes of fluid and solid body. This validity Archimedes, too, had only implicitly hypothesized in his treatise on floating bodies. Further, Archimedes, as well as the young Galileo, could only prove that bodies specifically lighter than water will not wholly submerge. They could not specify a condition for the equilibrium of the floating body (see Table 1).

Now, in presenting the second step in Galileo's strategy of explanatory unification, I wish to develop a fascinating insight by William Shea. He realized that the paradox Galileo advertised in the *Discourse* was not an account of an empirical discovery "crying out for mathematical interpretation", but rather the very product of mathematical

[47] In this lemma, Galileo proves that the weights of solids have the ratio compounded of the ratio of the specific weights and the ratio of their volumes. *Opere*, IV, p. 74.

[48] *Opere*, IV, p. 76.

Table 1. Archimedes's and Galileo's theorems on bodies specifically less heavy than water

Archimedes(Commandino's edition) (cf. Fig. 1, first row)	Galileo's *De motu* (cf. Fig. 2, second row)
Proposition IV. Any solid magnitude whatever, which is lighter than water, when placed in water, will not wholly submerge, but a part will rise above the surface of water.[49]	Second demonstration, in which it is proven that the things that are less heavy than water cannot wholly submerge.[50]

investigation, steering Galileo's mind towards "extraordinary phenomena".[51] In fact, more radically, I believe that since the 1590s Galileo had been literally seduced by the intricacies of the interaction body/ fluid, and in the 1610s he finally pushed the boundaries of the theory of buoyancy to the limits of paradox.

What was the *accidente ammirando*, the solution to which Galileo strategically advertised in the manifesto statement of the *Discourse*? We have seen that Galileo tells us that the *accidente ammirando* is a great paradox, as "that a smallest quantity of water might with its own modest weight raise a body, a thousand times heavier than it [... *quale sarebbe che una piccolissima quantità d' acqua potesse col suo lieve peso sollevare e sostenere un corpo solido, cento e mille volte più grave di lei*]", which he explains in detail a little further on in the *Discourse*.

Indeed, the *great paradox* has an antecedent in *De motu*. It has never been recognized as such, however, perhaps because its structure is the inverse, so to speak, of the structure of the *great paradox*. The latter, as I shall explain in detail below, shows that a quantity of water, however small, is sufficient to keep a body afloat whose specific gravity is smaller than that of water, if certain geometrical conditions obtain and mechanical principles are applied. The former shows that not even the whole sea could keep a pebble afloat, whose specific gravity is greater than that of water, because the explanatory mechanism of Galileo's theory of Archimedean buoyancy hinges on the principle of the equality of the volumes of the interacting bodies. Here the paradoxicality lies in the principle itself. Why, asks Galileo, must this principle hold true, in the first place? Why should the comparison be made between the volume of the pebble and an equal volume of water, instead of the whole sea?[52] His answer is tantalizingly cryptic. Not that he shies away from it, he simply asserts that the set of proofs, given in the *De motu*, and showing which bodies float and which sink, holds the key to the resolution of the paradox.[53]

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[49] Solidarum magnitudinum, quaecunque levior humido fuerit, demissa in humidum non demergetur tota, sed aliqua pars ipsius ex humidi superficie extabit (Archimedes 1565, p. 3 recto).

[50] Secunda demonstratio, in qua probatur, ea quae leviora sunt ac aqua non posse demergi tota (*Opere*, I, p. 256).

[51] Shea 1972, p. 22.

[52] *Opere*, I, pp. 254–257, 347.

[53] *Opere*, I, pp. 257, 364.

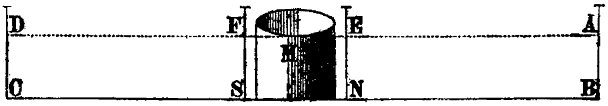

Fig. 8. Galileo's figure illustrating the great paradox. The cylinder at the centre is immersed in a small vessel FSNE whose width could be extended indefinitely, for example, as far as DCBA

Two decades later Galileo displayed more steadfastness. In the 1590s he had to grapple with some paradoxical issues inherent in the Archimedean tradition, now he proceeded to create an even more bizarre paradox, all of his own.

