- independent of coordinate system
- easy + consistent!

Let q be a generalized coordinate, for example angle of pendulum or displacement of a spring. Let $\dot{q}$ be its derivative,

The Euler Lagrange equations say

$$\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}}{\partial q}}$$

They minimize the Lagrangian,

we will not prove this, but it is straight forward.

~~A~~ Example 1 pendulum

$\cancel{x = l\cos\theta,\ y = l\sin\theta}$

$x = l\sin\theta \quad y = -l\cos\theta$ Always True

$$K.E. = \frac{m\dot{x}^2}{2} + \frac{m\dot{y}^2}{2} = \frac{m|V|^2}{2}$$

$P.E. = mgY$ ← Depends on problem

$\dot{x} = l\cos\theta\,\dot{\theta} \quad \dot{y} = l\sin\theta\,\dot{\theta}$

$\dot{x}^2 + \dot{y}^2 = l^2\dot{\theta}^2$ $\boxed{KE = \frac{ml^2\dot{\theta}^2}{2}}$

$$PE = -mgl\cos\theta$$

$$\mathcal{L} = \frac{m}{2}l^2\dot{\theta}^2 + mgl\cos\theta$$

KE − PE

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}$$

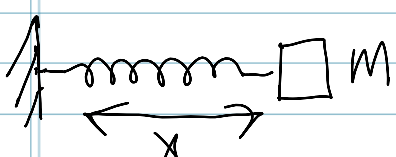
$$\frac{d}{dt}\frac{4ml^2\dot{\theta}}{2} = -mgl\sin\theta$$

$$\Rightarrow ml^2\frac{d^2\theta}{dt^2} = -mgl\sin\theta$$

$$\Rightarrow \boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta}$$

Wasn't That Easy

<u>Example 2</u> Spring mass



$$PE = \frac{k(x-x_0)^2}{2}$$

Linear spring, x_0 = rest length

$$KE = \frac{m\dot{x}^2}{2}$$

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - \frac{k(x-x_0)^2}{2}$$

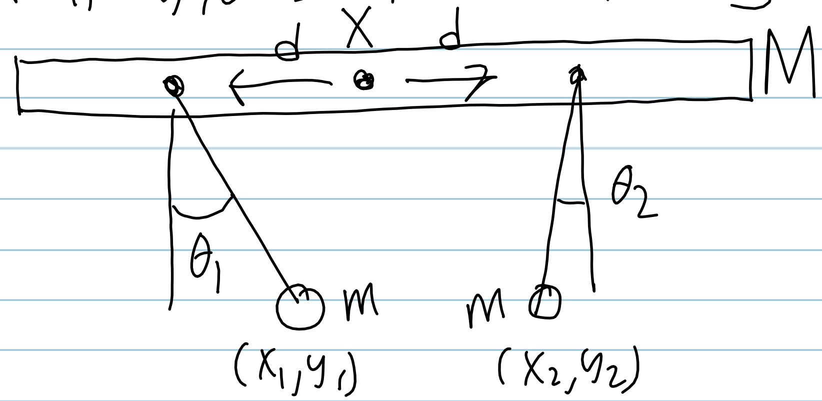
$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x}$$

$$\frac{d}{dt} m\dot{x} = -k(x - x_0)$$

$$m\ddot{x} = -k(x - x_0)$$

Example 3 SUPER COOL

Huygens pendulums



$$P.E. = mg(y_1 + y_2) = -mgl[\cos\theta_1 + \cos\theta_2]$$

$$x_1 = X - d + l\sin\theta_1$$

$$x_2 = X + d + l\sin\theta_2$$

$$KE = \frac{m}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2) + \frac{M\dot{X}^2}{2}$$

$$= \left(\frac{M}{2} + m\right)\dot{X}^2 + \frac{1}{2}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2)$$
$$+ m\dot{X}l[\cos\theta_1\dot{\theta}_1 + \cos\theta_2\dot{\theta}_2]$$

