LECT 2

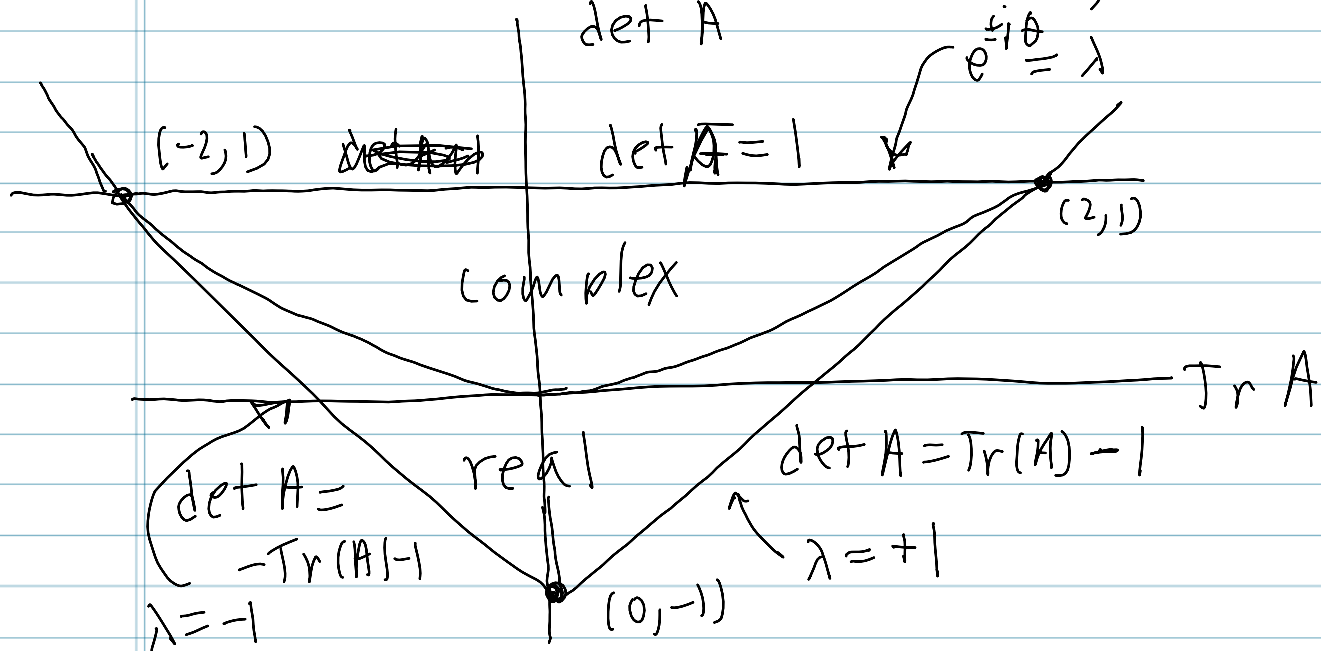
Let's recall

$$X_{n+1} = A X_n$$

we want to know the fate of X_n. This is determined by eigenvalues of A

for A 1×1 This is just a

for A 2×2 $\lambda^2 - Tr(A)\lambda + det(A)$



inside triangle $|\lambda| < 1$ outside triangle $|\lambda| > 1$

so we can always det fate.

Example 1 Romeo + Juliet

Let R_n be romeo's love or hate of juliet on day n

Let J_n " Juliet's " " " Romeo on day n

$R_n > 0$ Love $R_n < 0$ Hate $R_n = 0$ neutral

$$R_{n+1} = a_R R_n \qquad J_{n+1} = a_J J_n$$

Assume $a_{R,J} > 0$ (so don't have daily mood swings!)

$a > 1$ then whatever it is it gets more passionate!

How do they interact?

