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Appendix A

Selected Hints and Solutions

1.1.1; Hint: What if one basis was smaller than another?

1.1.3; (a) Follows from direct verification.

(b) Follows from (a). If the norm is known to be induced by an inner product, then (a) shows how to uniquely calculate the inner product.

(c) Suppose $||x|| = (\sum_{k=1}^{n} |x_k|^p)^{1/p}$.

($\Leftarrow$) If $p = 2$, then $\langle x, y\rangle = \sum_{k=1}^{n} x_k y_k$ induces the norm.

($\Rightarrow$) If the norm is induced by an inner product, then from (a)

$$\langle x, y\rangle = \frac{1}{4}\left((\sum_{k=1}^{n} |x_k + y_k|^p)^{2/p} - (\sum_{k=1}^{n} |x_k - y_k|^p)^{2/p} \right).$$

Take $x = (1, 0, 0, \ldots, 0)$, and $y = 0, 1, 0, \ldots, 0)$. Then $\langle x, x\rangle = 1$, $\langle x, y\rangle = 0$, and $\langle x, x+y\rangle = \frac{1}{4}\left((2^p+1)^{2/p} - 1\right)$. Since for an inner product, $\langle x, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle$, it must be that $(2^p + 1)^{2/p} = 5$. Since $(2^p+1)^{2/p}$ is a monotone decreasing function of p which approaches 1 for large p and is unbounded at the origin, the solution of $(2^p+1)^{2/p} = 5$ at $p = 2$ is unique. We conclude that $p = 2$.

1.1.4; $\langle F, F\rangle = 0$ does not imply that $f(x) = 0$.

1.1.8; Observe that with $\beta = \langle x, y\rangle/||y||^2$, $x - \beta y$ is orthogonal to y, so that

$$||x - \alpha y||^2 = ||x - \beta y||^2 + ||(\alpha - \beta)y||^2,$$

which is minimized when $\alpha = \beta$. Clearly, if $x = \alpha y$, then

$$|\langle x, y\rangle|^2 = ||x||^2 ||y||^2.$$

If so, then we calculate directly that $||x - \beta y||^2 = 0$, so that $x = \beta y$.

1.1.9; (a) $\phi_0(x) = 1, \phi_1(x) = x, \phi_2(x) = x^2 - \frac{1}{3}, \phi_3(x) = x^3 - \frac{3}{5}x, \phi_4(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}$.

(b) $\phi_0(x) = 1, \phi_1(x) = x, \phi_2(x) = x^2 - \frac{1}{2} = \frac{1}{2}\cos(2\cos^{-1}x), \phi_3(x) = x^3 - \frac{3}{4}x = \frac{1}{4}\cos(3\cos^{-1}x), \phi_4(x) = x^4 - x^2 + \frac{1}{8} = \frac{1}{8}\cos(4\cos^{-1}x)$.

(c) $\phi_0(x) = 1, \phi_1(x) = x - 1, \phi_2(x) = x^2 - 4x + 2, \phi_3(x) = x^3 - 9x^2 + 18x - 6, \phi_4(x) = x^4 - 48x^3 + 72x^2 - 96x + 24$.

(d) $\phi_0(x) = 1, \phi_1(x) = x, \phi_2(x) = x^2 - \frac{1}{2}, \phi_3(x) = x^3 - \frac{3}{2}x, \phi_4(x) = x^4 - 3x^2 + \frac{3}{4}$.

1.1.10; $\phi_0(x) = 1, \phi_1(x) = x, \phi_2(x) = x^2 - \frac{1}{3}, \phi_3(x) = x^3 - \frac{9}{10}x, \phi_4(x) = x^4 - \frac{33}{28}x^2 + \frac{27}{140}, \phi_5(x) = x^5 - \frac{1930}{1359}x^3 + \frac{445}{1057}x$.

1.2.1; (a) Relative to the new basis, $A = \begin{pmatrix} 2 & 3 & 3 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$.

(b) Set $C = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 2 & 3 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$. Then the representation of A in the new basis is

$$A' = D^{-1}CAC^{-1}D = \begin{pmatrix} \frac{53}{6} & -\frac{19}{3} & -4 \\ \frac{13}{12} & -\frac{5}{6} & -1 \\ \frac{45}{4} & -\frac{17}{2} & -5 \end{pmatrix}.$$

1.2.2; (c) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and $B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$ have the same determinant but are not equivalent.

1.2.3; (a) Notice that if $ABx = \lambda x$, then $BA(Bx) = \lambda(Bx)$.

(b) If $AA^*x = \lambda x$, then $\lambda\langle x, x\rangle = \langle AA^*x, x\rangle = \langle A^*x, A^*x\rangle \geq 0$.

1.2.4; If $Ax = \lambda x$ then $\lambda\langle x, x\rangle = \langle Ax, x\rangle = \langle x, A^T x\rangle = -\langle x, Ax\rangle = -\overline{\lambda}\langle x, x\rangle$.

1.2.5; (a) $R(A) = \{(1,1,2)^T, (2,3,5)^T\}, N(A) = 0, R(A^*) = I\!R^2, N(A^*) = \{(1,1,-1)^T\}$.

(b) $R(A) = R(A^*) = I\!R^3, N(A) = N(A^*) = 0$.

1.2.6; (a) $T = \begin{pmatrix} 1 & \frac{1}{5} & 0 \\ \frac{1}{3} & \frac{3}{5} & 1 \\ 0 & 1 & 0 \end{pmatrix}, T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix}$

(b) $T = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, T^{-1}AT = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$.

(c) $T = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \end{pmatrix}, T^{-1}AT = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

(d) T does not exist.

(e) $T = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & -\sqrt{3} & \frac{2}{\sqrt{3}} \end{pmatrix}, T^{-1}AT = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$.

1.2.7; $T = \begin{pmatrix} 1 & 2 \\ -2 & -1 \end{pmatrix}, T^{-1}AT = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$.

1.2.8; If x is in M, then $Px = x$, and if x is in the orthogonal complement of M, the $Px = 0$. Therefore, P has two eigenvalues, $\lambda = 0, 1$.

1.2.10; (d) Examine the eigenvectors of A and A^* where $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

1.3.1; Hint: Minimize $\langle Ax, x\rangle$ with a vector of the form $x^T = (1, -1, z, 0)$.

1.3.2; (a) Prove that if the diagonal elements of a symmetric matrix are increased, then the eigenvalues are increased (or, not decreased) as well.

(b) Find the eigenvalues and eigenvectors of $B = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & -4 \\ 4 & -4 & 8 \end{pmatrix}$, and use them to estimate the eigenvalues of A.

1.3.3; The matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 7 \end{pmatrix}$ has a positive, zero, and negative eigenvalue. Apply 1.3.2a.

1.4.1; (a) b must be orthogonal to $(1, 1, -1)^T$, and the solution, if it exists, is unique.

(b) The matrix A is invertible, so the solution exists and is unique.

1.4.2; b must be in the range of P, namely M.