Euler Lagrange

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = (M+2m)\dot{x} + ml[\cos\theta_1 \dot{\theta}_1 + \cos\theta_2 \dot{\theta}_2]$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = ml\dot{x}\cos\theta_1 + ml^2 \dot{\theta}_1$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = ml\dot{x}\cos\theta_2 + ml^2 \dot{\theta}_2$$

$$A = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = ml\ddot{x}\cos\theta_1 - ml\dot{x}\dot{\theta}_1 \sin\theta_1 + ml^2 \ddot{\theta}_1$$

$$B = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2} = ml\ddot{x}\cos\theta_2 - ml\dot{x}\dot{\theta}_2 \sin\theta_2 + ml^2 \ddot{\theta}_2$$

$$C = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = (M+2m)\ddot{x} + ml[\cos\theta_1 \ddot{\theta}_1 + \cos\theta_2 \ddot{\theta}_2 - \sin\theta_1 \dot{\theta}_1^2 - \sin\theta_2 \dot{\theta}_2^2]$$

$$\boxed{C = 0} \quad \text{since } \frac{\partial \mathcal{L}}{\partial x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = k \text{ constant} \Rightarrow$$

$$\boxed{k = (M+2m)\dot{x} + ml[\cos\theta_1 \dot{\theta}_1 + \cos\theta_2 \dot{\theta}_2]}$$

Make it easy suppose start with

$\dot{\theta}_1 = \dot{\theta}_2 = \dot{X} = 0 \Rightarrow K = 0$

$$\Rightarrow \dot{x} = -\frac{ml\,[\cos\theta_1\,\dot{\theta}_1 + \cos\theta_2\,\dot{\theta}_2]}{M+2m}$$

$$\boxed{\ddot{x} = \frac{ml}{M+2m}\left[-\sin\theta_1\,\dot{\theta}_1^2 - \sin\theta_2\,\dot{\theta}_2^2 + \cos\theta_1\,\ddot{\theta}_1 + \cos\theta_2\,\ddot{\theta}_2\right]}$$

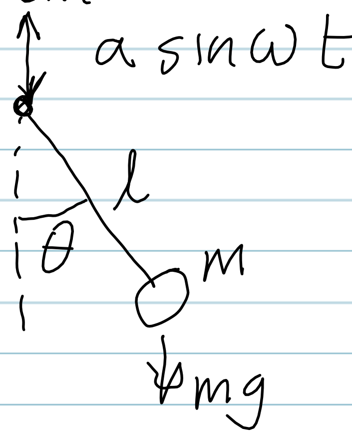
Can express A_3 in terms of θ_1, θ_2, etc

Then use, e.g. Maple to find

$\ddot{\theta}_1, \ddot{\theta}_2$ in terms of $\dot{\theta}_1, \dot{\theta}_2, \theta_1, \theta_2$!

~~as $M \gg 1$ (as is Decme in Huygens clocks)~~

Example ~~Pendulum plus~~ Moving pendulum



$$x = l \sin\theta \quad , \quad y = a \sin\omega t - l\cos\theta$$

$$PE = mg\,y = mg\,(a\sin\omega t - l\cos\theta)$$

$$KE = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2\right) = \frac{m}{2}\left[(l\cos\theta\,\dot\theta)^2 + (\omega a\cos\omega t + l\sin\theta\,\dot\theta)^2\right]$$

$$= \frac{m}{2} l^2 \dot\theta^2 + \frac{m}{2}\omega^2 a^2 \cos^2\omega t + m\omega a\cos\omega t\, l\sin\theta\,\dot\theta$$

$$\mathcal{L} = KE - PE$$

Euler-Lagrange:

$$\frac{\partial\mathcal{L}}{\partial\theta} = -mgl\sin\theta + m\omega a\cos\omega t\, l\cos\theta\,\dot\theta$$

$$\frac{\partial\mathcal{L}}{\partial\dot\theta} = ml^2\dot\theta + m\omega a\cos\omega t\, l\sin\theta$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\theta} = ml^2\ddot\theta - m\omega^2 a\sin\omega t\, l\sin\theta + m\omega a l\cos\omega t\cos\theta\,\dot\theta$$

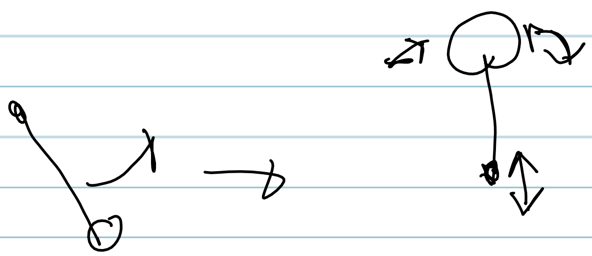
$$= -mgl\sin\theta + m\omega a\cos\omega t\, l\cos\theta\,\dot\theta$$

$$\Rightarrow \boxed{ml^2\ddot\theta = -mgl\sin\theta + m\omega^2 a l\sin\omega t\sin\theta}$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \omega^2 a \sin\omega t \sin\theta - \nu\dot{\theta}$$