Figure 8 shows Galileo's illustration of the *great paradox* in the *Discourse*. The cylinder at the centre is immersed in a small vessel FSNE whose width could be extended indefinitely, for example, as far as DCBA.

> . . . if a solid [specifically] less heavy than water is placed in a vessel of any size whatever, and water is poured around it, up to such a height that a volume of water equal to the submerged portion of the solid weighs absolutely as much as the entire solid, the latter will be supported by water, regardless of whether the quantity of water poured in is immense or negligible [. . .]. While solid M rises, its ascent has the same ratio to the descent of the circumfused water, ENSF, as the surface of water to the surface, or base, of solid M. This base has the same ratio to the surface of water, AD, as the descent of water AC to the ascent of solid M. Thus, by perturbed proportion[54], while solid M rises, the descent of water ABCD has the same ratio to the descent of water ENSF as the surface of water EF to the surface of water AD, that is as the entire volume of water ENSF to the entire volume ABCD, given that they have the same height. It is manifest, therefore, that while the solid is raised and pushed, the speed of motion of water ENSF overcomes water ABCD as much as the latter overcomes the former as to quantity: so that their *momenti* are equal in this operation.[55]

Regardless of the quantity of water of which the vessel is capable, indeed regardless of the size of the vessel, the equilibrium condition of mechanical buoyancy requires that water must be circumfused around the cylinder up to a height such that a volume of water equal to the submerged portion of the cylinder would weigh as much as the cylinder. Yet, crucially, not all that water must be present, as the case of the narrow vessel, FSNE, almost lining the cylinder on the outside, clearly shows. What remains constant in this case of equilibrium are simply the *momenti* of water and solid.

Later on, some time between the first and the second edition of the *Discourse*, which was published at the end of 1612, Galileo found another spectacular way to visualize the great paradox.

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[54] Cf. Euclid 1956, II, p. 115.

[55] *Opere*, IV, p. 76–77. "*Momento*" is defined by Galileo at the beginning of the *Discourse* as follows. "Momento, according to the mechanicians, means that virtue, that force, that efficacy, with which the motor moves and the moved resists; this virtue depends not only on simple gravity, but on the speed of motion, and on the different slopes on which motion occurs, because a heavy body while descending produces more impetus along a more inclined space than along a less inclined one" (*Opere*, IV, p. 68). On Galileo's use of *momento*, cf. Settle 1966, pp. 201–242, and Galluzzi 1979.

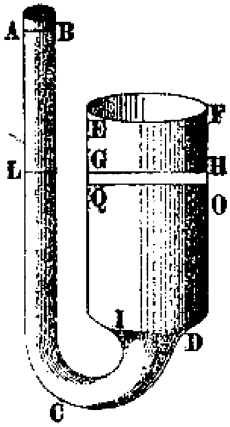

Fig. 9. A graphic instance of mechanical buoyancy. This figure was published by Galileo only in the second edition of the *Discourse*

A large vessel, EIDF, is connected with a very narrow cannula, ABLC (Fig. 9). When water is poured in, it will level at LGH. Why so, asks Galileo, given that the quantity of water in the vessel is much greater than that in the cannula? Why, in other words, is the area of the cross-sections of vessel and cannula totally irrelevant for equilibrium, so that water levels at the same height in both of them? Note that here a small quantity of water is shown in the cannula balancing a much greater quantity of water in the large vessel. The equilibrium condition of the theory of mechanical buoyancy requires that when the water in the cannula reaches a height equal to that of the water in the vessel, the *momenti* of the two volumes of water be equal, regardless of their absolute weight. If one imagines the cannula becoming larger and larger, one will at some point find the volume of water in it equalling, and finally surpassing that of the vessel. The roles of cannula and vessel are, so to say, swapped at that point. This gradual transformation has not altered the condition of the equality of the *momenti*. It has thus further illustrated an extraordinary instance of the phenomenon of mechanical buoyancy.