1.4.3; ($\Rightarrow$) Suppose A is invertible, and try to solve the equation $\sum_i \alpha_i\phi_i = 0$. Taking the inner product with ϕ_j, we find $0 = A\alpha$, so that $\alpha = 0$, since the null space of A is zero.

($\Leftarrow$) Suppose $\{\phi_i\}$ form a linearly independent set and that $Ax = 0$. Then $\langle x, Ax\rangle = \langle \sum_i x_i\phi_i, \sum_j x_j\phi_j\rangle = 0$, so that $\sum_i x_i\phi_i = 0$, implying that $x = 0$, so that A is invertible (by the Fredholm alternative).

1.4.4; Since $\langle Ax, x\rangle = \langle x, A^*x\rangle > 0$ for all $x \neq 0$, the null spaces of A and A^* must be empty. Hence, $\langle b, x\rangle = 0$ for all x in $N(A^*)$ so that $Ax = b$ has a solution. Similarly, the solution is unique since the null space of A is empty.

1.5.1; (a) $A' = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

(b) $A' = \frac{1}{24}\begin{pmatrix} 7 & 4 & 1 \\ 1 & 4 & 7 \end{pmatrix}$

(c) $A' = \frac{1}{12}\begin{pmatrix} 3 & 1 \\ 3 & 5 \\ 0 & -4 \end{pmatrix}$

(d) $A' = \frac{1}{2}\begin{pmatrix} -6 & 2 & 2 \\ -5 & 4 & 1 \\ -4 & 2 & 2 \end{pmatrix}$

1.5.3; (a) Take $u = x - y$.

1.5.4; $Q = (\phi_1, \phi_2, \phi_3)$, where $\phi_1 = \frac{1}{\sqrt{101}}\begin{pmatrix} 2 \\ 4 \\ 9 \end{pmatrix}, \phi_2 = \frac{1}{\sqrt{505}}\begin{pmatrix} 9 \\ 18 \\ -10 \end{pmatrix}, \phi_3 = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$, and $R = \begin{pmatrix} \sqrt{101} & \frac{19}{\sqrt{101}} & \frac{28}{\sqrt{101}} \\ 0 & \frac{35}{\sqrt{505}} & \frac{25}{\sqrt{505}} \\ 0 & 0 & \sqrt{5} \end{pmatrix}$.

1.5.5; Use that

$$A = \frac{1}{\sqrt{10}}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{10} & 0 \\ 0 & 2\sqrt{10} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

or

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & -3 \end{pmatrix}.$$

1.5.6; Use that $A = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 5 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$, so that $A' = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}$.

1.5.8; For $A = \begin{pmatrix} 1.002 & 0.998 \\ 1.999 & 2.001 \end{pmatrix}$, singular values are $\sqrt{10}$ and $\epsilon\sqrt{10}$ with $\epsilon = 0.001\sqrt{10}$, $A' = \begin{pmatrix} 0.1 + \frac{2}{\epsilon\sqrt{10}} & 0.2 - \frac{1}{\epsilon\sqrt{10}} \\ 0.1 - \frac{2}{\epsilon\sqrt{10}} & 0.2 + \frac{1}{\epsilon\sqrt{10}} \end{pmatrix} = \begin{pmatrix} 200.1 & -99.8 \\ -199.8 & 100.2 \end{pmatrix}$.

Using instead singular values $\sqrt{10}$ and 0, $A' = \begin{pmatrix} 0.1 & 0.2 \\ 0.1 & 0.2 \end{pmatrix}$.

1.5.10; $A = U\Sigma V$ where $U = \frac{1}{\sqrt{5}}\begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & -1 & -2 \\ 0 & -2 & 1 \end{pmatrix}$, $\Sigma = \begin{pmatrix} 2\sqrt{10} & 0 \\ 0 & 3\sqrt{10} \\ 0 & 0 \end{pmatrix}$, $V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix}$, so that $A' = \begin{pmatrix} \frac{\sqrt{5}}{20} & \frac{1}{30} & \frac{1}{15} \\ -\frac{\sqrt{5}}{20} & \frac{1}{30} & \frac{1}{15} \end{pmatrix}$.

2.1.3; For $x_n = \sum_{k=1}^{n} \frac{1}{k!}$, the difference $|x_n - x_m| = \sum_{k=n+1}^{m} \frac{1}{k!} \le \frac{1}{(n+1)!} \sum_{k=0}^{m-1} \frac{1}{n^k}$ $\le \frac{1}{(n+1)!} \frac{1}{1-\frac{1}{n}} < \frac{2}{(n+1)!}$ is arbitrarily small for m and n large.

2.1.4; The functions $\{\sin n\pi x\}_{n=1}^{\infty}$ are mutually orthogonal and hence linearly independent.

2.1.5; $\max_t |f_n(t) - f_m(t)| = \frac{1}{2}(1 - \frac{n}{m})$ if $m > n$, which is not uniformly small for m and n large. However, $\int_0^1 |f_n(t) - f_m(t)|^2 dt = \frac{(n-m)^2}{12nm^2} < \frac{1}{12n}$.

2.1.6; (a) Hint: Show that $|(|u| - |v|)| \le |u - v|$.

2.1.8; $\int_0^1 \mathcal{X}(t) = 0$.

2.1.11; $\int_0^1 \left(\int_0^1 f(x,y)dx\right) dy = -\int_0^1 \left(\int_0^1 f(x,y)dy\right) dx = -\frac{\pi}{4}$. Fubini's theorem fails to apply because $\int_0^1 \left(\int_0^1 |f(x,y)|dx\right) dy = \int_0^1 \left(\int_0^1 \frac{1}{x^2+y^2} dx\right) dy = \int_0^1 \frac{1}{y} \tan^{-1} \frac{1}{y} dy$ does not exist.

2.2.1; With $w(x) = 1, p(x) = \frac{15}{16}x^2 + \frac{3}{16}$, with $w(x) = \sqrt{1-x^2}, p(x) = \frac{8}{15\pi}(6x^2 + 1)$, and with $w(x) = \frac{1}{\sqrt{1-x^2}}, p(x) = \frac{2}{3\pi}(4x^2 + 1)$.

2.2.2; Define the relative error e_k by $e_k^2 = 1 - \frac{\|f_k\|^2}{\|f\|^2}$, where $f_k(x)$ is the kth partial sum, $f_k(x) = \sum_{n=0}^{k} a_n \phi_n(x)$. Then

(a) $H(x) - 2H(x-\pi) = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \sin(2n+1)x$, $e_5 = 0.201$,

(b) $x - \pi = -2 \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$, $e_5 = 0.332$,

(c) $\begin{cases} x & 0 < x < \pi \\ 2\pi - x & \pi < x < 2\pi \end{cases} = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x$, $e_5 = 0.006$,

(d) $x^2(2\pi - x)^2 = \frac{8\pi^4}{15} - 48 \sum_{n=1}^{\infty} \frac{1}{n^4} \cos nx$, $e_3 = 0.002$.