FRICTION

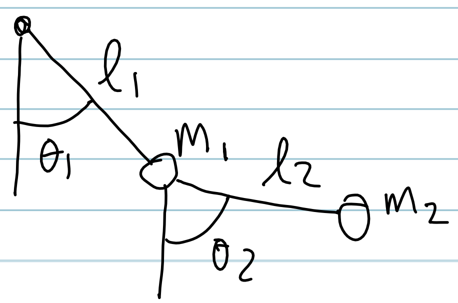
FUN TO simulate This. Can you make the pendulum stand up?!



Stabilize inverted pendulum

This is the key to juggling!

Double pendulum:



$x_1 = l_1 \sin\theta_1 \quad x_2 = x_1 + l_2 \sin\theta_2$

$y_1 = -l_1 \cos\theta_1 \quad y_2 = y_1 \cancel{+} - l_2 \cos\theta_2$

PE $= m_1 g y_1 + m_2 g y_2$

To make your life & my life simpler we assume that $l_1 = l_2 = l$ & $m_1 = m_2 = m$

so PE $= -mgl\,[2\cos\theta_1 + \cos\theta_2]$

$\dot{x}_1 = l\cos\theta_1\,\dot{\theta}_1$, $\dot{y}_1 = l\sin\theta_1\,\dot{\theta}_1$

$\dot{x}_2 = l\cos\theta_1\,\dot{\theta}_1 + l\cos\theta_2\,\dot{\theta}_2$, $\dot{y}_2 = l\sin\theta_1\,\dot{\theta}_1 + l\sin\theta_2\,\dot{\theta}_2$

$$\dot{x}_1^2 + \dot{y}_1^2 + \dot{x}_2^2 + \dot{y}_2^2 =$$

$$2l^2\dot{\theta}_1^2 + l^2\theta_2^2 + 2l^2[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2]\dot{\theta}_1\dot{\theta}_2$$

$$= l^2[2\theta_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2]$$

so $KE = \frac{ml^2}{2}\left[2\dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\cos(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2\right]$

$$\mathcal{L} = KE + mgl\,[2\cos\theta_1 + \cos\theta_2]$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = 2ml^2\dot{\theta}_1 + ml^2\cos(\theta_1-\theta_2)\dot{\theta}_2$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\theta}_2} = ml^2\dot{\theta}_2 + ml^2\cos(\theta_1-\theta_2)\dot{\theta}_1$$

$$\frac{\partial\mathcal{L}}{\partial\theta_1} = -2mgl\sin\theta_1 - ml^2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2$$

$$\frac{\partial\mathcal{L}}{\partial\theta_2} = -mgl\sin\theta_2 + ml^2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2$$

so

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\theta}_1} = 2ml^2\ddot{\theta}_1 + ml^2\cos(\theta_1-\theta_2)\ddot{\theta}_2 - ml^2\sin(\theta_1-\theta_2)\dot{\theta}_2(\dot{\theta}_1-\dot{\theta}_2)$$

$$= -2mgl\sin\theta_1 - ml^2\sin(\theta_1-\theta_2)\dot{\theta}_1\dot{\theta}_2$$

$$\Rightarrow \boxed{2\ddot{\theta}_1 + \cos(\theta_1-\theta_2)\ddot{\theta}_2 = \underbrace{-\sin(\theta_1-\theta_2)\dot{\theta}_2^2 - 2\frac{g}{l}\sin\theta_1}_{F_1}}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta_2} = m\ell^2 \ddot\theta_2 + m\ell^2 \cos(\theta_1-\theta_2)\,\ddot\theta_1 - m\ell^2 \sin(\theta_1-\theta_2)\,\dot\theta_1\,[\dot\theta_1-\dot\theta_2]$$

$$= \underbrace{-mg\ell \sin\theta_2 + m\ell^2 \sin(\theta_1-\theta_2)\,\dot\theta_1\dot\theta_2}_{F_2}$$

$$\Rightarrow \boxed{\ddot\theta_2 + \cos(\theta_1-\theta_2)\,\ddot\theta_1 = -\frac{g}{\ell}\sin\theta_2 + \sin(\theta_1-\theta_2)\,\dot\theta_2^{2}}$$

$$\begin{bmatrix} 2 & \cos\theta_1-\theta_2 \\ \cos(\theta_1-\theta_2) & 1 \end{bmatrix}\begin{bmatrix}\ddot\theta_1 \\ \ddot\theta_2\end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

