

1. A chemostat is a device that continuously pumps nutrient into a tank where bacteria use it to grow takes place and then they are removed for use. The equations for this are:

$$\begin{aligned} S' &= F(S_0 - S) - K_1NS/(K_2 + S) \\ N' &= -FN + \gamma K_1NS/(K_2 + S) \end{aligned}$$

$S$  has units of moles/liter,  $N$  has number of bacteria per liter. Time is in hours. Given this, (a) What are the dimensions of the parameters,  $F, S_0, K_1, K_2, \gamma$ ? (b) By rescaling  $N, C, t$  you should be able to reduce this to only two dimensionless parameters. Start with letting  $N = An, S = Ns, t = CT$ . There are at least two choices to make here, so I will force your hand. Let  $A = K_2$  and proceed from there and having made the correct scaling you should have:

$$\begin{aligned} s' &= (a - s) - ns/(1 + s) \\ n' &= -n + bns/(1 + s) \end{aligned}$$

where, you should tell me what  $a, b$  are in terms of the original parameters. (c) Find the equilibria and determine their stability. (d) Assume that  $a > 1$  and  $b > 1/(a - 1)$  and sketch the nullclines in the  $s, n$  plane along with some representative trajectories. Use  $a = 2, b = 2$  for simplicity. (e) Now, for this part, you have an equilibrium,  $n$  in terms of the parameters,  $a, b$ . Get  $N$  in terms of the original parameters. Now, the rate of bacteria production is  $FN$  (since that is what is collected) What value of  $F$  maximizes this rate?

2. (a) Consider the simple epidemiological model that we discussed in class:

$$\begin{aligned} S' &= -\beta SI + \delta(N - S - I) \\ I' &= I(\beta S - \gamma) \end{aligned}$$

Find all the equilibria and stability of this. Sketch the phase plane and the nullclines for  $\beta N/\gamma > 1$ . Use the phase-plane to sketch the solution to the differential equation given  $S(0) = N$  and  $I(0)$  is a small positive number (b) Compute the sensitivity of nonzero equilibrium for the infecteds,  $I$  with respect to the contact rate,  $\beta$ .

3. Recall from homework #1 that you fit some data to the logistic function by replotting numerical derivatives and using least squares on a linear set of equations. Here we you should do it using the way that we discussed in class. First, recall:

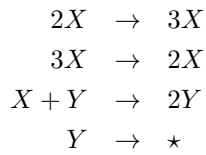
$$N(t, a, K, N_0) = \frac{KN_0}{\exp(-at)[K - N_0] + N_0}$$

- (a) Compute the sensitivity of  $N(t)$  with respect to  $a$  and with respect to  $K$  the carrying capacity (b) Suppose that we know that  $K = 5$  and

$x_0 = 1$ . We given  $(t_j, N_j)$  data as  $(t_1, N_1) = (2, 1.9)$ ,  $(t_2, N_2) = (4, 3.5)$ ,  $(t_3, N_3) = (6, 4)$  and  $(t_4, N_4) = (8, 4.5)$ . Compute the best fit for the parameter  $a$  by minimizing:

$$\sum_{j=1}^4 (N(t_j, a, 5, 1) - N_j)^2.$$

4. Consider the following chemical reaction system:



with the first three rate constants equal to 1 and the last,  $d$ . (a) Using the Laws of Mass Action write the differential equations for  $x, y$ . (b) Assume,  $d$  and find the equilibria and their stability. You can ignore the  $(x, y) = (0, 0)$ . (c) Set  $d = 0.75$  and draw the phase-plane with nullclines in the region  $-0.05 < x < 1.1$  and  $-0.05 < y < 1.1$ . Sketch a few trajectories either using the computer (XPP code can be found on the website) or by hand. (d) Let  $d = 0.55$ , describe what happens for  $(x(0), y(0)) = (1.0, 0.25)$  (e) do the same thing for  $d = 0.4$ .