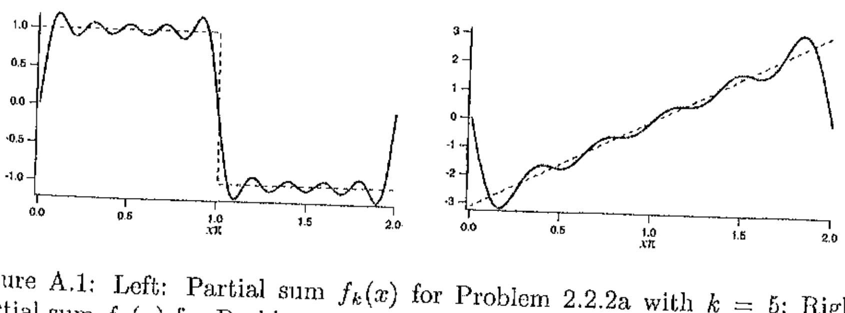


Figure A.1: Left: Partial sum $f_k(x)$ for Problem 2.2.2a with $k = 5$; Right: Partial sum $f_k(x)$ for Problem 2.2.2b with $k = 5$.

2.2.3; $f(x) = x(x-\pi)(x-2\pi) = 12 \sum_{n=1}^{\infty} \frac{1}{n^3} \sin nx$.

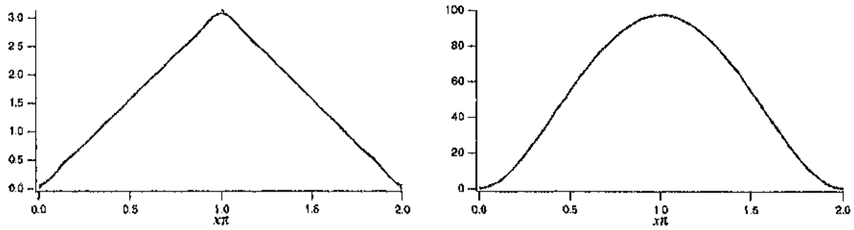


Figure A.2: Left: Partial sum $f_k(x)$ for Problem 2.2.2c with $k = 5$; Right: Partial sum $f_k(x)$ for Problem 2.2.2d with $k = 3$.

2.2.4; With the additional assumption that the function $f(x)$ to be fit intersects the straight line at the two points $x = x_1$ and $x = x_2$, $x_1 < x_2$, we must have that $x_2 - x_1 = \frac{1}{2}$, and $x_2 + x_1 = 1$, so that $\alpha = \frac{3}{2}f(\frac{1}{4}) - \frac{1}{2}f(\frac{3}{4}), \beta = 2f(\frac{3}{4}) - 2f(\frac{1}{4})$.

2.2.6; (a) You should have a procedure that generates the Legendre polynomials from Problem 1.1.9, or use Maple, which knows about Legendre polynomials.

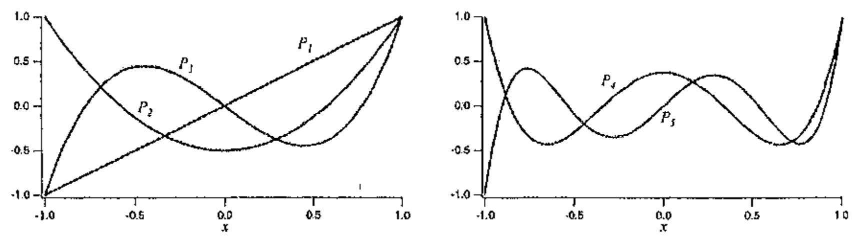


Figure A.3: Left: Legendre polynomials $P_1(x), P_2(x)$ and $P_3(x)$; Right: Legendre polynomials $P_4(x), P_5(x)$.

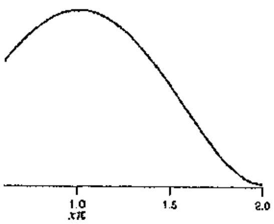
(b) $g(x) = ax + bx^3 + cx^5$, where $a = \frac{105}{8\pi^6}(\pi^4 - 153\pi^2 + 1485) = 3.10346$, $b = -\frac{315}{4\pi^6}(\pi^4 - 125\pi^2 + 1155) = -4.814388$, $c = \frac{693}{8\pi^6}(\pi^4 - 105\pi^2 + 945) = 1.7269$.

(c) $g(x) = 3.074024x - 4.676347x^3 + 1.602323x^5$. A plot of $g(x)$ is barely distinguishable from $\sin \pi x$ on the interval $-1 \le x \le 1$.

2.2.7; Use integration by parts to show that the Fourier coefficients for the two representations are exactly the same for any function which is sufficiently smooth.

2.2.9; (b) Write $\phi_{n+1} - A_n x \phi_n = \sum_{k=0}^{n} \beta_k \phi_k$, and evaluate the coefficients by taking inner products with ϕ_j, and using part (a), and the fact that $||\phi_k|| = 1$. Show that $B_n = -A_n \langle x\phi_n, \phi_n \rangle$, $C_n = A_n / A_{n-1}$.

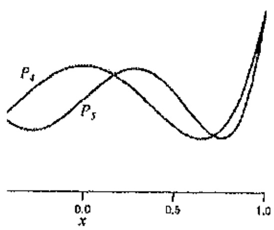
2.2.14; Direct substitution and integration yields $h(t) = 2\pi \sum_{k=-\infty}^{\infty} f_k g_k e^{ikt}$.



:.2.2c with $k = 5$; Right:

on $f(x)$ to be fit intersects $x = x_2$, $x_1 < x_2$, we must t $\alpha = \frac{3}{2}f(\frac{1}{4}) - \frac{1}{2}f(\frac{3}{4}), \beta =$

es the Legendre polynomi- ch knows about Legendre



and $P_3(x)$; Right: Legen-

$\cdot\, 153\pi^2 + 1485) = 3.10346$, 38, $c = \frac{693}{8\pi^5}(\pi^4 - 105\pi^2 +$

^{5}. A plot of $g(x)$ is barely $-1 \le x \le 1$.

ier coefficients for the two ıction which is sufficiently

valuate the coefficients by), and the fact that $||\phi_k|| =$ $ı-1$.

$= 2\pi \sum_{k=-\infty}^{\infty} f_k g_k e^{ikt}$.

2.2.15; Suppose the transforms of f, g, and h are $\widehat{f}, \widehat{g}$, and $\widehat{h}$, respectively. Use direct substitution and the fact that $\frac{1}{N}\sum_{j=0}^{N-1} e^{2\pi ijk/N} = 1$ if k is an integer multiple of N (including 0), and $= 0$ otherwise to show that $\widehat{h}_j = \widehat{f}_j\widehat{g}_j$.

2.2.16; Rewrite the definition of the discrete Fourier transform as a matrix multiplication. Show that the matrix is orthogonal.

2.2.19; See Fig. A.4.