The crafting of the *great paradox* and its resolution transformed Galileo's mind. With the stupor of paradox resolution he paved the way to a profound change in his belief system. With the achievement of explanatory unification, he sealed the fixation of belief in the theory of mechanical buoyancy. Galileo may also have thought that he had eventually equalled, if not surpassed, Archimedes himself, whom, in his juvenile work on Hiero's crown, he had referred to as a "divine man", to whose genius all other intellects must needs be inferior, so that little hope remains that results similar to his will ever be found.[56]

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[56] "... dalle quali pur troppo chiaramente si comprende, quanto tutti gli altri ingegni a quello di Archimede siano inferiori, e quanta poca speranza possa restare a qualsisia di mai poter ritrovare cose a quelle di esso simiglianti". *Opere*, I, pp. 215–216.

c. Paradox and persuasion

Stillman Drake has furnished a convincing reconstruction of the basic events concerning the 1610s controversy on the role of shape in buoyancy, which I will follow.[57] The controversy was occasioned by a discussion on the nature of floating bodies at a gathering of literati, probably at Filippo Salviati's house in Florence, in the summer of 1611. According to Drake, Vincenzio di Grazia, professor of philosophy at Pisa University, noted that one of the effects of cold is to produce condensation, as one can see in the formation of ice. Galileo replied that he thought that ice was not condensed water, but rather rarefied water, since ice floats in water, which, in accord with Archimedes' theory of buoyancy, means that it must be lighter than water. The fact that ice floats, Galileo was rebuked, depends on its having a large shape incapable of "pushing through the resistance of water", not on its being lighter than water.[58] A few days later, Galileo was told by Vincenzio di Grazia that someone he had encountered was able to prove experimentally that figure plays an "important role in the floating of bodies".[59] The man was Ludovico delle Colombe, a philosopher who was already known to Galileo, at least since the former attacked the Copernican system, earlier in 1610 or 1611.[60] Colombe, according to Drake, began to show publicly the substance of his experiments, in which pieces of ebony (a type of wood whose specific weight is slightly greater than water) shaped like thin laminae could be floated while others shaped like, for example, spheres sunk. Galileo and Colombe agreed to set a later date for a presentation of the experiments. But Colombe, for whatever reason, did not turn up. Subsequently, Colombe made a new appointment, at Salviati's house. There "he appeared with a number of followers [. . .]. By then, however, the controversy had become notorious, and Galileo's foes had been using it to discredit him at court with his august employer, Cosimo II".[61] Thus, Drake concludes, warned by the Grand Duke not to engage in public disputes, Galileo "refused to be drawn into argument with Colombe [. . .]; rather, he said, he would write out his arguments".[62]

The anomalous experience produced by Ludovico delle Colombe consisted of a simple lamina of ebony which floated because of surface tension when it was gently placed on water. The only reason for the lamina to float should be its being lighter than water in terms of specific weight. Galileo came up with an analysis of the anomalous

[57] Drake 1970. Biagioli 1993, pp. 159–209, furnishes interesting insights on the social context of the controversy, none of which, however, calls into question Drake's account of the basic facts. See also De Ceglia 1999 and Camerota 1995.

[58] Drake 1981, p. 22. Cf. also Drake 1995, 169ff. In *De Caelo*, at 313a-313b, Aristotle mentioned the shape of bodies as being actually responsible not for their upward or downward motion, but for their motion being faster or slower. Although he referred to the floating of flat objects made of iron or lead, Aristotle appears to have been more interested in attacking the atomistic theories of Democritus (Aristotle 1939, pp. 366–368).

[59] Drake 1970, p. 159.

[60] Colombe's *scrittura* was commented by Galileo and has been printed together with Galileo's notes, in *Opere*, III, pp. 251–290.

[61] Drake 1970, pp. 160–161.

[62] Drake 1970, p. 161.