Assume linear:

$$R_{n+1} = a_R R_n + p_R J_n$$

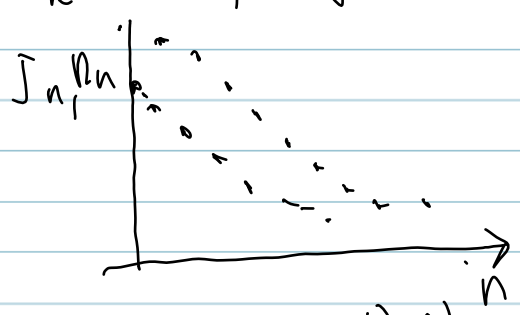
$$J_{n+1} = a_J J_n + p_J R_n$$

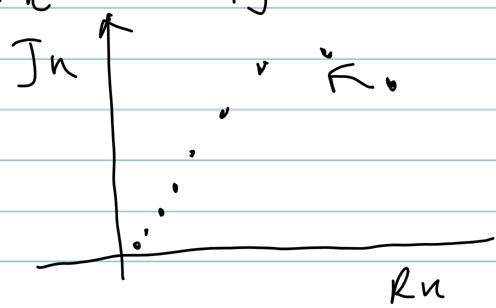
$R_0 = J_0 = 1$ (Both love each other)

4 styles of romance

Easy to simulate

(i) $a_R = 0.5$, $a_J = 0.7$ $p_R = 0.2$ $p_J = 0.5$





Peters out!

$\mathrm{Tr}(A) = a_R + a_J \quad \det(A) = a_R a_J - p_R p_J$

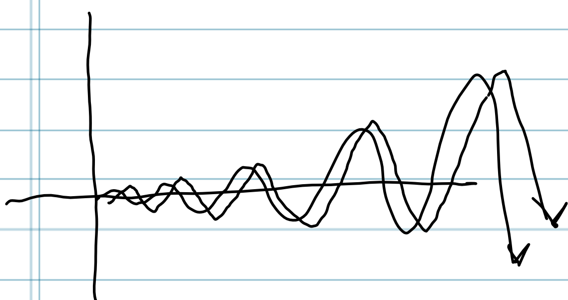
(i) $\mathrm{Tr}(A) = 1.2 \quad \det(A) = .35 - .1 = .25$

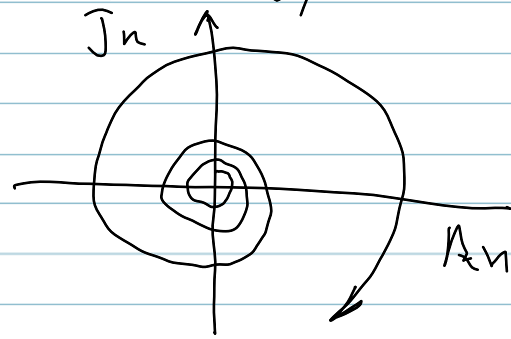
in the middle of stab. triang so peters out

$|\lambda_j|^n \to 0$

(ii) $a_R = 1,\ a_J = 1 \quad p_R = 0.2 \quad p_J = -0.2$

Fickle if Romeo hates juliet then juliet increases love of Romeo + vice versa



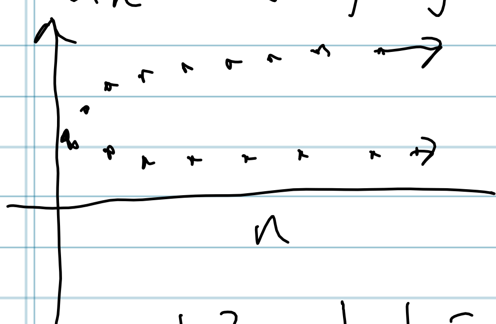


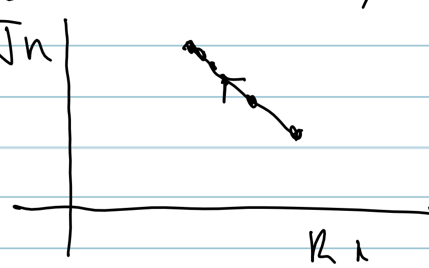
$|\lambda_j^n| \to \infty$

$\mathrm{Tr}(A) = 2 \quad \det(A) = 1 + .04 = 1.04$

complex growing out of triangle.

(iii) $a_R = 0.5,\ a_J = 0.8 \quad p_R = 0.2,\ p_J = 0.5$





$\mathrm{Tr} = 1.3 \quad \det = 0.4 - 0.1 = 0.3$

$\det = \mathrm{Tr} - 1 \Rightarrow \lambda_1 = +1 \quad \lambda_2 = 0.3$

$\lambda_1^n \to 1 \quad \lambda_2^n \to 0$

will the PLANTS survive?

