Homework 8

1. A chemostat is a device that continuously pumps nutrient into a tank where bacteria use it to grow takes place and then they are removed for use. The equations for this are:

$$\begin{aligned} S' &= F(S_0 - S) - K_1 NS/(K_2 + S) \\ N' &= -FN + \gamma K_1 NS/(K_2 + S) \end{aligned}$$

S has units of moles/liter, N has number of bacteria per liter. Time is in hours. Given this, (a) What are the dimensions of the parameters, F, S_0, K_1, K_2, γ? (b) By rescaling N, C, t you should be able to reduce this to only two dimensionless parameters. Start with letting $N = An, S = Bs, t = CT$. There are at least two choices to make here, so I will force your hand. Let $B = K_2$ and proceed from there and having made the correct scaling you should have:

$$\begin{aligned} s' &= (a - s) - ns/(1 + s) \\ n' &= -n + bns/(1 + s) \end{aligned}$$

where, you should tell me what a, b are in terms of the original parameters. (c) Find the equilibria and determine their stability. (d) Assume that $b > 1$ and $a > 1/(b - 1)$ and sketch the nullclines in the s, n plane along with some representative trajectories. Use $a = 2, b = 2$ for simplicity. (e) Now, for this part, you have an equilibrium, n in terms of the parameters, a, b. Get N in terms of the original parameters. Now, the rate of bacteria production is FN (since that is what is collected) What value of F maximizes this rate?

2. Suppose I have two candidates in an election and since it is a competition, their dynamics is determined by the following equations:

$$\begin{aligned} x' &= x(1 - x - ay) \\ y' &= y(1 - y - bx) \end{aligned}$$

where a, b are positive. Assume that $ab \neq 1$. (a) Show that the equilibria are $(0,0)$, $(1,0)$,$(0,1)$, and (x_0, y_0). Determine the stability of these four equilibria and conditions under which each of these is stable. (b) Draw the xy phaseplane and some different trajectories in the window $[-.05, 1.05] \times [-.05, 1.05]$ for $a = .5, b = .5$, $a = .55, b = 2$, $a = 2, b = 2$. For each of these three cases enumerate all the stable equilibria. Which of these three cases seems to best model a competition? Are there any cases in which there is more than one stable equilibrium? If so, then how can you decide which of these two stable equilibria you will go to?

3. Now suppose I have three candidates, x, y, z with

$$\begin{aligned} x' &= x(1 - x - ay - bz) \\ y' &= y(1 - y - az - bx) \\ z' &= z(1 - z - ax - by) \end{aligned}$$

and $0 < b < 1 < a$. Assume that x, y, z are nonegative. In a contest between x, y $(z = 0)$ use the results of problem 1 to determine who wins when $x(0) > 0, y(0) > 0$. Similarly in a contest between (y, z) $(x = 0)$ who wins? In a context between (x, z) $(y = 0)$ who would win? So, now, suppose $a = 2, b = 0.55$. What do you think happens when $x(0), y(0), z(0)$ are all positive? Simulate this system with $x(0) = 0.4, y(0) = 0.45, z(0) = 0.5$ for $0 \le t \le 1000$ with a time step of 0.05 and report your findings. I have provided XPP and Matlab code.

4. In class, we derived the dimensionless waterwheel equations:

$$\begin{aligned} x' &= -\nu x + y \\ y' &= xz - y \\ z' &= -xy - z + q \end{aligned}$$

Assume that $\nu, q > 0$. (a) Find the equilibria. There will be two types of equilibria, $(0, 0, \hat{z})$ corresponding to the waterwheel not moving and $(\pm\bar{x}, \pm\bar{y}, \bar{z})$ corresponding to constant motion of the wheel clockwise or counterclockwise. (b) What relationship between q and ν is required for the second type of equilibrium to exist (that is, $\bar{x}, \bar{y}$ are real). (c) Compute the Jacobian matrix for the system and evaluate it at the motionless case $(0, 0, \hat{z})$. It is a simple block diagonal matrix, with an upper 2x2 block. (d) Use the determinant-trace condition on this 2x2 block to determine the stability of this equilbrium as a function of q, ν. Do you see a connection between the answer to (d) and the answer to (b)? (e) I will save you a bunch of work. If I substitute the equilibria of the second type $(\pm\bar{x}, \pm\bar{y}, \bar{z})$ into the Jacobian and compute the characteristic polynomial for this I get the nasty cubic:

$$P(s) = s^3 + (2 + \nu)s^2 + [(\nu^2 + q)/\nu]s + 2(q - \nu).$$

The roots of $P(s)$ are the eigenvalues of the matrix. But, we don't care about the eigenvalues; all we care about is if they have negative real parts. So, recall for a two-d system, we could just look at the trace and the determinant. So, for three d systems, we have the following result: *The roots of $s^3 + a_2 s^2 + a_1 s + a_0$ have negative real parts if and only if $a_0 > 0, a_2 > 0, a_1 a_2 - a_0 > 0$. Furthermore, if $a_2 > 0, a_0 > 0$ and $a_1 a_2 = a_0$ then there are imaginary eigenvalues.* This is called the **Routh-Hurwitz** criterion. Use this to find the range of values of q so that the equilibria, $(\pm\bar{x}, \pm\bar{y}, \bar{z})$ are stable. (b) Pick $\nu = 3$ and $q = 60$. Are these stable? What about $\nu = 3, q = 70$? Simulate the equations for 50 time units with $x(0) = y(0) = z(0) = 1$ and $\nu = 3, q = 70$. Plot the solutions in the x, z plane. This is the famous Lorenz butterfly! It is a chaotic solution. I have provided the matlab and xpp codes for you, ww.ode and ww.m.