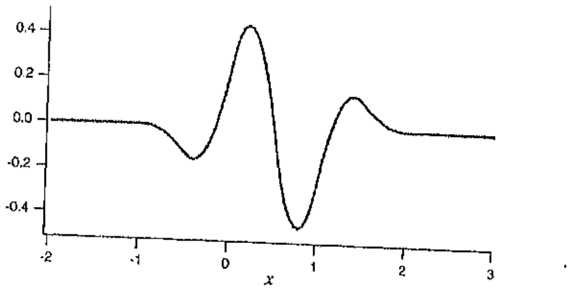


Figure A.4: The wavelet generated by the B-spline $N_3(x)$, for Problem 2.2.19.

2.2.24; Solve the equation $A\alpha = \beta$ where

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{pmatrix}, \qquad \beta = -3\begin{pmatrix} f_0 - f_1 \\ f_0 - f_2 \\ f_1 - f_2 \end{pmatrix}.$$

The solution is

$$\alpha = \begin{pmatrix} -\frac{5}{4} & \frac{3}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{4} & -\frac{3}{2} & \frac{5}{4} \end{pmatrix}\begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix}.$$

2.2.26; (a) Solve $A\alpha = \beta$ where

$$a_{ij} = \int_0^1 \psi_i''\psi_j''dx, \qquad \beta_j = -\sum_{i=0}^{N} f_i \int_0^1 \phi_i''\psi_j''dx.$$

Use Problem 2.2.23 to evaluate the coefficients, and find that the ith equation is $\alpha_{i-1} + 4\alpha_i + \alpha_{i+1} = \frac{3}{h}(f_{i+1} - f_{i-1})$, for $i \neq 0, N$.

(b) Solve $A\alpha = \beta$ where

$$a_{ij} = \int_0^1 \psi_i'\psi_j'dx, \qquad \beta_j = -\sum_{i=0}^{N} f_i \int_0^1 \phi_i'\psi_j'dx.$$

Use Problem 2.2.23 to evaluate the coefficients, and find that the ith equation is $-\alpha_{i-1} + 8\alpha_i - \alpha_{i+1} = \frac{3}{h}(f_{i+1} - f_{i-1})$, for $i \neq 0, N$.

(c) Require $\alpha_{i-1} + 4\alpha_i + \alpha_{i+1} = \frac{3}{h}(f_{i+1} - f_{i-1})$, for $i \neq 0, N$.

(d) Solve $A\alpha = \beta$ where

$$a_{ij} = \int_0^1 \psi_i''(x)\psi_j''(x)dx,$$

$$\beta_j = \int_0^1 g(x)\psi_j(x)dx - \sum_{i=0}^{N} f_i \int_0^1 \phi_i(x)\psi_j(x)dx.$$

2.2.27; Observe that the equations for $\alpha_1, \dots, \alpha_{N-1}$ from Problem 2.2.26a and c are identical.

3.1.1; Use Leibniz rule to differentiate the expression $u(x) = \int_0^1 y(x-1)f(y)dy + \int_x^1 x(y-1)f(y)dy$ twice with respect to x.

3.2.1; Find a sequence of functions whose L^2 norm is uniformly bounded but whose value at zero is unbounded. There are plenty of examples.

3.2.2; The proof is the same for all bounded linear operators; see page 107.

3.2.3; (a) The null space is spanned by $u = 1$ when $\lambda = 2$, therefore solutions exist and are unique if $\lambda \neq 2$, and solutions exist (but are not unique) if $\lambda = 2$ and $\int_0^{1/2} f(t)dt = 0$.

(b) The null space is spanned by $u = x$ when $\lambda = 3$, therefore solutions exist and are unique if $\lambda \neq 3$, and solutions exist, but are not unique, if $\lambda = 3$ and $\int_0^1 tf(t)dt = 0$.

(c) The null space is spanned by $\phi(x) = \cos jx$ if $\lambda = \frac{j}{\pi}$. Therefore, if $\lambda \neq \frac{j}{\pi}$ for $j = 1, \cdots, n$, the solution exists and is unique, while if $\lambda = \frac{j}{\pi}$ for some j, then a solution exists only if $\int_0^{2\pi} f(x)\cos jx dx = 0$.

3.3.1; $u(x) = f(x) + \lambda \int_0^{2\pi} \sum_{j=1}^{n} \frac{1}{j-n\pi} \cos jt \cos jx f(t)dt = \sin^2 x - \frac{\lambda}{2}\frac{\pi}{2-\lambda\pi}\cos 2x$, provided $\lambda \neq \frac{2}{\pi}$. For $\lambda = \frac{2}{\pi}$, the least squares solution is $u(x) = \frac{1}{2}$.

3.3.2; $u(x) = \frac{1}{3-6\lambda}P_0(x) + \frac{3}{3-2\lambda}P_1(x) + \frac{10}{15-6\lambda}P_2(x)$, provided $\lambda \neq \frac{3}{2}, \frac{5}{2}$. It is helpful to observe that $x^2 + x = \frac{1}{3}P_0(x) + P_1(x) + \frac{2}{3}P_2(x)$.

3.4.1; (a) Eigenfunctions are $\phi_n(x) = \sin n\pi x$ for $\lambda_n = \frac{1}{n^2\pi^2}$.

(b) $f(x) = \frac{1}{\pi^2}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin(2n-1)\pi x$.

3.4.2; (b) $\phi_1(x) = \sin x, \lambda_1 = \frac{\pi}{2}, \phi_2(x) = \cos x, \lambda_2 = \frac{\alpha\pi}{2}$.

(c) There are no eigenvalues or eigenfunctions. Remark: The existence of eigenfunctions is only guaranteed for self-adjoint operators.

(d) $\phi_n(x) = \sin a_n x, \lambda_n = \frac{1}{a_n^2}$, where $a_n = \frac{2n+1}{2}$.

3.4.3; $\lambda_n = \frac{8}{n^2\pi^2}$ is a double eigenvalue with $\phi_n(x) = \sin\frac{n\pi x}{2}$, $\psi_n(x) = \cos\frac{n\pi x}{2}$ for n odd.

3.4.4; $k^*(x,y) = k(y,x)\frac{w(y)}{w(x)}$.

3.5.1; (a) $u(x) = f(x) + \int_0^x e^{x-t}f(t)dt = e^x$ when $f(x) = 1$.

(b) $u(x) = f(x) + \int_0^x \sin(t-x)f(t)dt = \cos x$ when $f(x) = 1$.

(c) $u(x) = f(x) + \int_0^x \sin(x-t)f(t)dt = e^x$ when $f(x) = 1 + x$.

3.5.2; (a) $u(x) = f(x) + \frac{\lambda}{1-\lambda} + \int_0^1 f(t)dt = x + \frac{\lambda}{2(1-\lambda)}$ when $f(x) = x$, provided $\lambda \neq 1$.

(b) $u(x) = f(x) + \frac{3}{5}\int_0^1 xtf(t)dt = x$ when $f(x) = \frac{5x}{6}$.

(c) $u(x) = p(x) + a(x)\int_{x_0}^x \exp\left(-\int_x^y b(t)a(t)dt\right) b(y)p(y)dy$.