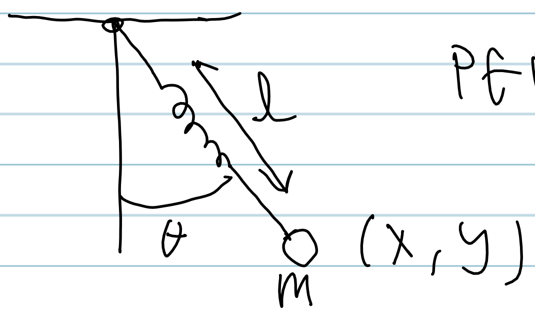
$$\boxed{\begin{aligned}\ddot\theta_1 &= \frac{F_1 - \cos(\theta_1-\theta_2)F_2}{2-\cos^2(\theta_1-\theta_2)} - \nu\dot\theta_1 \\ \ddot\theta_2 &= \frac{2F_2 - \cos(\theta_1-\theta_2)F_1}{2-\cos^2(\theta_1-\theta_2)} - \nu\dot\theta_2\end{aligned}}$$

Fake Friction

Let XPP solve this for you
Look at the animation!

HW problems   Due Friday Oct 8

(1)



PENDULUM / SPRING

l & θ can move so that l & θ are functions of time.

$$P.E. = mgy + \frac{k}{2}(l - l_0)^2$$

k is the spring constant

(a) Express x, y in terms of θ, l

(b) $K.E. = \frac{m}{2}(\dot{x}^2 + \dot{y}^2)$. Express K.E. in terms of $l, \theta, \dot{l}, \dot{\theta}$

(c) Express PE in terms of (l, θ)

(d) $\mathcal{L} = K.E. - P.E.$ ~~Write the~~

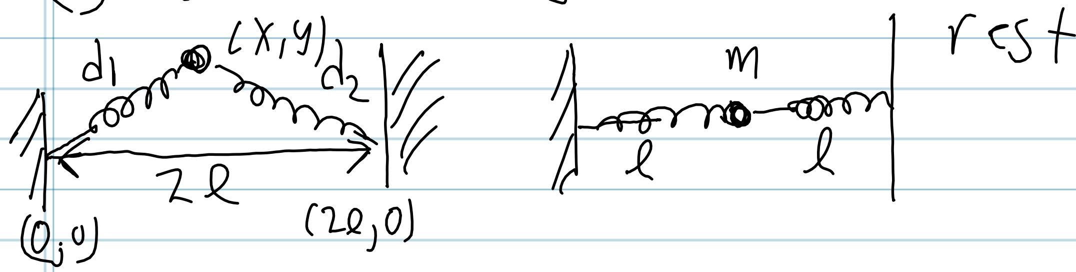
compute $\frac{\partial \mathcal{L}}{\partial \dot{l}}, \frac{\partial \mathcal{L}}{\partial \dot{\theta}}, \frac{\partial \mathcal{L}}{\partial l}, \frac{\partial \mathcal{L}}{\partial \theta}$

(e) Write equations of motion

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{l}} = \frac{\partial \mathcal{L}}{\partial l}, \quad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = \frac{\partial \mathcal{L}}{\partial \theta}$$

(, Show you recover linear spring if $\theta(0) = \dot{\theta}(0) = 0$

(2) "Guitar string"



(a) compute d_1, d_2 as function of (x,y)

$$PE = \frac{k}{2}\left[(d_1-\ell)^2 + (d_2-\ell)^2\right]$$

$$KE = m\,\frac{\dot{x}^2+\dot{y}^2}{2}$$

(b) $\mathcal{L} = KE - PE$

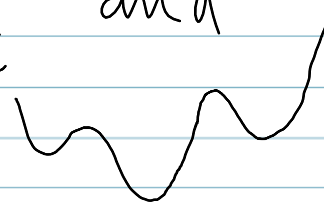
compute equations of motion

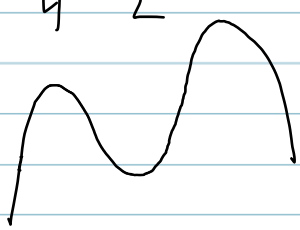
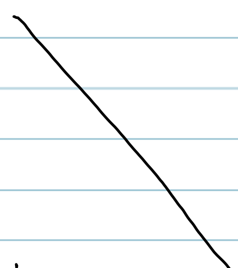
(c) Show that if $y(0)=0$ ~~Pem~~ + $\dot{y}(0)=0$
Then this is just a linear
spring on the line

(3) Sketch the phase plane for the following

$\ddot{x} = -f(x)$ where $F(x) = \frac{dF}{dx}$ and

$F(x)$ is:

(a) $\frac{x^4}{4} - \frac{x^2}{2}$ (b) $\cos x$ (c) 

(d)  (e) 

MORE HW Next page!!