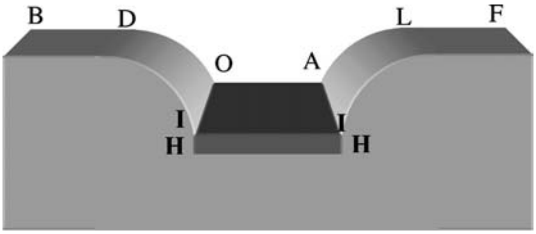

Fig. 10. A reconstruction of Galileo's figure accompanying his explanation of the phenomenon of the floating laminae

phenomenon which allowed him to reconceptualize the entire situation in terms of his theory of mechanical buoyancy. This reconceptualization took the form of an analysis of the workings of the small banks [*arginetti*] forming around the rim of the floating ebony lamina (Fig. 10).

The ebony lamina OAIH floats because small banks BD, LF form around its rim, thus creating a cavity within which air descends. The combined specific weight of the lamina and the air inside the cavity is, according to Galileo, less than that of water. Therefore the phenomenon does not violate the explanatory principles of the theory of mechanical buoyancy. On his analysis of the small banks, and on his unified theory of buoyancy, Galileo founded the paradoxes that he disseminated in the *Discourse*. Let us see how.

Figure 8 has a powerfully suggestive visual structure. The actual quantity of water does not matter as long as just enough of it is present to make the cylinder wet up to the minimum height. The latter determines what I call, for a reason that will be clear in a moment, the principle of the "virtual equality of volumes". This principle explains the floatation of bodies under the idealized condition of being *just wet*. One such body submerges in *just enough water* for the water level to reach a height such that a volume of water equal to the submerged portion of the body, let it be called the *virtual volume*, would weigh as much as the whole body. This *just enough water* cannot be specified independently of the geometry of vessel and body. Since the *virtual volume* does not have to be actually present in the vessel, and all that is needed for the body's floatation is *just enough water*, I have chosen the adjective *virtual*.

Here the unification of explanations is graphically conveyed in the picture by Galileo's artifice of suggesting the variable geometry of the vessel. If the vessel is extended indefinitely, possibly as far as covering the earth's entire circumference, the diagram will continuously metamorphose into a depiction of floatation on the spherical shell of water, where always enough water is present to equal the virtual volume, and the change in water level is all but negligible. But if it is horizontally contracted enough, the diagram will continuously metamorphose into a depiction of floatation where the change in water level is highly significant, the mechanical principles obtain, and *just enough water* wets the body. As a matter of fact, the vessel's walls could even be eliminated from the diagram, and the cylinder be simply represented together with the horizontal water level, on the assumption that *just enough water* wets the cylinder. Such a diagram could be regarded as a unified pictorial representation of buoyancy phenomena.

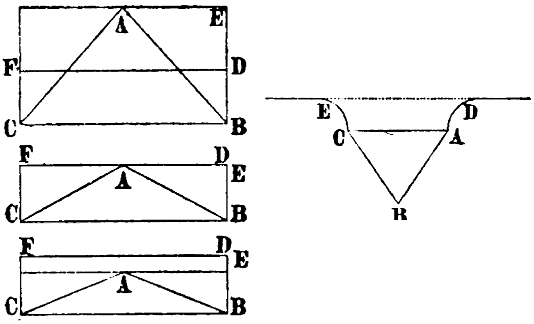

Fig. 11. A cone or pyramid that does not sink when place on water can be formed of any material (*Opere*, IV, p. 114)

The paradoxes invented by Galileo to defend the claim of the independency of buoyancy from shape were based on unified pictorial representations of buoyancy phenomena, under the idealized condition that *just enough water* wets the bodies. The paradoxes flesh out Galileo's discussion of what he considered blatant counter-examples to the Aristotelian claim. Among them the case stands out of cones and pyramids, specifically heavier than water, which under certain surface conditions float point down, yet sink when floated point up. Galileo was evidently delighted in bringing into prominence such counter-examples as, in his eyes, negated the Aristotelian claim to the role of shape in buoyancy.