3.6.1; (a) Show that $||u - u_n|| < \frac{||\lambda K||^{n+1}||f||}{1-||\lambda K||}$.

(b) Use that $||K|| \leq \frac{1}{3}$ to conclude that $n \geq 2$. The exact solution is $u(x) = \frac{3x}{4}$, and some iterates are $u_0(x) = 0$, $u_1(x) = 1$, $u_2(x) = \frac{2x}{3}$, $u_3(x) = \frac{7x}{9}$.

3.6.2; If $f(x^*) = 0$, for convergence to x^*, require $|1 - f'(x^*)| < 1$ or $0 < f'(x^*) < 2$.

3.6.3; For convergence to x^*, require $|\frac{f(x^*)f''(x^*)}{(f'(x^*))^2}| < 1$, which if $f(x^*) = 0$, and $f'(x^*) \neq 0$ is always satisfied.

3.6.4; (a) Let $x_{n+1} = D^{-1}b - D^{-1}Rx_n$.

3.6.5; Show that $u_n(x) = \sum_{k=0}^n \frac{x^k}{k!}$.

3.7.1; $y = y(x)$ where $a^2x = \sin^{-1}\sqrt{y} + y\sqrt{1-y^2}$, $a = \frac{\pi T}{\sqrt{2g}}$.

3.7.2; (a) A straight line.

(b) $y = y(x)$ where $x = \frac{1}{2}y\sqrt{y^2-1} - \frac{1}{2}\ln(y + \sqrt{y^2-1})$, provided $y \geq 1$, which means this problem is not physically meaningful, since y cannot reach zero.

3.7.3; (a) $T(p) = \int_0^{z(p)} \frac{1}{c(z)\sqrt{1-c^2(z)p^2}}dz$.

(b) Let $y = 1/c^2(z)$, then use Abel's technique to show that $z(c) = -\frac{1}{\pi}\int_{1/c_0}^{1/c} \frac{cx(p)dp}{\sqrt{c^2p^2-1}}$.

(c) $T(x) = \frac{2}{b}\ln\sqrt{1+y^2} - y$, where $y = \frac{bx}{2a}$.

4.1.1; (a) Use that $S_k(x) = \frac{1}{2}\frac{\sin(k+\frac{1}{2})\pi x}{\sin\frac{\pi x}{2}}$ for $-1 < x < 1$, and then observe that $\frac{\sin\frac{\pi x}{2}}{\frac{\pi x}{2}}S_k(x) = (k+\frac{1}{2})\,\mathrm{sinc}((k+\frac{1}{2})x)$ is a delta sequence.

4.1.4; Observe that $\chi = \int_{-\infty}^{x} \psi(x)dx$ is a test function, $\chi(0) = 0$, so that $\chi = x\phi$ for some test function ϕ. Hence, $\psi(x) = \frac{d}{dx}(x\phi(x))$.

4.1.5; $u(x) = c_1 + c_2H(x) + c_3\delta(x)$. Show that a test function ψ is of the form $\psi = \frac{d}{dx}(x^2\phi)$ if and only if $\int_{-\infty}^{\infty} \psi dx = \int_0^{\infty} \psi dx = \psi(0) = 0$.

4.1.6; $u(x) = \delta(x) + c_1x + c_2$.

4.1.7; Set $u = xv$, so that $x^2v' = 0$, and then $u(x) = c_1x + c_2xH(x)$ (using that $x\delta(x) = 0$).

4.1.8; $u(x) = c_1 + c_2H(x)$.

4.1.9; $\delta(x^2 - a^2) = 2|a|(\delta(x - a) + \delta(x + a))$.

4.1.10; In the sense of distribution, $\chi'(x) = \delta(x) - \delta(x - 1)$, since $\langle\chi'(x)\phi(x)\rangle = -\langle\chi(x), \phi'(x)\rangle = -\int_{\infty}^{\infty} \chi(x)\phi'(x)dx = -\int_0^1 \phi'(x)dx = \phi(0) - \phi(1)$.

4.1.11; (a) For distributions f and g, define $\langle f * g, \phi\rangle = \langle g, \psi\rangle$, where $\psi(t) = \langle f(x), \phi(x + t)\rangle$.

(b) $\delta * \delta = \delta(x)$.

4.2.1; $g(x, y) = -x$ for $0 \le x \le y, g(x, y) = g(y, x)$.

4.2.2; $U(x) = x$ is a solution of the homogeneous problem. There is no Green's function.

4.2.3; $g(x, y) = \frac{\cos\alpha(\frac{1}{2} - |x-y|)}{\sin\frac{\alpha}{2}}$ provided $\alpha \ne 2n\pi$.

4.2.4; $g(x, y) = \begin{cases} -(1-y)^2x & \text{for } 0 \le x < y \le 1 \\ (2-y)yx - y & \text{for } 0 \le y < x \le 1 \end{cases}$.

4.2.5; $g(x, y) = \ln y$ for $x < y$, $g(y, x) = g(x, y)$.

4.2.6; $g(x, y) = \begin{cases} -\frac{x}{5}(3y^{5/2} + 2) & \text{for } 0 \le x < y \\ -\frac{y^{3/2}}{5}(3x + 2x^{-3/2}) & \text{for } x > y \end{cases}$.

4.2.7; $u(x) = -\frac{1}{2\alpha}\int_{-\infty}^{\infty} e^{-\alpha|x-\xi|}f(\xi)d\xi$.

4.2.8; The functions $\sin 2x$ and $\cos 2x$ are elements of the null space of the operator.

4.2.9; $u(x) = \int_0^1 g(x, y)f(y)dy - \lambda\int_0^1 g(x, y)u(y)dy + \alpha(1-x) + \beta x$, where $g(x, y) = x(y - 1)$ for $0 \le x < y \le 1, g(x, y) = g(y, x)$.

4.2.10; $u(x) = \int_0^1 g(x, y)f(y)dy - \lambda\int_0^1 g(x, y)u(y)dy$ where $g(x, y) = \frac{1}{3}(x+1)(y-2)$ for $0 \le x < y$, and $g(x, y) = g(y, x)$.

4.2.11; $u(x) = \int_0^1 g(x, y)f(y)dy - \lambda\int_0^1 g(x, y)u(y)dt$ where $g(x, y) = \frac{1}{2n}x^n(y^n - y^{-n})$ for $0 \le x < y \le 1, g(x, y) = g(y, x)$.

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$= \langle g, \psi\rangle$, where $\psi(t) =$

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4.2.12; $g(x,y) = -\frac{1}{2}e^{-|y-x|}$, $u(x) = \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{-|y-x|}u(y)dy - \int_{-\infty}^{\infty}\frac{1}{2}e^{-|y-x|}f(y)dy$.

4.2.13; $u(x) = \cos x + \lambda\int_0^x p(\xi)\sin(x-\xi)u(\xi)d\xi$.

4.2.14; $u(x) = 1 + \int_0^x \xi\ln(\frac{x}{\xi})u(\xi)d\xi$.

