Mathematical Modelling

GOAL: To get students to take practical and vague ideas & put them into the language of mathematics.

While many of the examples come from Biology, the ideas that we use will be applicable to many other areas. We will include examples from physics, economics, chemistry, and the humanities!

The class will be organized according to methods used, That is, we will start with discrete time systems, then go to diff equs & move onto random or stochastic models. We will do some PDE models & some parameter fitting if time permits.

HW: Always Due the Friday, a week after assigned

Project: You will have to do a project which can be done on your own or with a few (say ≤ 4) other people.

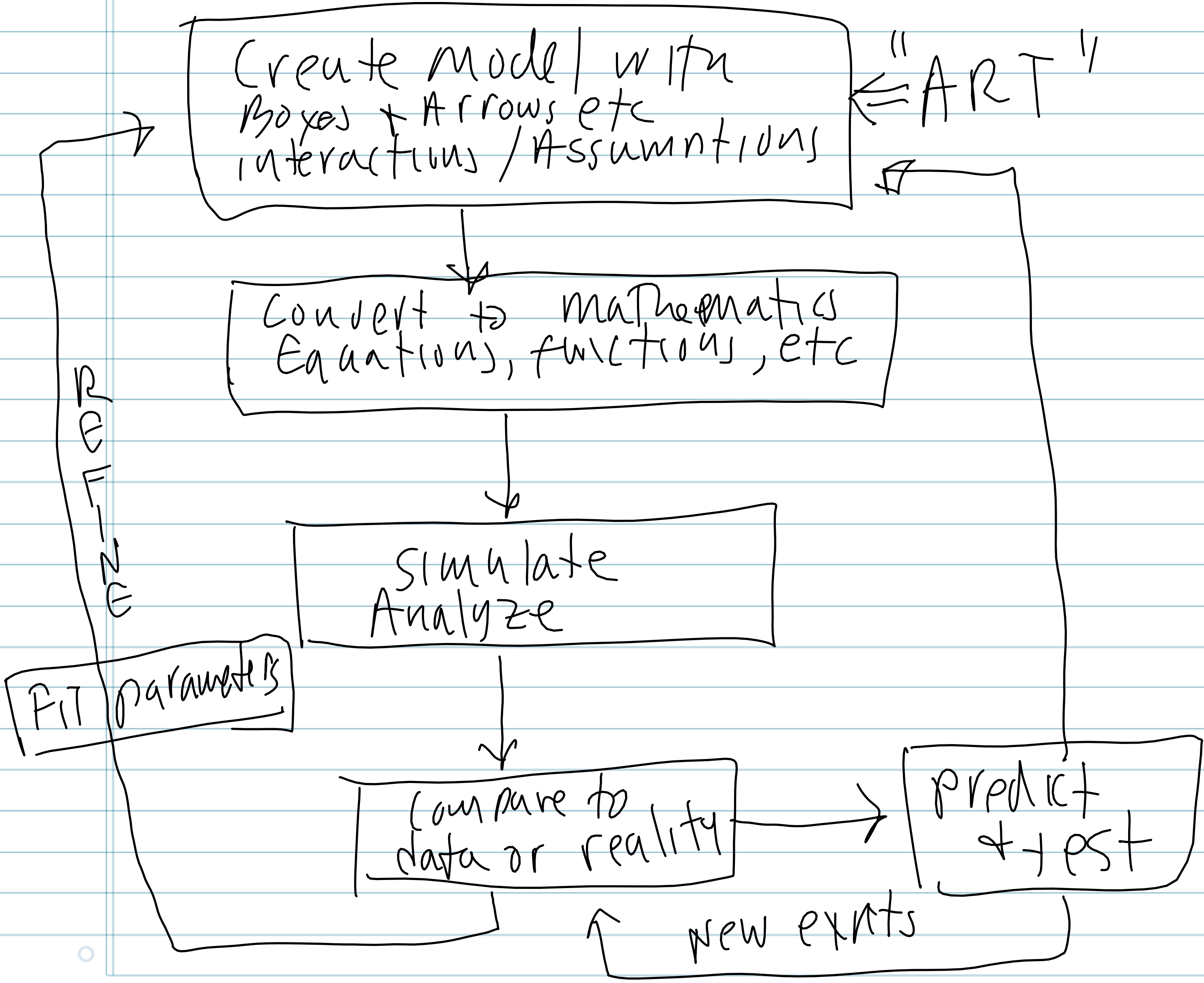
Project involves making a model of something of interest, creating the equations, solving the equations, & interpreting the results. You will have to do an oral & written part.

MODELING IS AS MUCH AN ART AS IT IS A SCIENCE!

