Homework 8

1. Problems 6.16,6.17,6.19 in Teschl (sections 6.5,6.6)

2. The following odes have Hamiltonian structure,that,is $f_x+g_y = 0$ for there is a function $H(x, y)$ such that $f = H_y$ and $g = -H_x$. Find $H(x, y)$ for the following systems

   (a) $x' = x^2y + y^2$, $y' = -xy^2 + x^3$

   (b) $x' = q(y)$, $y' = -r(x)$

   (c) Sketch the phase-space picture for $x' = y$, $y' = x(1 - x^2)$.

3. Use $V = x^2 + y^2$ to analyze the following planar ODEs. What stability conclusions can be drawn.

   (a) $x' = -x^3 + 2y^3$, $y' = -2xy^2$

   (b) $x' = -x^3 + 2xy^2$,$y' = -2x^2y - y^3$

   (c) $x' = y - x^3$, $y' = -x$

   (d) $x' = -y$,$y' = x + y^5 - 2y$

   Determine the stability of the origin for the system, $x' = y^2 - x^2$, $y' = xy$.

4. Analyze $y'' + f(y)y' + h(y)$ where $f(y) > 0$ and $yh(y) > 0$ for $y \neq 0$ and such that f, g are continuous. (Hint: convert this to a system of 2 odes and find a suitable Lyapunobv function.) Additionally, show that if

$$\lim_{|y|\to\infty} \int_0^y h(s)\, ds = +\infty$$

   then all solutions to this ODE are bounded.

5. The Lorenz equation has the form:

$$\begin{aligned} x' &= \sigma(y - x) \\ y' &= rx - y - xz \\ z' &= xy - bz \end{aligned}$$

   where σ, r, b are positive parameters. Show that $V = x^2 + \sigma y^2 + \sigma z^2$ is a strict Liapunov function for $r < 1$ and thus the origin is globally aymptotically stable.