$P_{n+1} = \alpha\sigma\gamma P_n + \beta\sigma(1-\alpha) S_n'$

$S_{n+1}' = \sigma\gamma P_n$

WANT TO BE OUT OF Triangle

$$A = \begin{bmatrix} \alpha\sigma\gamma & \beta\sigma(1-\alpha) \\ \sigma\gamma & 0 \end{bmatrix}$$

$Tr = \alpha\sigma\gamma$

$Det = -\beta\sigma^2\gamma(1-\alpha)$

For example if $Det < -1$ then always out

$\Rightarrow \beta\sigma^2\gamma(1-\alpha) > 1$ or

$\det A < -Tr(A) - 1$ ☆

$< Tr A - 1$

Since $Tr(A) < 0$ if $\det < -Tr(A) - 1$

it is also $< Tr(A) - 1$

so $-\beta\sigma^2\gamma(1-\alpha) < \pm Tr(A) - 1$

if < -1 then ☆ done or

$-\beta\sigma^2\gamma(1-\alpha) < \alpha\sigma\gamma - 1$

~~pretty easy~~

$\Rightarrow 1 < \gamma[\beta\sigma^2(1-\alpha) + \sigma\alpha]$

$\Rightarrow \gamma > 1/(\beta\sigma^2(1-\alpha) + \sigma\alpha)$

EXPR 355

IUBOOK

Nonlinear difference equations

dw - we know that nothing can keep

EX1 growing forever, so the equation for cells dividing, eg:

$$M_{n+1} = a M_n$$

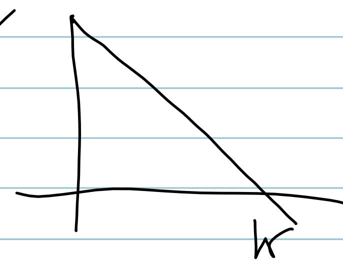
is probably only good for M_n small if $a>1$

Instead, there can be crowding or resource use to lead to:

$$M_{n+1} = \cancel{a} f(M_n) M_n \qquad f(0)=1$$

$f(M_n) \le 1$ is a funct that accounts for slowing of growth as M_n gets larger $\qquad f'(M)<0$

simplest $f(M) = \cancel{a}(1-\frac{M}{k})$



k is called carrying capacity

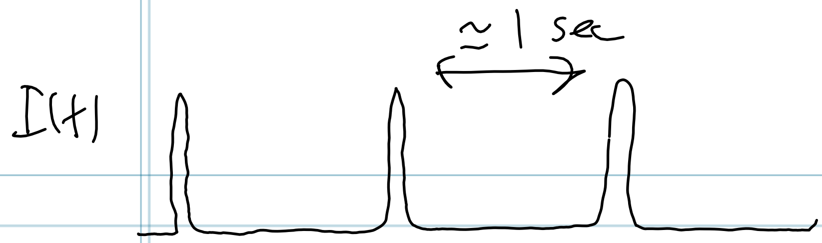
$$M_{n+1} = a\left(1-\frac{M_n}{k}\right)M_n$$

Let $x_n = \frac{M_n}{k}$ then $\frac{M_{n+1}}{k} = \frac{a}{\cancel{k}}\left(1-\frac{M_n}{k}\right)\frac{M_n}{k}$

$$x_{n+1} = a(1-x_n)x_n \qquad \underline{\text{WLOG}}$$

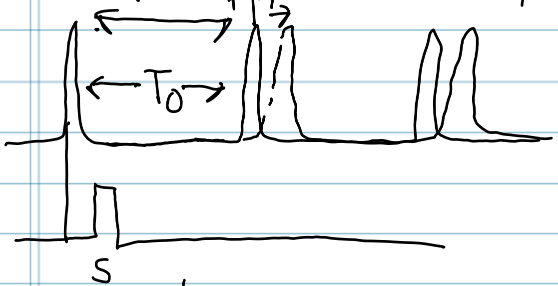
"Logistic equation"

EX2 Fireflies. S.E. Asian Pteroptyx Malaccae flashes rhythmically



Thousands congregate in trees & synchronize!

If you flash a pulse of light this will shift the time of the next flash:



$T' = T'(s)$

Define $\Delta(s) = T_0 - T'(s)$

Called phase resetting curve

For P. M.

$$\Delta(s) \approx -\alpha \sin \frac{2\pi s}{T_0}$$

Lets flash light with period τ & ask what this does to the time of the flash.