4.3.1; $L^*v = v'' - (a(x)v)' + b(x)v$, with boundary conditions $v(0) + v'(1) - a(1)v(1) = 0, v(1) + v'(0) - a(0)v(0) = 0$.

4.3.2; $L^*v = -(p(x)v')' + q(x)v$ with $p(0)v(0) = p(1)v(1)$, and $p(0)v'(0) = p(1)v'(1)$.

4.3.3; $L^*v = v'' - 4v' - 3v$ with $v'(0) = 0, v'(1) = 0$.

4.3.4; (a) $g^*(y,x)w(y) = \langle g^*(\xi,x), \delta(\xi - y)\rangle_\xi = \langle g^*(\xi,x), L_\xi g(\xi,y)\rangle_\xi$
$= \langle L_\xi g^*(\xi,x), g(\xi,y)\rangle_\xi = \langle \delta(\xi - x), g(\xi,y)\rangle_\xi = g(x,y)w(x)$.

(b) $u(x)w(x) = \langle \delta(\xi - x, u(\xi)\rangle_\xi = \langle L_\xi g^*(\xi,x), u(\xi)\rangle_\xi = \langle g^*(\xi,x), f(\xi)\rangle_\xi$
$= w(x)\int_a^b g(y,x)f(x)dx$.

4.3.6; This follows from Problem 4.3.2.

4.3.7; $M^*y = \frac{dy}{dt} + A^T(t)y$, with the vector $\begin{pmatrix} y(t_1) \\ y(t_2) \end{pmatrix} \in N^{\perp}([L, R])$.

4.3.8; (a) The operator is formally self-adjoint, but not self-adjoint.
(b) Require $u'(0) = u'(1) = 0$, for example.

4.3.9; Require $\int_0^{2\pi} f(x)\sin x dx = \alpha$, and $\int_0^{2\pi} f(x)\cos x dx = \beta$.

4.3.10; Require $\int_0^1 f(x)dx = -\beta$.

4.3.11; Require $\int_0^{\frac{1}{2}} f(x)\sin\pi x dx = \beta + \pi\alpha$.

4.3.12; Require $\int_0^1 f(x)dx = \alpha - \beta$.

4.4.1; $g(x,y) = yx + \frac{1}{2}(y-x) + \frac{1}{12} - \frac{1}{2}(x^2+y^2)$, for $0 \le x < y \le 1, g(x,y) = g(y,x)$.

4.4.2; $g(x,y) = (\frac{1}{2} + y^2(4y-3))(x - \frac{1}{2}) + \frac{1}{4} - \frac{y}{2} - \frac{x^2}{2} - xH(y-x)$.

4.4.3; $g(x,y) = -\frac{1}{8\pi^2}\cos 2\pi(x-y) - \frac{x-y}{2\pi}\sin 2\pi(x-y) - \frac{1}{4\pi}\sin 2\pi(y-x)$.

4.4.4; $g(x,y) = \frac{2x}{\pi}\cos x\sin y + \frac{2y}{\pi}\sin x\cos y - \frac{1}{\pi}\sin x\sin y - 2H(y-x)\sin x\cos y - 2H(x-y)\cos x\sin y$.

4.4.5; $g(x,y) = \frac{9}{5}xy - x - \frac{xy}{2}(x^2+y^2)$ for $x < y, g(x,y) = g(y,x)$.

4.4.6; $g(x,y) = \frac{1}{2}\ln(1-x) + \frac{1}{2}\ln(1+y) + \frac{1}{2}$ for $-1 \le x < y \le 1, g(x,y) = g(y,x)$.

4.4.7; $u(x) = \frac{1}{8}\cos 2x + (\beta - \alpha)\frac{x^2}{2\pi} + \alpha x - \frac{\pi^2}{3}(\alpha + \frac{\beta}{2})$.

4.4.8; $u(x) = -\frac{3}{\pi}x\cos x + \cos x + \frac{1}{32}\sin 3x - \frac{3}{2\pi}\sin x$.

4.4.9; $u(x) = 0$.

4.5.1; $u(x) = \frac{\alpha-\beta}{6} - \frac{\alpha}{2} + \alpha x + \frac{\beta-\alpha}{2}x^2 + \sum_{n=1}^{\infty}\frac{b_n}{n^2\pi^2}\cos n\pi x$, where $b_n = -2\int_0^1 (f(x)+\alpha-\beta)\cos n\pi x dx$.

4.5.2; Use Fourier series on $[0, 2\pi]$.

4.5.3; Use Fourier series on $[0, 1]$.

4.5.4; No eigenfunction expansion solution exists.

4.5.5; $u(x) = \alpha x + \beta - \alpha\pi + \sum_{n=1}^{\infty} a_n \cos(2n-1)\frac{x}{2}$, where $a_n = -\frac{8}{\pi(2n-1)^2}\frac{1-\cos(2n-1)\frac{\pi}{2}}{(2n-1)^2/2-1}$.

4.5.6; $u(x) = -\frac{c}{9}(\frac{3}{2}x^2 - \frac{1}{2}) - \frac{b}{2}x$. Solution is exact if $a + \frac{c}{3} = 0$.

4.5.7; $u(x) = -cL_2(x) + (b+4c)L_1(x)$, where $L_1(x) = 1 - x$, and $L_2(x) = \frac{1}{2}(x^2 - 4x + 2)$ are Laguerre polynomials.

4.5.8; Use Hermite polynomials: $u(x) = (-\frac{c}{5}x - \frac{bx}{3} - a - \frac{4c}{5})e^{x^2/2}$.

4.5.9; Eigenfunctions are $\phi_n(x) = \sin\frac{n\pi}{2}(x+1), \lambda = \frac{n^2\pi^2}{4}$, so $a_n = 0$ for n even, $a_n = -\frac{1}{\pi}\frac{4n}{n^2-4}$ for n odd.

4.5.11; $\phi_n^{(k)} = \cos(\frac{(n-\frac{1}{2})k\pi}{N-1}), \lambda_k = -4\sin^2\frac{k\pi}{2(N-1)}, k = 1, 2, \ldots, N-1$.

4.5.12; For $\lambda = 4\pi^2$, eigenfunctions are $1, \cos 2\pi x, \sin 2\pi x$. For $\lambda = 4\pi^2 n^2$ with $n > 1$, eigenfunctions are $\cos 2\pi nx$ and $\sin 2\pi nx$.

4.5.14; With $u(x_j) = u_j$, require $u_{j-1} - 2u_j + u_{j+1} = h^2\int_{x_{j-1}}^{x_{j+1}}\phi_j(x)f(x)dx$.

5.1.1; (a) $(\frac{y'}{x^2})' = 0$.
(b) $y' - y = -\cos x$.
(c) $y' - y = -e^x$.

5.1.2; $y(x) = \frac{1}{2}(x^2 - 3x + 1)$.