ASIDE

(4)

If you're down + out + your life's a bust
It's 'cause you didn't learn the calculus

It's better'n drugs, It's better'n sex
Get to know du/dx

Your momma + your poppa are a buncha old farts
If they can't integrate by parts

I tell you right now, I pity you fool
If you didn't learn the chain rule

CHAPT 4 Homework also Due Oct 8

#3 page 152

#11 page 154

#14 a, b, c, d

#22 a, b, f

Analysis of 1 degree of Freedom systems.

Suppose $\mathcal{L} = \mathcal{L}(x, \dot{x})$ is independent of t.

consider

$$\mathcal{L} - \frac{dx}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = A$$

$$\frac{dA}{dt} = \frac{\partial \mathcal{L}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathcal{L}}{\partial \dot{x}}\frac{d^2x}{dt^2} - \frac{d^2x}{dt^2}\frac{\partial \mathcal{L}}{\partial \dot{x}} - \frac{dx}{dt}\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}$$

$$= \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}}\right)\frac{dx}{dt} = 0$$

$\Rightarrow$ $A =$ constant

$$\boxed{\mathcal{L} - \frac{dx}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0}$$

This is conservation of energy

To see this!

consider simple particle in potential

$$KE = \frac{m\dot{x}^2}{2} \qquad PE = F(x)$$

$$\mathcal{L} = \frac{m\dot{x}^2}{2} - F(x)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x} \qquad \mathcal{L} - \frac{dx}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{m\dot{x}^2}{2} - F(x) - m\dot{x}^2$$

$$= -\left(\frac{m\dot{x}^2}{2} + F(x)\right) = \text{constant}$$

$$\Rightarrow \frac{m\dot{x}^2}{2} + F(x)$$
$$K.E + PE = \underline{\text{constant}}$$

Analysis of these equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} \qquad f(x) = \frac{dF}{dx}$$

$$\boxed{m\ddot{x} = -f(x)}$$. Set $m = 1$ WLOG

$$\boxed{\begin{aligned}\dot{x} &= v \\ \dot{v} &= -f(x)\end{aligned}}$$

Equilibria: $v = 0 \quad f(x) = 0$

Stability $\begin{pmatrix} 0 & 1 \\ -f'(\bar{x}) & 0 \end{pmatrix} \quad \lambda^2 + f'(\bar{x}) = 0$

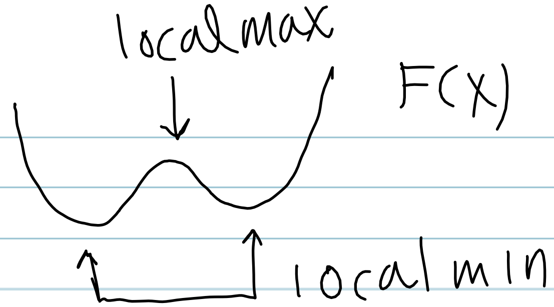
$\lambda = \pm\sqrt{-f'(\bar{x})}$ if $f'(\bar{x}) > 0 \quad \lambda = \pm i\omega$

if $f'(\bar{x}) < 0$ ~~there~~ 1 + eigenv, 1 − eign
saddle point.

Interpretation $f(\bar{x}) = 0$ means $\frac{dF}{dx} = 0$ so $\bar{x}$ is extremal of the potential.