Define phase, θ = time after ~~last~~ pulse called phase

$0 \le \theta < T$. Let θ_n = phase at moment of Light pulse!

$$\theta_{n+1} = \theta_n + \tau \qquad \text{mod } T_0$$

with no stimulus. PRC. But:

$$\boxed{\theta_{n+1} = \theta_n + \tau - \alpha \sin \frac{2\pi \theta_n}{T}}$$

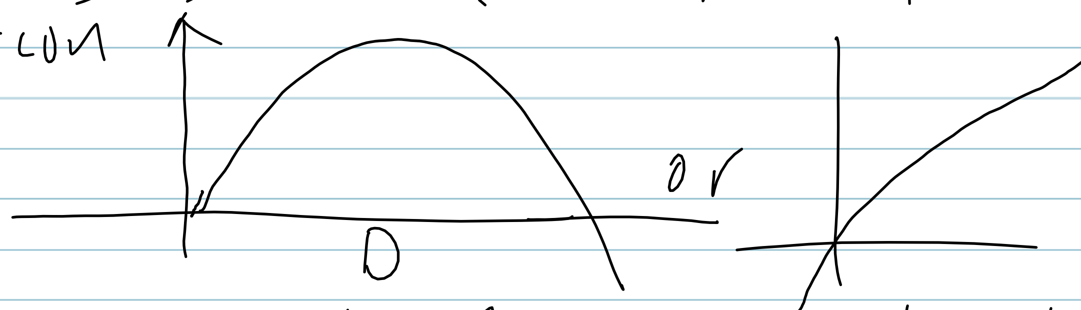
Nonlinear difference equation

Economic model with consumer sentiment

Let $y(t)$ be income at time t (say month)

D $y(t-1) - y(t-2)$ is the income difference from previous period.

Let $I(D)$ be income investment function



A consumption function & sentiment

$$C(y,D) = a + y\left(b + \frac{c}{1+\varepsilon e^{-(y(t-2)-y(t-1))}}\right)$$

If $y(t-2) - y(t-1) \to \infty$ sentiment $\to c$

$y(t-2) - y(t-1) \to -\infty$ sent $\to 0$

Saving function

$$d(y(t-2) - y(t-1)) + m \text{ Sentiment}$$

$$y(t) = a + d(y(t-2) - y(t-1)) + b\,y(t-1) + I(y(t-1) - y(t-2)) + \frac{c\,y(t-1) + m}{1+\varepsilon e^{-(y(t-2)-y(t-1))}}$$

a = autonomous expenditures like rent & food ~~+ stuff~~ c is trend toward consumption, m is trend toward saving

$0 < b < 1$ pessimistic $1 < b$ optimism

spend like heck!

$I(D) = vD - wD^3$ e.g.

Keynes model

$$\boxed{y(t) = d(y(t-2) - y(t-1)) + a + b\,y(t-1)}$$

Hicks-Samuelson model:

$$\boxed{y(t) = (1-v-s)\,y(t-1) - v\,y(t-2)}$$

Both linear.

General Nonlinear difference equation:

$$X_{n+1} = F(X_n)$$

Equilibrium $X_{n+1} = X_n \Rightarrow$

$$\underline{X} = F(X).$$

Let $\bar{X}$ be equilibrium $\bar{X} = F(\bar{X})$

want to know, e.g. is it stable.

$X_n = \bar{X} + Y_n \quad X_{n+1} = \bar{X} + Y_{n+1} = F(\bar{X} + Y_n)$

$F(\bar{X}+Y_n) = F(\bar{X}) + AY_n + \cdots$

where A = matrix of partial derivatives of F w.r.t X.

$$\boxed{Y_{n+1} = AY_n}$$

$$F = \begin{bmatrix} f_1(x_1, \ldots, x_m) \\ \\ f_m(x_1, \ldots, x_m) \end{bmatrix}$$

$$A = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ \vdots & & & \\ \frac{\partial f_m}{\partial x_1} & & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix}\right|_{(x_1, \ldots, x_m) = \bar{X}}$$

called "Linearization"

$\bar{X}$ is stable if all eigenvalues of A are st $|\lambda_j| < 1$. Just like linear Diff Eqn.