EINSTEIN: "Everything should be made as simple as possible but no simpler"

Like OCCAM's Razor – but with the caveat that your model should not <u>oversimplify</u>

procedure:



In addition to The Ideas in Book, we will also introduce some useful concepts from chemistry (Law of MASS action), mechanics (Euler-Lagrange equations), stochastics (Markov chains) & some parameter fitting.
☆ LINEAR DIFF EQNS ☆

EXAMPLE (Fibonacci's rabbits.

1175-1250 Leonardo of Pisa. Let N_t be # pairs of rabbits at year t. Each pair produces a new pair for two successive seasons & Then retires (hopefully to The stew pot)

$$\underset{\text{New season}}{N_{t+1}} = \underset{\text{Last season}}{N_t} + \underset{\text{Season before}}{N_{t-1}}$$

$N_0 = 0$
$N_1 = N_2 = 1$

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89...

Very Famous sequence appears everywhere sunflowers, pine cones, Finger joints

Example 2 Cell Growth.

Let M_n = # cells in each generation

$$\underset{\#\text{ in next}}{M_{n+1}} = \underset{\#\text{ current}}{M_n} + \underset{\text{change}}{\Delta M_n}$$

$\Delta M_n = \text{Growth}_n - \text{Death}_n$

Suppose fraction d die & fraction f divide, Then

$$\Delta M_n = (f-d)\, M_n \equiv b M_n$$

$$M_{n+1} = (1+b)\, M_n \equiv a M_n$$

$M_1 = a M_0 \quad M_2 = a^2 M_0 \quad M_3 = a^3 M_0 \cdots$

$\boxed{M_n = a^n M_0}$ if $|a| < 1$ die

$|a| = 1$ constant

$|a| > 1$ increase

If $a < 0$ Then not really valid since M changes sign!!

~~Class~~ Example 3 Classic compound interest equation. P = principle, r = rate of interest

Q = payment per month.

Let L_n = amount left on loan at month n

$L_0 = P$

$$L_{n+1} = (1+r)\, L_n - Q$$

- Pay off in N months, what is Q
- How much total interest
-

Example 4 Insect growth

a_n = # adults female, p_n = # progeny,
m = mortality of young aphids, $f = \frac{\# \text{prog}}{\text{female}}$
r = ratio of female to total adults

$p_{n+1} = f a_n$ $\quad$ $1-m$ survive & r are female
so $a_{n+1} = r(1-m) p_{n+1}$

Thus $a_{n+1} = \boxed{\boxed{r(1-m)} f} a_n \equiv \beta a_n$

So $\boxed{a_n = \beta^n a_0}$

Example 5 Annual plants produce seeds at end of summer. Some seeds survive winter & sprout, others die & others remain dormant & sprout the next year. After that, suppose all dead

γ = # seeds per plant in August
α = fraction of 1 winter seeds germ in May
β = " " 2 winter " " " May
σ = " " surviving winter

April-May $\quad$ Aug $\quad$ Winter $\quad$ Apr $\quad$ Aug $\quad$ Win $\quad$ Apr $\quad$ Aug

$$\begin{tikzcd}
P_n \arrow[r, "\gamma"] & S_n^0 \arrow[r, "\sigma"] & S_{n+1}^1 \arrow[d, "\alpha"] \arrow[rr, "S_{n+1}^1\ \sigma"] & & {} \arrow[d, "\beta"] & \\
 & & P_{n+1} \arrow[r, "\gamma"] & S_{n+1}^0 \arrow[r, "\sigma"] & \text{plants} \arrow[r] & S_{n+2}^0 \\
 & & & & \alpha \arrow[r] & {}
\end{tikzcd}$$

p_n = # plants in generation n
S_n^1 = 1 yr old seeds in April
S_n^2 = 2 yr old " " Apr
$\bar{S}_n^1$ = 1 yr old left after germ.
$\bar{S}_n^2$ = 2 " " " " "
S_n^0 = # new seeds in August

We will get rid of most of these

plants from 1 + 2 yr old seeds

$$P_n = \alpha S_n^1 + \beta S_n^2$$

seeds left = (fract not germ) x (# seeds in april)

$$\bar{S}_n^1 = (1-\alpha) S_n^1 \qquad \bar{S}_n^2 = (1-\beta) S_n^2$$

New ~~type~~ seeds $S_n^0 = \gamma P_n$

$$S_{n+1}^1 = \sigma S_n^0 \qquad S_{n+2}^2 = \sigma \bar{S}_n^1$$

overwinter

$$S_{n+1}^1 = \sigma \gamma P_n \qquad S_{n+1}^2 = \sigma (1-\alpha) S_n^1$$

$$\boxed{\begin{aligned} P_{n+1} &= \alpha S_{n+1}^1 + \beta S_{n+1}^2 = \alpha \sigma \gamma P_n + \beta \sigma (1-\alpha) S_n^1 \\ S_{n+1}^1 &= \sigma \gamma P_n . \end{aligned}}$$

