

Midterm – due Oct 26th in class – each problem is 20 points

1. **Cruising speed of birds.** You can find a data set in excel format available on my website that contains the weight (in kilograms), cruising speed (in m/s) and the surface area of the wings (in m<sup>2</sup>) for over 125 species of birds. You will need it for this problem. (a) Let  $\rho$  denote the density of air,  $S$  the surface area,  $U$  the velocity, and  $L$  the lift force. Use the dimensions of these variables and the Buckingham Pi theorem to derive the following

$$L = C\rho U^2 S$$

for the lift force, where  $C$  is a dimensionless constant. (b) Let's now figure out how the cruising speed of a bird depends on its mass. Assume that to get off the ground, the lift force should balance the force due to gravity,  $mg$  where  $g$  is gravity and  $m$  is mass. Assume that birds are geometrically similar and that the constant  $C$  is the same for all birds. Use this to derive a scaling:

$$U \sim m^p$$

where you should determine  $p$ . (c) Use the data that I have given you to first check the geometric similarity assumption, by estimating how the surface area of the wing depends on the mass. What exponent do you get from the data? What exponent do you expect? (d) Now estimate the exponent  $p$  for the birds cruising speed. Report the slope and intercept of your fit. (You have done this for other data). Is it greater or less than the theory? Is it due to the assumption about geometric scaling? (e) Monty Python problem. The European swallow typically weighs 20 grams. How much much faster must it cruise in order to carry a 2 kilogram coconut? (Hint: Use  $mg = L$ )

2. Bard needs to study his math and his physics and up his GPA. His GPA is proportional to the number of hours he spends studying math times his ability to absorb the material and similarly with physics. He estimates that his concentration in studying math falls from 100% by 4% for every hour he studies math and by 3% for every hour he studies physics. Similarly, he finds his ability to concentrate on physics falls by 2% for each hour of studying math and by 5% for each hour of studying physics. Furthermore he finds that his GPA improves by an additional 10% for every hour of the day that he does not do math or physics. (a) Find the optimal number of hours that he should study math and physics to maximize his expected GPA. Set the problem up and define all the variables and find the answer. (b) Examine the sensitivity of your answer to the additional 10% benefit he gets from the time not studying. How would increasing the effect of this benefit affect his GPA? (c) He takes up lawn bowling and with eating, sleeping, he finds he needs at least 15 hours of free time per day. What is your answer now and how does this affect his GPA? You could solve this with Lagrange multipliers, but an easier method is to

turn it into a one variable optimization problem, by noting that the total number of hours studying should be constrained.

3. The Weedwacker Company makes gas and electric trimmers. They have been asked to supply 30000 electric and 15000 gas trimmers. However, they are limited in their production, assembly, and packaging with the following hours needed per trimmer:

	Electric	Gas	Hours Avail
Production	0.20	0.40	10000
Assembly	0.30	0.50	15000
Packaging	0.10	0.10	5000

They make the electric trimmers in house for \$55 and the gas trimmers for \$85. They can also outsource and purchase them for \$67 for electric and \$95 for gas. How many gas and electric trimmers should it make and how many should it buy in order to fulfill its contract in the least costly manner? (a) Formulate the problem as a linear programming question - this means I want you to write down all your variables and all the constraints. (b) Create a spreadsheet model, and solve it (c) What is the optimal solution? Also answer the following:

- Use the spreadsheet program to create a sensitivity analysis. How much would the electric trimmers have to cost in order for the company to consider buying them instead of making them?
  - If the cost to make a gas trimmer rose to \$90, how would that change the optimal solution?
  - How much should the company be willing to pay to acquire additional capacity for assembly? For production?
  - Suppose the hours available for production could vary between 9500 and 10500. How does this change the optimal solution for every 100 hour change in production capacity?
4. The Allegheny River Beach is a popular summer destination for thousands of people. Each summer the city hires temporary lifeguards. They are assigned to work 5 consecutive days each week with two days off. However, they must also have *at least* the following numbers of lifeguards on each day: Sunday(18), Monday(17), Tuesday(16), Wednesday(16), Thursday(16), Friday(14), Saturday(19). The city manager wants to hire the minimal number of lifeguards for this. (a) Formulate an integer programming model for this. That is, define all your variables and write down your constraints. (b) Solve it using excel or some other software (c) What is the optimal solution? (d) Several lifeguards would like the weekend off. What is the maximal number of lifeguards that can be off without affecting the total number hired?

5. A group of us are working on the development of robotic tracking devices that can follow odor trails. The device samples in discrete time at sampling intervals of  $h$  seconds. It moves at a fixed velocity,  $v$  and has two odor sensors located on “antenna” that have length,  $L$  and are at angles,  $\pm\phi$  from the body. See the picture. The robot changes its orientation,  $\theta$  according to the difference between the concentrations of the odor on its left and right sensors,  $\beta(C_l - C_r)$ . So, we will see if this robot can find and follow a trail.

- If  $(x, y)$  is the position of the head of the robot, show that

$$\begin{aligned}x_l &= x + L \cos(\theta + \phi) \\y_l &= y + L \sin(\theta + \phi) \\x_r &= x + L \cos(\theta - \phi) \\y_r &= y + L \sin(\theta - \phi)\end{aligned}$$

- The odor concentration is independent of  $y$  since we are assuming that it is just an infinite trail along the  $y$ -axis. The profile of the odor concentration is given by  $C(x) = e^{-x^2}$ . This means that  $C_l(x, \theta) = C(x_l)$  and  $C_r(x, \theta) = C(x_r)$ . Let  $x_n, y_n, \theta_n$  denote the position of the robot after the  $n^{\text{th}}$  sniff. Show that

$$\begin{aligned}x_{n+1} &= x_n + hv \cos(\theta_n) \\y_{n+1} &= y_n + hv \sin(\theta_n) \\\theta_{n+1} &= \theta_n + h\beta(C_l(x_n, \theta_n) - C_r(x_n, \theta_n))\end{aligned}$$

Notice that since  $y_n$  does not appear in the  $x_n, \theta_n$  equations, we don't have to worry about it, so for the rest of the problem, we will look at

$$\begin{aligned}x_{n+1} &= x_n + hv \cos(\theta_n) \\\theta_{n+1} &= \theta_n + h\beta(C_l(x_n, \theta_n) - C_r(x_n, \theta_n))\end{aligned}$$

- Show that  $x = 0, \theta = \pi, 3\pi/2$  are the equilibria for this system. Interpret what they mean in terms of the robot. For example, is the robot on the trail? If so, which direction is it moving?
- Set  $\phi = \pi/4, L = \sqrt{2}, h = 1$ . Use these values to determine conditions on  $v, \beta$  so that the equilibrium is stable.
- Set  $\beta = 0.5, v = 0.5$  and the other parameters as above. Let  $\theta_0 = \pi/2$  and let  $x_0 = 1, -1, 2$  be three starting positions. By iterating the equations (say, 200 iterates) determine which of these three values will find the trail. If you found that one of these initial conditions did not find the trail (that is, it did not converge to the equilibrium), then change  $\beta$  so that it will.