First order difference equations

$x_{n+1} = f(x_n)$

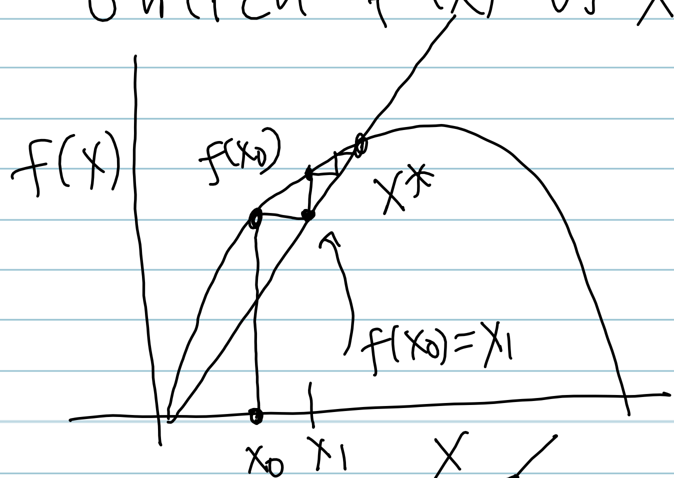
$x^* = f(x^*) \equiv$ equilibrium

$f'(x^*) = \beta$ if $-1 < \beta < 1$ x^* is stable

That is if we start near x^* we stay there and actually move to x^*

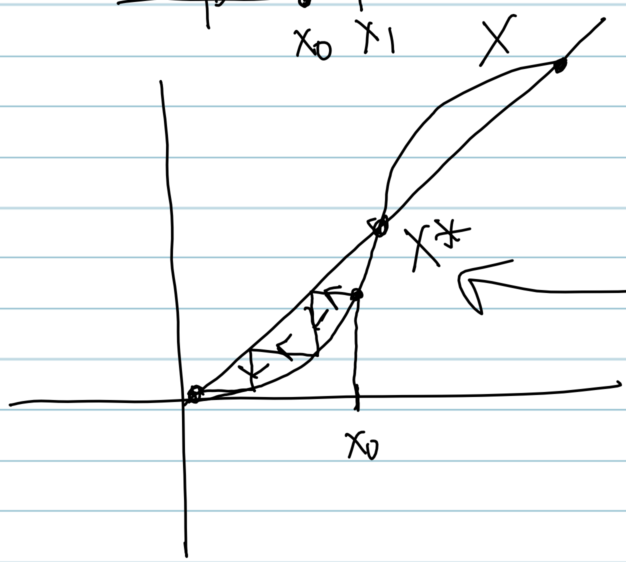
Cobwebbing

Sketch $f(x)$ vs x Draw $y = x$

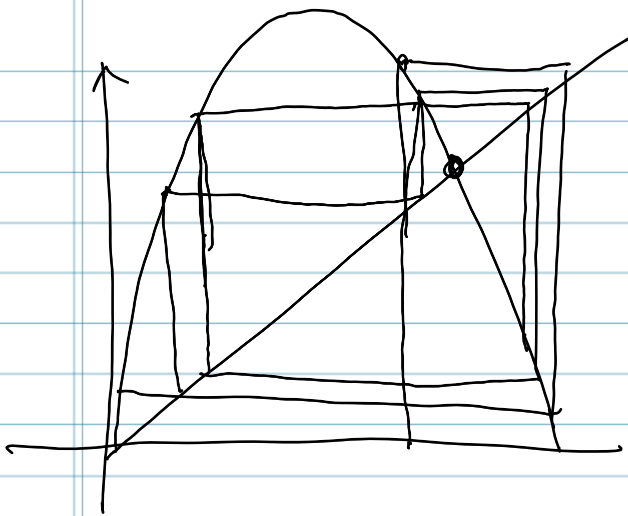


Where $y = x = y = f(x)$ we have $x = f(x)$ A FIXED PT!

X^* is stable!



X^* is UNSTABLE



This is called cobwebbing!