According to Galileo (cf. Fig. 11, on the left),

> . . . it is possible to form of any material a pyramid or cone upon any base, which, placed on water, is not submerged, or wetted except on the base. Let the maximum possible height of the ridge be line DB, and let the diameter of the base of the cone, of any given material, be line BC at right angles to DB, and let the ratio of specific weight of the material of this pyramid or cone be in the same ratio to the specific weight of water as the [maximum] height of ridge DB is to one-third the height of the pyramid or cone ABC, whose base has the diameter BC. Then I say that the cone ABC, and any other lower than it, will rest on the surface of the water BC without being submerged.[63]

He now claims that

> there is no material so heavy, even gold itself, from which it is not possible to form all sorts of shapes that, by virtue of the adherent air above them (and not through resistance of water to penetration), remain sustained so that they do not sink to the bottom. Moreover, to remove a certain error, I shall show how a pyramid or cone placed point down in water will rest without going down, while the same placed base down cannot be made to float – though just the opposite should happen if difficulty in fending the water were what

[63] Drake 1981, pp. 131–136. This result and those that follow depend on a preliminary theorem proven by Galileo. This theorem asserts that "solids whose volumes are inversely proportional to their specific weights are equal in absolute weight" (*ibid.*, 131). The proof is not too difficult. It is based on the Euclidean theory of proportions. See Drake 1981, pp. 136ff.

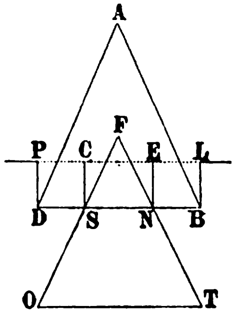

Fig. 12. A cone of any material cannot float base down. Note that Galileo implicitly assumes that the small banks are now rectangular. Cf. *Opere*, IV, p. 116

> impeded descent, since the same cone is much better adapted to fend and penetrate with its very sharp point than with its broad and spacious base.[64]

Let us consider cone CBA in Fig. 11, on the right. The level of water is ED. Let the cone's specific weight be double that of water. Let its height be triple the maximum possible height of the small banks. The cylinder of air within the cavity has a volume triple that of the cone. Thus the whole volume of the mixed solid formed by the cone and the air inside the cavity is double that of the cone itself. Since the cone's specific weight has been taken double that of water, then a volume of water equal to the volume of the mixed solid weighs as much as the cone. Therefore, Galileo concludes, the cone will not sink.[65] Note that Galileo simply indicates the level of water. In accordance with the principle of the virtual equality of volumes he only needs to assume that *just enough water* is available to wet the mixed body.

Let us now turn to Fig. 12 and examine the situation when the cone is placed base down. Since the cone's specific weight is double that of water its absolute weight will be double that of a volume of water equal to the volume of the cavity (remember that the volume of the cavity is equal to that of the cone since the height of the cone has been taken triple the maximum possible height of the small banks). Under these circumstances the cone cannot float.

In all these cases, all Galileo needs to depict in the diagrams is the idealized body and the level of water. According to the theory of mechanical buoyancy, the principle of the "virtual equality of volumes" fully explains these extraordinary instances of buoyancy.

Galileo's fascination with paradoxes and belief change did not end with cones and pyramids, however. We have a confirmation of the import of his methodology on his persuasion techniques in a letter on buoyancy phenomena written to Tolomeo Nozzolini, a professor at Pisa University. Some time during the dispute, Monsignor Alessandro Marzimedici, Archbishop of Florence, asked Nozzolini to read Galileo's *Discourse* and to write a report.[66] Nozzolini's report, later on brought to Galileo's attention, was very

[64] Drake 1981, pp. 140–141.
[65] Drake 1981, p. 141.
[66] *Opere*, IV, p. 289.

