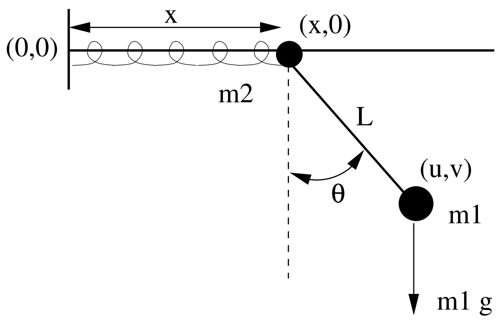
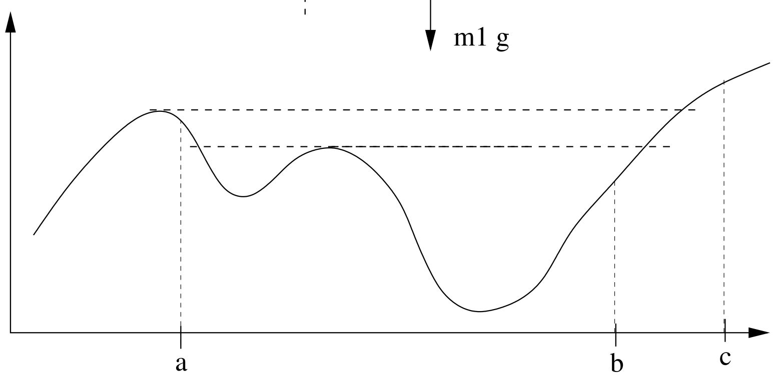
Final Exam 135 points Due Friday Dec 14 by 5 PM EST





1. (20 pts) In the figure above, I depict a spring that is threaded through a rod with the left side attached to the wall. The right end of the spring is attached to a ball of mass m_2 that is free to move along the rod. Also attached to this ball is pendulum that can rotate freely in the plane. The mass of the bob of the pendulum is m_1. The spring has rest length a, and linear spring constant, k. There are two degrees of freedom: x, the distance of the spring from the wall, and θ, the angle of the pendulum with respect to the vertical axis.

   - (5 pts). Using the coordinate system given in the figure, show that the potential energy,

     $$P = (k/2)(x-a)^2 - m_1 g L \cos\theta$$

     and that the kinetic energy is

     $$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1/2)L^2\dot{\theta}^2 + m_1 L \dot{x}\dot{\theta}\cos\theta$$

     Write down the Lagrangian, $B = T - P$.

   - (10 pts). Use the Euler-Lagrange equations to write down the differential equations for θ, x. They will be intermixed, that is there will be terms that mix $\ddot{\theta}, \ddot{x}$. You don't have to separate them out.
   - (5 pts) Find the equilibria for this model.

- (Bonus – 5 pts) Simulate the model using the following parameters: $m_1 = m_2 = 1, k = 1, g = 1, a = 1, L = 2$. Start with $\theta = 0$ and $x = 2$. You will have to solve for $\ddot{x}, \ddot{\theta}$ for this.

2. (10 pts) At the bottom of the above figure, I have drawn a potential energy function, $F(x)$ for a one-degree of freedom system. The equations are

$$m\ddot{x} = -F'(x)$$

   - Set $m = 1$ and sketch the phase plane in the $(x, \dot{x})$ phase plane and identify all saddle points, centers. Sketch relevant trajectories, including all the separatrices for the saddle points. Add arrows to indicate direction as well.
   - Sketch the three trajectories such that $x(0) = x_0$ and $\dot{x}(0) = 0$ where x_0 corresponds to the three points, a,b,c, shown in your phase-plane. Also sketch $x(t)$ for enough time so I can see what is going on. Finally, identify which of the trajectories is periodic.

3. (25 points) Consider the following chemical reactions:

$$\begin{aligned} A &\to X \\ X + B &\to Y \\ 2Y + X &\to 3Y \\ Y &\to D \end{aligned}$$

each having rate 1. Assume A, B are parameters that are positive. (a) (5 points) Show the equations for X, Y are :

$$\begin{aligned} X' &= A - BX - XY^2 \\ Y' &= BX + XY^2 - Y \end{aligned}$$

Make sure you tell me which chemical reaction is responsible for each term in the equations. (b) (2 points) Find the equilibria (Hint: Add the two equations); (c) (5 points) Write down the Jacobian at the equilibria and show (i) the determinant is always positive; (ii) find values of B as a function of A when the trace is positive, negative, or 0. (d) (13 points) Choosing $A = 0.5$, sketch the phase-plane (including the nullclines) for $B = 0.2$ and for $B = 0.05$. In each case, determine if the equilibrium is stable and also describe the behavior for initial conditions, $X = 0, Y = 0$. For this last part, you can use the XPP or Matlab code that I provide. (Solve the equations for 50 time units)

4. (10 points) In the chemical model of the previous problem, write a Gillespie algorithm for simulating this equation for a finite number of each species. If I want the number of molecules to be roughly 100 for each species, how should I choose the parameters, A, B, k_1, k_2, k_3, k_4? You do not have to simulate this; I just want an algorithm. (Although it is pretty cool to watch.)