Analyze the Logistic equation

$$x_{n+1} = a\,x_n(1-x_n)$$

$x^* = a\,x^*(1-x^*)$ $\quad x^* = 0,\ x^* = 1-\frac{1}{a}$

$f(x) = ax(1-x)$ $\quad f'(x) = a - 2ax$

$f'(0) = a$ $\quad f'(1-\frac{1}{a}) = a - 2a(1-\frac{1}{a})$

$\qquad\qquad\qquad\qquad = 2-a$

Stability iff $1 < f'(x^*) < -1$

$1 < a < -1$ $\quad 0$ is stable

$1 < 2-a < -1 \Rightarrow 1 \not< a \not< 3 \Rightarrow 1-\frac{1}{a}$ is stable

What happens for $a \geq 3$?

We will look at this shortly

Linear Diff Equns examples

$X_{n+1} = X_n + X_{n-1}$ $\qquad X_0 = 0 \quad X_1 = 1$

Fibonacci.

~~X~~ General solution is $X_n = C\tau^n$

$\tau^{n+1} = \tau^n + \tau^{n-1} \Rightarrow \tau^2 = \tau + 1$

$\Rightarrow \tau^{\pm} = \dfrac{+1 \pm \sqrt{5}}{2} = \dfrac{+1}{2} \pm \dfrac{\sqrt{5}}{3}$ GOLDEN MEAN!

$X_n = A(\tau^+)^n + B(\tau^-)^n$

$X_0 = A + B = 0 \Rightarrow A + B = 0 \Rightarrow A = -B$

$X_1 = A[\tau^+ - \tau^-] = 1 \Rightarrow A\left(\frac{+1}{2} + \frac{\sqrt{5}}{2} \cancel{+} \frac{1}{2} + \frac{\sqrt{5}}{2}\right) = 1$

$\Rightarrow A = \dfrac{1}{\sqrt{5}}$

$$X_n = \frac{1}{\sqrt{5}}\left(\left(+\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^n - \left(\frac{+1}{2} \cancel{+} \frac{\sqrt{5}}{2}\right)^n\right)$$

For n large $\left(+\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^n \to 0$

so $X_n \sim \dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n$

$$\frac{X_{n+1}}{X_n} \sim \left(\frac{1+\sqrt{5}}{2}\right) = 1.6180$$

$\frac{8}{5} = 1.6$, $\frac{55}{34} = 1.6176$, $\frac{89}{55} = 1.6181$

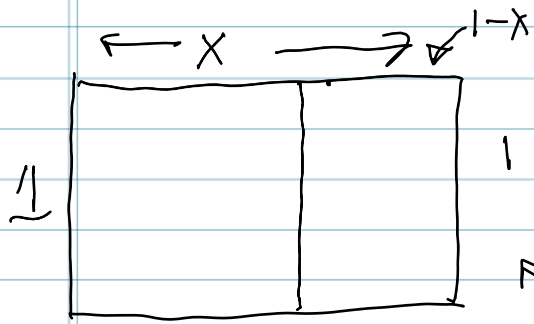
Remarks on Golden mean:

"Continued Fraction"

$$- \; x = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} \Rightarrow x = \frac{1}{1+x} \Rightarrow x^2 + x - 1 = 0$$

$x = \frac{-1 \pm \sqrt{5}}{2}$ since $x > 0$

Take + root



$$\frac{x}{1} = \frac{1}{1-x} \Rightarrow \cancel{x^2+x-}$$

$$x - x^2 = 1$$

$$\Rightarrow x^2 - x + 1 = 0 \qquad x = \frac{1+\sqrt{5}}{2}$$

Architecture

Solve

(a) $x_n - 5x_{n-1} + 6x_{n-2} = 0 \qquad x_0 = 2 \quad x_1 = 5$

$\lambda^2 - 5\lambda + 6 = 0 \quad (\lambda - 2)(\lambda - 3)$

$x_n = c_1 2^n + c_2 3^n \qquad c_1 + c_2 = 2$

$2c_1 + 3c_2 = 5$

$\Rightarrow c_1 = c_2 = 1$

$$\boxed{x_n = 2^n + 3^n}$$

(b) $x_n + \frac{1}{2}x_{n-1} + x_{n-2} = 0$

$\lambda^2 + \frac{1}{2}\lambda + 1 = 0 \qquad \lambda = \dfrac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} - 4}}{2} = -\dfrac{1}{4} \pm \dfrac{i\sqrt{15}}{4}$

$\approx e^{i1.8234}$

$\lambda^n = \cos(1.823n) \pm i\sin(1.823n)$