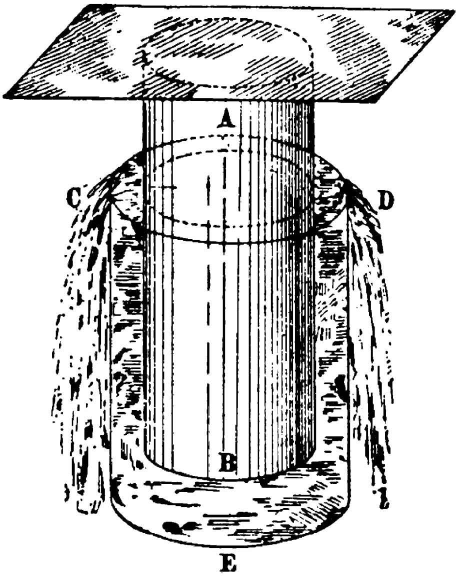

Fig. 13. Galileo's explanation to Nozzolini of the paradox of the unchanging weight of a vessel of water in which a fixed solid gradually submerges making water spill out (cf. Opere, IV, 307)

appreciative of Galileo's explanation of the strange behaviour of the small banks. In exchange, Galileo wrote to Nozzolini a mini-tract in the form of a letter, once again pushing the boundaries of his theory to the limits of seemingly unexplainable phenomena. Galileo tickled Nozzolini's curiosity with the following paradox (Fig. 13).[67]

If a person holding a vessel filled up with water were to raise the vessel (CED) against a solid cylinder (AB), somehow fixed and immobile, then that person would not feel any decrease in weight while the vessel's water is gradually expelled by the solid. This holds true no matter how much water is spilled, as long as a little quantity of it remains in the vessel. Why? In Galileo's words,

> ... the power supporting the solid in A, while the latter was outside of water, felt less weight than after the solid submerged in water; for, there is no doubt that if I hold a stone in air with a rope, I will feel more weight than if someone put a vessel of water below the stone, within which the stone immerges. Thus, since the fatigue of the virtue holding the solid AB decreases, while the latter submerges in the water of vessel CDE, which gradually moves toward it, and it being impossible that the weight of the solid disappears, then the weight of the solid must lie on water, and in consequence on vessel CDE, and thus on him who holds it. Since we know that a solid submerging in water gradually loses as much weight as the weight of a volume of water equal to the volume of the immersed

[67] Galileo begins the explanation of the paradox with the following words, in which Galileo promises a solution, while at the same time emphasizing how one has to experience the marvelous nature of the phenomenon: *Ma per ben dichiarare il tutto, ed insieme accrescer la meraviglia...* (*Opere*, IV, p. 306–7).

> portion of the solid, it will be easy to understand that the fatigue of the virtue holding solid AB in A will decrease as much as water decreases the weight of the solid. Thus solid AB weighs on the force holding vessel CDE as much as the weight of a volume of water equal to the volume of the submerged portion of the solid. But to the volume of the submerged portion of the solid is equal the volume of the water spilled out of the vessel. Therefore, because of this spilling of water the weight lying on the virtue holding the vessel will not decrease.[68]

To sum up, in order to persuade his interlocutors through the same cognitive experience that had led him to belief change, Galileo relied on the wonder raised by paradoxes, and on his theory's power to rationalize such apparent violations in the order of nature. Thus paradoxes played a very special role in Galileo's methodology. They shaped his search for a new theory, and above all constituted a form of cognitive probing, which brought to light the unification of theories, and ultimately the turning of the mind leading to belief fixation.

4. Conclusion

As is well known, Galileo became completely blind toward the end of his life, sadly before he could see a printed copy of his celebrated *Two New Sciences*. In a few letters he exchanged with friends around that time, we find moving comments on his irremediably impaired capacity for deep thought, and especially on his compromised ability to pursue the search for geometrical proofs owing to the loss of sight. Those comments have never been considered in relation to Galileo's revolutionary methodology. They stress the indispensable role that the sense of sight plays, in connection with the visuo-spatial information afforded by the diagrams on which proportional reasoning is based. I now wish to present those remarks briefly, before drawing the conclusions. In response to an inquiry by his pupil, Benedetto Castelli, Galileo wrote

> ... and if I could regain a less troubled condition, I would explain to you my concept; but since it is a very complex excogitation, or structure, difficult to elucidate, especially with naked words, it being impossible for a blind person to draw a diagram, I am unable to say anything more specific, except that my strategy depend on a proposition by Euclid.[69]

To the friend, G. Battista Baliani, who had sent him a new book on motion, Galileo, in 1639, replied

> ... although I have been unable to understand the proofs clearly, since I could not correlate them with the diagrams.[70]

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[68] *Opere*, IV, pp. 307–308.