5. (25 pts) In this problem, you will explore several variants of the Hawk-Dove game. I introduce several new strategies. The first is the bully who threatens a dove but runs from a hawk. Here is his table with a dove and a hawk:

	B	H
B	(V-D)/2	0
H	V	(V-C)/2

	B	D
B	(V-D)/2	V
D	0	V/2

Here V is the payoff in territory and C is the cost for fighting like a hawk and D is the damage to the two bullies incur for fighting dirty. We assume V, D, C are nonnegative.

- (10 pts) Let x be the population of bullies and $1 - x$ be the dove or hawk. Write down the equations for the dynamics of the bully when paired with the hawk and with the dove. Determine which states are stable (pure bully,hawk,dove, or mixed) as a function of (V, C, D).
- (15 pts) Consider a three strategy game with hawks,doves, and bullies. Fill in the table, given that V is the gain from territory and C is the cost for fighting honorably and D for fighting dirty:

	B	H	D
B			
H			
D			

  Let x be the number of bullies and y be the number of hawks with $1 - x - y$ the number of doves. Write down the equations for the dynamics of this system. Fix $V = 1$ and consider the following three examples ($C = .5, D = .75$), $C = 1.5, D = .75$, $C = 1.5, D = 2$. Draw the x, y phaseplane and nullclines in both cases. (I would use the computer for this). For each of the three cases, what are the fractions of Bullies, Hawks, and Doves?

6. (25 points) We will play a really limited version of *Chutes and Ladders.* The board has 9 squares labeled 1-9. You start at 1 and move the number of spaces given by the single 3 sided die. You must land on 9 exactly, otherwise you stay where you are (e.g if you are at 8, you can only go to 9 if you roll a 1). In addition, if you land on 5, you immefiately go to 8 (the ladder) and if you land on 7, you immediately go to 2 (the chute). This game is a Markov Chain with seven realizable states (you will never be at 5 or 7 since you leave instantly on that move, so don't include them in your diagram). (a) (5 points) Draw the state transition diagram and the transition matrix. (b) (5 points) If you iterate this, what is the expected steady state (note you dont need to do any calculations; just think, where do you ultimately end up?) Suppose you land at square 8. What is the expectwd number of tries you need to take to land on 9? (Hint: what is

the probability of landing exactly at the m^{th} try - that is you have $m-1$ failures and then a success; this is prob of landing on the m^{th} try) (c) (8 points) Starting at square 1, simulate this game for 10000 times and find the average amount of time it takes to finish it. Here is some pseudocode (I can give you matlab code if you want):

```
tsum=0 /* for averaging
for j=1 to 10000
X=1
t=0
while not done
t=t+1
k=1+floor(rand()*3)
Y=X+k
if Y==5 then Y=8  /* ladder
if Y==7 then Y=2 /* chute
if Y>9 then Y=X  /* passed the end
X=Y
if X==9 exit
end while
tsum = tsum + t
end for
tbar=tsum/10000  /* here is the average
```

(d) (7 points) Now that you have simulated this, we can actually get the answer exactly. Your transition matrix P (without 5 and 7) is 7×7. The last row and last column in the matrix corresponds to the final state. Form a 6×6 matrix, T by deleting the last row and column from P. Use the computer to compute the matrix $F = (I - T)^{-1}$ where I is the 6×6 identity matrix. The sum of the first row of F is the expected time to complete the game! What do you get from this calculation? How does it compare to your simulation?

7. (20 points) The math department has 4 staff positions. Recently, there has been a lot of flux. Each person seems to quit at a rate of about once a year and the new position is filled at on average after 8 months due to the University bureaucracy. (a) (5 points) Create a model for the number of staff people we have using a Markov process, where X_t is the number of staff people we have at time t. Write down the Master equation for the $p_j(t)$ the probability of having j staff at time t.(b) (15 points) Find the steady states and compute the probability that we are unstaffed and that we have a fully staffed department. What is the average number of staff? Suppose if the staffing levels fall to 2 or fewer, they can replace in 4 months instead of 8 months. What is the effect on the average, the probability of full stagfiing, and probability of no staff?