5.1.4; Find the increase in energy due to rotation and determine how this affects the moving velocity. Then determine how this affects the fastest path. Show that the rotational kinetic energy of a rolling bead is 2/5 that of the its translational kinetic energy. Minimize $T = \int_0^{x_1}\sqrt{\frac{7(1+y'^2)}{10g(y_0-y)}}dx$.

5.1.6; Maximize $\int_0^1 y(x)dx$ subject to $\int_0^1\sqrt{1+y'^2}dx = l - a - b$, $y(0) = a, y(1) = b$.

5.1.7; The arc of a circle.

$1x.$

$_x \cos n\pi x$, where

$\frac{x}{2}$,

t if $a + \frac{c}{3} = 0$.

$L_1(x) = 1 - x$, and $L_2(x) =$

$\frac{bx}{3} - a - \frac{4c}{5})e^{x^2/2}$.

$= \frac{n^2\pi^2}{4}$, so $a_n = 0$ for n even,

$= 1, 2, \ldots, N-1$.

, $\sin 2\pi x$. For $\lambda = 4\pi^2 n^2$ with $2\pi n x$.

$_1 = h^2 \int_{x_{j-1}}^{x_{j+1}} \phi_j(x) f(x) dx$.

and determine how this affects w this affects the fastest path. a rolling bead is 2/5 that of the $T = \int_0^{x_1} \sqrt{\frac{7(1+y'^2)}{10g(y_0-y)}} dx$.

$^2 dx = l - a - b$, $y(0) = a, y(1) =$

5.1.9; The Euler-Lagrange equations are $y'' = z, z'' = y$.

5.1.10; The Euler-Lagrange equation is $\frac{d^4y}{dx^4} - y = 0$.

5.1.12; $y(x) = 2 \sin k\pi x$.

5.2.1; $u_x = 0$ at $x = 0, l$.

5.2.2; $u_{xx} = 0$ and $\mu_1 u_{xxx} - \mu_2 u_x = 0$ at $x = 0, 1$.

5.2.3; If u is the vertical displacement of the string, then $T = \frac{1}{2}\int_0^l \rho u_t^2 dx + m u_t^2(0,t) + m u_t^2(l,t)$, $U = \frac{\mu}{2}\int_0^l (\sqrt{(1+u_x^2)} - 1) dx + \frac{k}{2}u^2(0,t) + \frac{k}{2}u^2(l,t)$. Then, require $\rho u_{tt} = \mu \frac{\partial}{\partial x}\frac{u_x}{\sqrt{1+u_x^2}}$ subject to the boundary conditions $m u_{tt} + ku = \mu \frac{u_x}{\sqrt{1+u_x^2}}$ at $x = 0$, and $m u_{tt} + ku = -\mu \frac{u_x}{\sqrt{1+u_x^2}}$ at $x = l$.

5.2.4; $\rho u_{tt} = \mu u_{xx}$ on $0 < x < \frac{l}{2}$ and $\frac{l}{2} < x < 1$, and $m u_{tt} = \mu u_x(\frac{x}{2}, t)|_{x=l^-}^{x=l^+}$.

5.2.5; Use that $T = \frac{1}{2} m l^2 \dot\theta^2 + \frac{m}{2}\Omega^2 l^2 \sin^2\theta$, $U = mgl(1 - \cos\theta)$.

5.2.6; $H = \frac{1}{2}m\dot x^2 + \frac{1}{2}k_1 x^2 + \frac{1}{4}k_2 x^4$. Hamilton's equations are $\dot p = -k_1 q - k_2 q^3, \dot q = \frac{p}{m}$, where $p = m\dot x, q = x$.

5.2.7; $T = \frac{1}{2}m_1 l_1 \dot\theta_1^2 + \frac{1}{2}(l_1^2\dot\theta_1^2 + 2 l_1 l_2 \cos(\theta_1 - \theta_2)\dot\theta_1\dot\theta_2 + l_2^2\dot\theta_2^2), U = m_1 l_1 g(1 - \cos\theta_1) + m_2 g(l_1(1 - \cos\theta_1) + l_2(1 - \cos\theta_2))$.

5.2.9; Set $x = p + \cos p, y = \sin p$, and then $T = m\dot p^2(1 - \sin p), U = mg \sin p$, so that $2\ddot p(1 - \sin p) - \dot p^2 \cos p - g \cos p = 0$.

5.3.1; (a) $u(x) = a(1 + \frac{72}{83}x + \frac{70}{83}x^2)$.

(b) $u(x) = a(\frac{249}{250} + \frac{108}{125}x + \frac{21}{25}x^2)$.

5.3.2; (a) $u(x) = a(1 + \frac{2250}{2221}x + \frac{945}{2221}x^2 + \frac{2485}{8884}x^3) = a(1.0 + 1.013x + 0.4255x^2 + 0.2797x^3)$.

(b) $u(x) = a(\frac{35536}{35537} + \frac{36000}{35537}x + \frac{15120}{35537}x^2 + \frac{9940}{35537}x^3) = a(0.99997 + 1.013x + 0.4255x^2 + 0.2797x^3)$.

5.3.3; $u' = \begin{cases} \frac{246}{199}, & 0 < x < \frac{1}{2} \\ \frac{414}{199}, & \frac{1}{2} < x < 1 \end{cases}$.

5.3.4; (a) $u(x) = 1 - x - \frac{65}{282}x(1-x) = 1 - x - 0.230x(1-x)$.

(b) $u(x) = 1 - x - \frac{5}{22}x(1-x) = 1 - x - 0.227x(1-x)$. The exact solution is $u(x) = -\frac{\sinh(x-1)}{\sinh(1)}$.

5.3.5; Using (non-orthogonal) basis functions $\phi_1(x) = x, \phi_2(x) = x^2, \phi_3(x) = x^3$, $u(x) = a(1 + \frac{30}{29}\phi_1(x) + \frac{45}{116}\phi_2(x) + \frac{35}{116}\phi_3(x))$. The Sobolev inner product does not give a useful answer.

5.4.1; Use the functional $D(\phi) = \int_0^1 p(x)\phi'^2(x)dx + \sigma p(1)\phi^2(1)$, $\sigma \geq 0$, subject to $H(\phi) = \int_0^1 w(x)\phi^2(x)dx = 1$ and

(a) $\phi(0) = \phi(1) = 0$,

(b) $\phi(0) = 0$,

(c) $\phi(0) = 0$, and $\sigma = 0$.

Show that $\lambda^{(a)} \geq \lambda^{(b)} \geq \lambda^{(c)}$.

5.4.2; Minimize $\int_0^1 (x\phi'^2(x) + \frac{n^2}{x}\phi^2(x))dx$, subject to $\int_0^1 x\phi(x)dx = 1$, with $\phi(0) = \phi(1) = 0$, or $\phi(0) = 0$ with $\phi(1)$ unspecified.

5.4.3; Minimize $D(\phi) = \int_0^1 \phi'^2(x)dx$ subject to $\int_0^1 \phi^2(x)dx = 1$. Use $\phi(x) = \sqrt{30}x(1-x)$ to find $D(\phi) = 10 \geq \pi^2$.