$f'(\bar{x}) > 0 \Rightarrow \frac{d^2F}{dx^2} > 0 \Rightarrow \bar{x}$ is local minimum

$f'(\bar{x}) < 0 \Rightarrow \frac{d^2F}{dx^2} < 0 \Rightarrow \bar{x}$ is local max

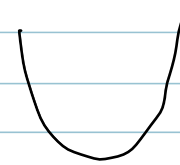


$m\dot{x}^2 + F(x) = \text{constant} \Rightarrow mv^2 + F(x) = \text{constant}$

$\Rightarrow v = \pm\sqrt{\frac{k - F(x)}{m}}$

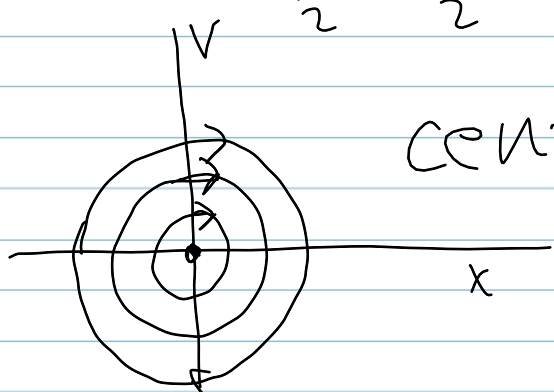
So solutions $(x(t), v(t))$ must be the curves $v = \pm\sqrt{\frac{k - F(x)}{m}}$

Example 1 $F(x) = \frac{x^2}{2}$



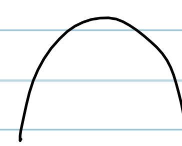
$\ddot{x} = -x$

$\frac{v^2}{2} + \frac{x^2}{2} = k$ circles

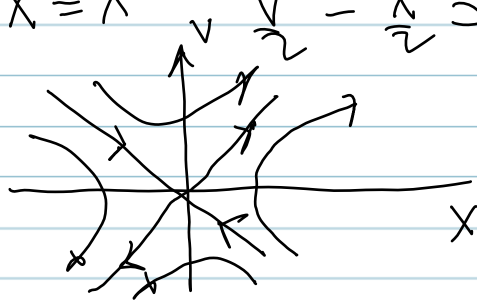


center

Example 2 $F(x) = -\frac{x^2}{2}$



$\ddot{x} = x$, $\frac{v^2}{2} - \frac{x^2}{2} = k$ Hyperbolas



saddle

Example 3 $F(x) = \frac{x^2}{2} - \frac{x^3}{3}$ $\frac{dF}{dx} = x - x^2 = f(x)$

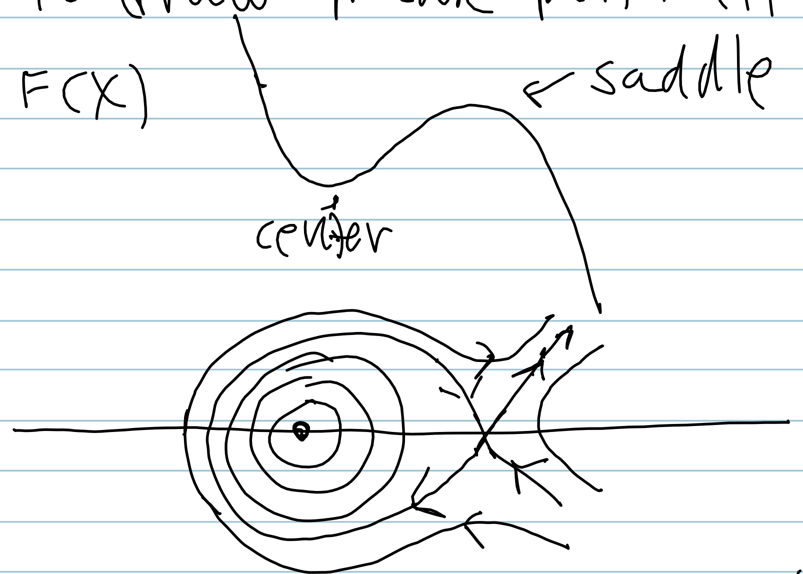
$\frac{v^2}{2} + \frac{x^2}{2} - \frac{x^3}{3} = k$

$\dot{x} = v \quad \dot{v} = -x + x^2$

(0,0) – center (1,0) saddle

How to draw phase portrait

(1) DRAW F(X)



(2) put X at saddles O at centers

(3) place little rolling ball at different point + figure out how it rolls

Example 4 $F(x) = 1 - \cos x$ (pendulum)

$f(x) = dF/dx = \sin(x)$

$\ddot{x} = -\sin(x)$

$\dot{x} = v \quad \dot{v} = -\sin(x)$ $x = 0, \pm 2\pi, \pm 4\pi$ minima

$x = \pi, \pm 3\pi, \pm 5\pi$ maxima

Windmilling

$V^2 = \pm\sqrt{4(K - \cos(x))}$ Hard to Draw!!

Example 5 $F(x) =$

No Equations!
Just Drawing!

Anyone can do this ———