Homework Due Friday Sept 17

#5 from previous HW
#7 " " HW

#2 page 61 in Book

8 b page 63 in Book

The tent map is defined as:

$$f(x) = \begin{cases} \mu x & \text{for } 0 \le x \le \frac{1}{2} \\ \mu(1-x) & \text{for } \frac{1}{2} \le x \le 1 \end{cases}$$

a) Sketch the graph of $f(x)$ for $\mu > 0$
b) Find equilibria + their stability
c) plot f for $\mu = 2$. Cob web for this value

Ventilation problem..

Let V_n be the volume at the nth breath and C_n be the CO_2 in the blood

$$C_{n+1} = C_n + \underbrace{m}_{CO_2 \text{ production}} - \underbrace{\mathcal{L}(V_n, C_n)}_{\text{Amount Lost}}$$

$$V_{n+1} = f(C_n) \leftarrow \text{sensitivity to } CO_2$$

Simple model: $\mathcal{L} = \beta V_n \qquad f = \alpha C_n$

$$C_{n+1} = C_n + m - \beta V_n$$
$$V_{n+1} = \alpha C_n \implies V_n = \alpha C_{n-1}$$

$$C_{n+1} = C_n + m - \beta \alpha C_{n-1}$$

How do we solve this — just like ODES particular + homogeneous.

Look for $C_n = \bar{C}$ constant!

$$\bar{C} = \bar{C} + m - \alpha\beta\bar{C} \implies$$

$$\bar{C} = m/(\alpha\beta)$$

Homogeneous problem:

$$C_{n+1} = C_n - \alpha\beta C_{n-1}$$

$$\lambda^2 = \lambda - \alpha\beta \qquad \lambda^2 - \lambda + \alpha\beta$$

roots are $\dfrac{1 \pm \sqrt{1 - 4\alpha\beta}}{2} \equiv \rho^+, \rho^-$

$$C_n = \frac{m}{\alpha\beta} + A(\rho^+)^n + B(\rho^-)^n$$

Assume $4\alpha\beta < 1 \Rightarrow$ roots are

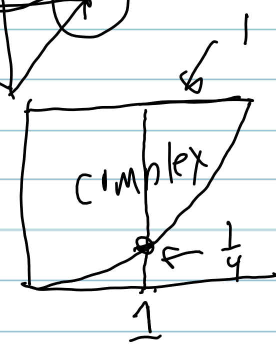
$$\frac{1}{2} + \frac{\sqrt{1-4\alpha\beta}}{2}, \quad \frac{1}{2} - \frac{\sqrt{1-4\alpha\beta}}{2}$$

Both roots are positive & both are less than 1

so $(\rho^+)^n \to 0$ $(\rho^-)^n \to 0$ &

$$\boxed{C_n \to \frac{m}{\alpha\beta}}$$

If $4\alpha\beta > 1$ Then complex roots



If $\cancel{4\alpha\beta}\ \frac{1}{4} < \alpha\beta < 1$

complex damped

If $\alpha\beta > 1$ Then complex undamped!

growing

Nonlinear diff equations continued:

$$X_{n+1} = \frac{k X_n}{b + X_n} \qquad b, k > 0$$

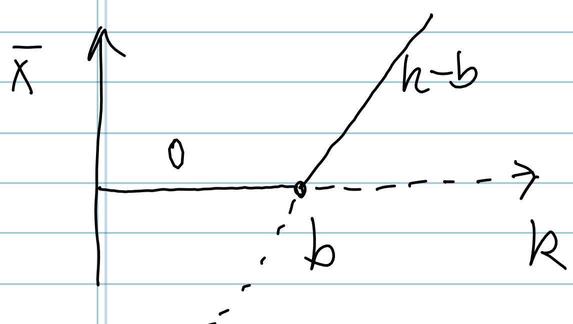
$$\bar{X} = \frac{k \bar{X}}{b + \bar{X}} \Rightarrow \bar{X} = 0 \text{ or } \bar{X} = k - b$$

Stability

$$f(x) = \frac{kx}{b+x} \qquad f'(x) = \frac{k}{b+x} - \frac{kx}{(b+x)^2} = \frac{kb}{(b+x)^2}$$

$$f'(0) = \frac{k}{b}, \quad f'(k-b) = \frac{b}{k}$$

If $k < b$ Then 0 is stable
If $k > b$ Then $k-b$ is stable



Bifurcation diagram

Logistic equation revisited:

$$X_{n+1} = a X_n (1 - X_n)$$

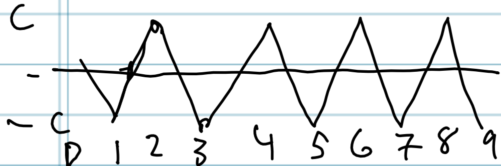
$\bar{X} = 0$ stable if $0 < a < 1$

$\bar{X} = 1 - \frac{1}{a}$ stable If $1 < a < 3$

What happens for $a > 3$

When $a=3$ $\lambda=-1$

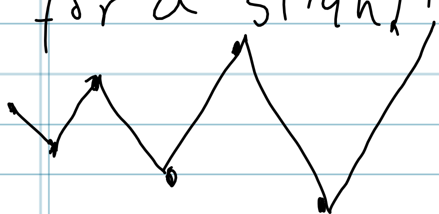
$y_{n+1}=-y_n$ $y_0=c$ $c,-c,c,-c$



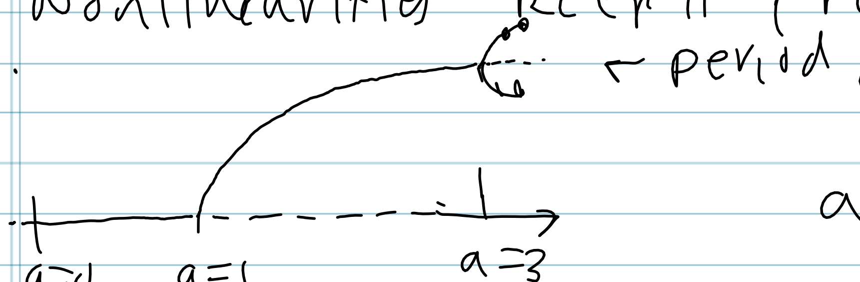
"period 2"

We say period k if $\boxed{x_{n+k}=x_n}$

for a slightly larger than 3 $\lambda<-1$



Nonlinearities keep it from blowing up.



For example if $a=3.2$ period 2

$x_n \to .799, .513, .799, .513 \ldots$

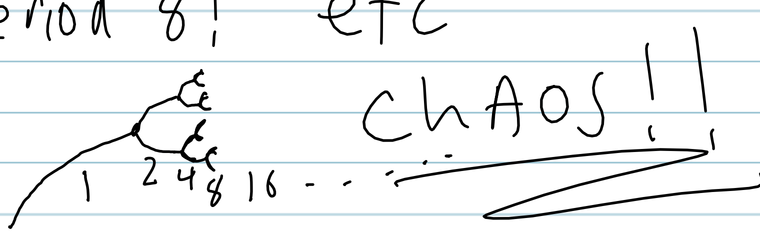
This persists until $a = \sim 3.449$

Then get period 4 point:

$x_{n+4}=x_n$

period 4 until $a \sim 3.564$ ~~period~~

Then period 8! etc



Systems of nonlinear diff equations

$m=2$ dimensions:

$$x_{n+1} = f(x_n, y_n)$$
$$y_{n+1} = g(x_n, y_n)$$

Steady states:

$$\bar{x} = f(\bar{x}, \bar{y})$$
$$\bar{y} = g(\bar{x}, \bar{y})$$

Linearization

$$A = \left.\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}\right|_{x=\bar{x},\, y=\bar{y}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Eigenvalues $\lambda^2 - \underbrace{(a_{11}+a_{22})}_{\text{TRACE}}\lambda + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{\text{DET}}$

Refer to stability triangle!

$$\boxed{2 > 1 + \det > |Tr|}$$

Example:

$$x_{n+1} = \alpha x_n - \beta x_n y_n$$
$$y_{n+1} = \gamma y_n + \delta x_n y_n$$

Assume $0 < \gamma < 1$, $\alpha > 1$ $\beta, \delta > 0$

Equilibrium:

$\bar{x} = \bar{x}(\alpha - \beta\bar{y})$

$\bar{y} = \bar{y}(\gamma + \delta\bar{x})$

$\boxed{\bar{x} = \bar{y} = 0}$ otherwise

$1 = \gamma + \delta\bar{x} \Rightarrow \boxed{\bar{x} = \frac{1-\gamma}{\delta} > 0}$

$1 = \alpha - \beta\bar{y} \Rightarrow \bar{y} = \frac{\alpha-1}{\beta} > 0$

STABILITY

$\alpha - \beta\bar{y} = 1$!

$f(x,y) = \alpha x - \beta xy$

$g(x,y) = \gamma y + \delta xy$

$$\begin{bmatrix} \alpha - \beta y & -\beta x \\ \delta y & \gamma + \delta x \end{bmatrix}$$

$\gamma + \delta x = 1$!

$(0,0)$ $\begin{bmatrix} \alpha & 0 \\ 0 & \gamma \end{bmatrix}$ eigenvalues $\alpha > 1$, $\gamma < 1$ UNSTABLE

$$A = \begin{bmatrix} 1 & -\frac{\beta(1-\gamma)}{\delta} \\ \frac{\delta}{\beta}(\alpha-1) & 1 \end{bmatrix}$$

$Tr(A) = 2$ $\det A = 1 + (1-\gamma)(\alpha-1) > 1$

SO ALWAYS UNSTABLE!!