1. A chemostat is a device that continuously pumps nutrient into a tank where bacteria use it to grow takes place and then they are removed for use. The equations for this are:

$$\begin{aligned} S' &= F(S_0 - S) - K_1 NS/(K_2 + S) \\ N' &= -FN + \gamma K_1 NS/(K_2 + S) \end{aligned}$$

S has units of moles/liter, N has number of bacteria per liter. Time is in hours. Given this, (a) What are the dimensions of the parameters, F, S_0, K_1, K_2, γ? (b) By rescaling N, C, t you should be able to reduce this to only two dimensionless parameters. Start with letting $N = An, S = Ns, t = CT$. There are at least two choices to make here, so I will force your hand. Let $A = K_2$ and proceed from there and having made the correct scaling you should have:

$$\begin{aligned} s' &= (a - s) - ns/(1 + s) \\ n' &= -n + bns/(1 + s) \end{aligned}$$

where, you should tell me what a, b are in terms of the original parameters. (c) Find the equilibria and determine their stability. (d) Assume that $a > 1$ and $b > 1/(a - 1)$ and sketch the nullclines in the s, n plane along with some representative trajectories. Use $a = 2, b = 2$ for simplicity. (e) Now, for this part, you have an equilibrium, n in terms of the parameters, a, b. Get N in terms of the original parameters. Now, the rate of bacteria production is FN (since that is what is collected) What value of F maximizes this rate?

2. (a) Consider the simple epidemiological model that we discussed in class:

$$\begin{aligned} S' &= -\beta SI + \delta(N - S - I) \\ I' &= I(\beta S - \gamma) \end{aligned}$$

Find all the equilibria and stability of this. Sketch the phase plane and the nullclines for $\beta N/\gamma > 1$. Use the phase-plane to sketch the solution to the differential equation given $S(0) = N$ and $I(0)$ is a small positive number (b) Compute the sensitivity of nonzero equilibrium for thr infecteds, I with respect to the contact rate, $/beta$.

3. Recall from homework #1 that you fit some data to the logistic function by replotting numerical derivatives and using least squares on a linear set of equations. Here we you should do it using the way that we discussed in class. First, recall:

$$N(t, a, K, N_0) = \frac{KN_0}{\exp(-at)[K - N_0] + N_0}$$

(a) Compute the sensitivity of $N(t)$ with respect to a and with respect to K the carrying capacity (b) Suppose that we know that $K = 5$ and

$x_0 = 1$. We given (t_j, N_j) data as $(t_1, N_1) = (2, 1.9)$, $(t_2, N_2) = (4, 3.5)$, $(t_3, N_3) = (6, 4)$ and $(t_4, N_4) = (8, 4.5)$. Compute the best fit for the parameter a by minimizing:

$$\sum_{j=1}^{4} (N(t_j, a, 5, 1) - N_j)^2.$$

4. Consider the following chemical reaction system:

$$\begin{aligned} 2X &\rightarrow 3X \\ 3X &\rightarrow 2X \\ X + Y &\rightarrow 2Y \\ Y &\rightarrow \star \end{aligned}$$

with the first three rate constants equal to 1 and the last, d. (a) Using the Laws of Mass Action write the differential equations for x, y. (b) Assume, d and find the equilibria and their stability. You can ignore the $(x, y) = (0, 0)$. (c) Set $d = 0.75$ and draw the phase-plane with nullclines in the region $-.05 < x < 1.1$ and $-.05 < y < 1.1$. Sketch a few trajectories either using the computer (XPP code can be found on the website) or by hand. (d) Let $d = 0.55$, descibe what happens for $(x(0), y(0)) = (1.0, 0.25)$ (e) do the same thing for $d = 0.4$.