[69] "... et io, se mai potessi ridurmi in stato men travaglioso, procurerei di significargli il mio concetto; ma perchè è una macchinazione o struttura assai grande e difficile a spiegarsi, e massime con nude parole senza poterne un cieco disegnare la figura, non posso per ora dir cosa essenziale, se non che il mio artifizio depende da una proposizione di Euclide". *Opere*, XVII, p. 360.

[70] "... anchorchè io non abbia potuto intendere distintamente le dimostrazioni, non potendo incontrarle con le figure". *Opere*, XVIII, p. 11.

To another correspondent, again in reference to Baliani's book, Galileo said

> . . . because I cannot see the diagrams and compare them with statement and proof, I remain doubtful in one or two places; since, I think, I failed to arrive with the imagination as far as the sense of sight would have allowed.[71]

Perhaps the most extraordinary of Galileo's comments on cognition and blindness concerns his own proofs. Here is what he told Baliani in a subsequent letter.

> Two other details which you mention in your letter I have been unable to compare with my writings, since there intervene linear diagrams and comparison of letters, which are impossible for me; and to my deep regret, I will forever be unable to understand even my own demonstrations, which contain diagrams and computations. . . [72]

When he succumbed to blindness, Galileo thus lost the ability to understand even his own previous proofs forever.[73] I believe that Galileo's comments indirectly reveal a profound dimension in his intellectual methodology, the relying on the visuo-spatial component of his argumentative strategies, i.e., typically, the diagram.

It was reasoning based on diagrams that motivated Galileo to call into question the explanatory potential of the theory of Archimedean buoyancy. Diagrams motivated him to focus on the relation between the geometry of the body and the vessel, and the varying configurations of body and water level in the processes of submersion and emersion.

On the other hand, it was proportional reasoning that allowed Galileo to exact precise items of information from the diagrams. Eventually, in the very specific case of cylinders and prisms, Galileo was able to replace the erroneous conclusion that the volume of water raised by the submerging body is equal to the volume of the body's submerged part with the correct proportionality associated with a geometrically simple instance of mersion. It was idealized phenomena that Galileo's methodology captured in simple diagrams. That simplicity, however, is deceiving. The cognitive processes underlying visual perception are irreplaceable. Once the sense of sight has been lost cognition itself is impaired forever, as, to his utter dismay, Galileo realized.

To conclude, argumentative strategies for theory unification, the cognitive dynamics of paradox building and resolution, the visual perception of diagrams, and proportional reasoning were inextricably linked in the creative process that led Galileo to the theory of

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71 ". . . per non poter veder le figure nè riscontrarle con la dichiaratione e dimostratione, mi lascia in qualche scrupolo in un luogo o due; credo, per non haver potuto arrivare con la immaginativa sin dove il senso della vista vi si ricerca di necessità". *Opere*, XVIII, p. 37.

72 "Due altri particolari che ella tocca nella sua lettera, non ho potuto riscontrarli in quello che scrivo, intervenendovi figure lineari e rincontri di caratteri, impossibili essere da me fatti, come per mia infelicità resto privo di poter mai più intendere le mie medesime dimostrazioni, dove intervengono figure e calculi . . . ". *Opere*, XVIII, p. 95.

73 On 2nd January 1638, Galileo indirectly hinted at the new conceptualization of space itself that blindness had required of him. He lamented with his correspondent: "Or pensi V. S. in quale afflizione io mi ritrovo, mentre che vo considerando che quel cielo, quel mondo e quello universo che io con mie maravigliose osservazioni e chiare dimostrazioni avevo ampliato per cento e mille volte più del communemente veduto da' sapienti di tutti i secoli passati, ora per me s'è si diminuito e ristretto, ch' e' non è maggiore di quel che occupa la persona mia". *Opere*, XVIII, p. 247.

mechanical buoyancy. The articulations of that link have the power to clarify the origins of his revolutionary methodology.

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