$S_n^1 = \sigma \gamma P_{n-1}$ so could write:

$$\boxed{P_{n+1} = \alpha \sigma \gamma P_n + \beta \sigma (1-\alpha) \sigma \gamma P_{n-1}}$$

~~Nonlinear difference equat~~

~~Analog~~

$x_{n+1} = a x_n + b x_{n-1}$

$x_{n+1} = y_n$

$y_{n+1} = x_{n+2} = a x_{n+1} + b x_n = a y_n + b x_n$

Any "k^{th}" order difference equation can be written as a system of k diff equation of first order:

★ $X_{n+1} = A X_n + b$ $\quad$ X is vector, A is matrix

Eg (1) $A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$

~~Fish plants~~ Fibonacci $\quad N_{t+1} = N_t + N_{t-1}$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

Equilibrium $\quad X_{n+1} = X_n \Rightarrow X = AX + b \Rightarrow$ $(I-A)X = b \Rightarrow X = (I-A)^{-1} b$. As long as $(I-A)$ is invertible. Write $X_{eq} = (I-A)^{-1} b$

Let $Y_n = X_n - X_{eq}$ Then:

$$\boxed{Y_{n+1} = A Y_n}$$

$Y_1 = A Y_0 \quad Y_2 = A^2 Y_0 \cdots$

$Y_n = A^n Y_0 \ldots$

Let $A\vec{v} = \lambda \vec{v}$ be eigenvector eigenvalue combination for A.

Suppose $Y_0 = c\vec{v}$

$Y_1 = A c\vec{v} = c A\vec{v} = c\lambda \vec{v}$

$Y_n = c\lambda^n \vec{v} \Rightarrow$ what's fate of Y_n

depends on λ^n

Suppose $\lambda =$ real $= r$ Then r^n

alternates if $r < 0$, decays if $|r| < 1$

+ grows if $|r| > 1$

So if $|\lambda| < 1$ Then $Y_n \to 0$ if $|\lambda| > 1$ $|Y_n| \to \infty$

Suppose $\lambda = \alpha + i\beta$ is complex.

Recall: $e^{i\theta} = \cos\theta + i\sin\theta$

so we can write $\alpha + i\beta = \rho e^{i\theta}$

$\rho = \sqrt{\alpha^2 + \beta^2} \quad \alpha = \rho\cos\theta, \ \beta = \rho\sin\theta$

$\Rightarrow \theta = \tan^{-1}(\beta/\alpha) \qquad \lambda = \rho e^{i\theta}$

$\lambda^n = (\alpha + i\beta)^n = \rho^n e^{in\theta} = \rho^n[\cos n\theta + i\sin n\theta]$

If $\rho > 1$ Then $|\lambda^n| \to \infty$ If $\rho < 1$ $|\lambda^n| \to 0$

Note That λ^n Then will "oscillate" around according to θ.

I will give some exercises so you can practice This.

Back to general problem. A is $m \times m$

$Y_n = A^n Y_0$. Suppose That

$Y_0 = c_1 \vec{V}_1 + \cdots + c_m \vec{V}_m$ where

$\vec{V}_1, \ldots, \vec{V}_m$ are eigenvectors & eigenvalues

For ease, we assume There are m lin ind eigenvectors. If NOT - This is pain in the butt!

$A Y_0 = c_1 \lambda_1 \vec{V}_1 + \cdots + c_m \lambda_m \vec{V}_m$

$A^n Y_0 = c_1 \lambda_1^n \vec{V}_1 + \cdots + c_m \lambda_m^n \vec{V}_m$

~~Exam~~ Example:

$X_{n+1} = \frac{1}{2} X_n + Y_n$

$Y_{n+1} = 2Y_n - X_n$

What happens

$A = \begin{bmatrix} \frac{1}{2} & 1 \\ -1 & 2 \end{bmatrix}$

Eigenvalues:

$$\lambda^2 - Tr(A)\lambda + \det A = 0$$

$Tr(A) = \frac{5}{2}$ Det $A = 2$

$$\lambda = \frac{\frac{5}{2} \pm \sqrt{\left(\frac{5}{2}\right)^2 - 8}}{2} = \frac{5}{4} \pm \sqrt{-\frac{7}{16}} = \frac{5}{4} \pm i\frac{\sqrt{7}}{4}$$

$|\lambda| > 1$ clearly so population will grow without bound.

2nd order diff equation.

$P_{n+1} = a P_n + b P_{n-1}$

could write as a system but easier to guess a solution. As above suppose $p_n = c\lambda^n$. Then:

$$c\lambda^{n+1} = ac\lambda^n + b\lambda^{n-1}c \Rightarrow (\text{divide by } c\lambda^{n-1})$$

$$\lambda^2 = a\lambda + b$$

$$\lambda^2 - a\lambda - b = 0 \qquad \boxed{\lambda = \frac{a \pm \sqrt{a^2+4b}}{2}}$$