5.4.4; The first eigenvalue is approximated by 10, from Problem 5.4.3. The second eigenfunction is approximated by $\phi_2(x) = \sqrt{840}x(x-1)(x-\frac{1}{2})$, and $D(\phi_2) = 42$. The exact values are $\lambda_1 = \pi^2$, $\lambda_2 = 4\pi^2$.

6.1.1; (a) $f(-3) = -i\sqrt{84}, f(\frac{1}{2}) = -\sqrt{\frac{7}{8}}, f(5) = -\sqrt{20}$.

(b) There are square root branch points at $z = \pm 1$ and there is a logarithmic branch point on the negative branch of the square root at $z = \frac{13}{12}$.

6.1.2; $(e^{2\pi i})^z$ is not single valued.

6.1.3; (a) $z = \frac{\pi}{2} + 2n\pi - i\ln(2 \pm \sqrt{3})$.

(b) $z = (2n+1)\pi - i\ln(\sqrt{2}+1), z = 2n\pi - i\ln(\sqrt{2}-1)$.

(c) No such values exist.

6.1.4; $i^i = e^{-(\pi/2+2n\pi)}$ for all integer n; $\ln(1+i)^{i\pi} = -\pi^2(\frac{1}{4}+2n) + \frac{i\pi}{2}\ln 2$; arctanh 1 has no value.

6.1.6; It is not true. For example, consider $z_1 = e^{\pi i/2}, z_2 = e^{3\pi i/2}$.

6.1.7; The two regions are $|z| < 1$ and $|z| > 1$; There are branch points at $w = \pm 1$.

6.2.1; $f(z) = \frac{15-8i}{4(z-2)^2(z-\frac{i}{2})}$.

6.2.2; $\int_C f(z)dz = -2\pi\sqrt{19}(15)^{1/3}e^{-i\pi/3}$.

6.2.3; The integral is independent of path. $\int_C z^{-1/3}dz = \frac{3}{2}z^{2/3}|_{1+i}^{1-i} = -3(2)^{1/3}e^{i\pi/}$

6.2.4; Find the real part of $f(z) = z^{1/2}$ and use that f is an analytic function.

6.2.5; (a) $|z-i| < \sqrt{2}, |z-i| > \sqrt{2}$.

(b) $|z-i| < 2, |z-i| > 2$.

$p(1)\phi^2(1)$, $\sigma \geq 0$, subject

o $\int_0^1 x\phi(x)dx = 1$, with
fied.

$^2(x)dx = 1$. Use $\phi(x) =$

rom Problem 5.4.3. The
$) = \sqrt{840}x(x-1)(x-\frac{1}{2})$,
, $\lambda_2 = 4\pi^2$.

$\sqrt{20}$.

$z = \pm 1$ and there is a log-
anch of the square root at

$\ln(\sqrt{2}-1)$.

$= -\pi^2(\frac{1}{4}+2n) + \frac{i\pi}{2}\ln 2$;

$^2, z_2 = e^{3\pi i/2}$.

here are branch points at

$z = \frac{3}{2}z^{2/3}|_{1+i}^{1-i} = -3(2)^{1/3}e^{i\pi/}$

; f is an analytic function.

(c) $|z-i| < 2, |z-i| > 2$.

(d) $|z-i| < 1, 1 < |z-i| < \sqrt{2}, |z-i| > \sqrt{2}$.

(e) $|z-i| < 1, 1 < |z-i| < \sqrt{2}, \sqrt{2} < |z-i| < 2, |z-i| > \sqrt{2}$.

6.2.6; $\int_{|z|=1/2} \frac{z+1}{z^2+z+1}dz = 0$.

6.2.7; $\int_{|z|=1/2} \exp[z^2 \ln(1+z)]dz = 0$ (There is a branch point at $z = -1$).

6.2.8; $\int_{|z|=1/2} \arcsin z dz = 0$ (There are branch points at $z = \pm 1$).

6.2.9; $\int_{|z|=1} \frac{\sin z}{2z+i}dz = \pi \sinh \frac{1}{2}$.

6.2.10; $\int_{|z|=1} \frac{\ln(z+2)}{z+2}dz = 0$.

6.2.11; $\int_{|z|=1} \cot z dz = 2\pi i$.

6.2.13; Hint: Use the transformation $z = \xi^\rho$ where $\rho = \frac{1}{\alpha}$ and apply the Phragmén-Lindelof theorem to $g(\xi) = f(z)$.

6.2.14; The function $G(z) = F(z)e^{i(\sigma+\epsilon)z}$ satisfies $|G(iy)| \leq A$ and $|F(x)| \leq \sup_{-\infty<x<\infty}|G(x)|$. Apply problem 6.2.13.

6.3.4; $\kappa = \frac{R-r_1}{r_2-R}$.

6.3.5; $\phi + i\psi = a - \frac{2}{3}(b-a) - \frac{4i(b-a)}{3\pi}\ln(\frac{z-1}{z+1})$

6.3.6; (b) $w(z) = Az^2 + (\frac{1}{z-i} - i)^2$, $F_x - iF_y = -8\rho\pi i A$.

(c) $F_x - iF_y = \rho\pi(4\gamma A - 8A^2 i)$.

6.3.7; The upper half ξ plane.

6.3.8; Flow around a corner with angle $\theta = \beta\pi$. This makes sense only for $\beta < 2$.

6.3.9; $F_x - iF_y = -2\pi\rho U^2 a \sin^2 \alpha e^{i\alpha}$.

6.3.11; $f(z) = U\sqrt{z^2+a^2}$.

6.3.12; Show that $\frac{dz}{d\omega} = \frac{1}{U}\left(e^{-\omega/2k} + \sqrt{1+e^{-\omega/k}}\right)$.

6.4.1; $\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \frac{2\pi}{\sqrt{4ac-b^2}}$.

6.4.2; $\int_0^{\infty} \frac{x \sin x}{a^2+x^2}dx = \frac{\pi}{2}e^{-|a|}$.

6.4.3; $\int_0^{\infty} \frac{dx}{1+x^k} = \frac{\pi}{k \sin \frac{\pi}{k}}$.

6.4.4; $\int_0^{\infty} \frac{dx}{(x+1)x^p} = \frac{\pi}{\sin \pi p}$.

6.4.5; $\int_1^{\infty} \frac{x}{(x^2+4)\sqrt{x^2-1}}dx = \frac{\pi}{2\sqrt{5}}$.

6.4.6; $\int_0^\infty \frac{\sqrt{x}}{x^2+1}dx = \frac{\pi}{\sqrt{2}}$.

6.4.7; $\int_{-\infty}^\infty \frac{e^{ikx}}{a+ix}dx = 2\pi H(ka)e^{-ka}$.

6.4.8; Consider $\int_C \frac{z\,dz}{a-e^{iz}}$ on the rectangular contour with corners at $z = \pm\pi$ and $z = \pm\pi - iR$, and let $R \to \infty$. Show that $\int_{-\pi}^{\pi} \frac{x\sin x}{a^2-2a\cos x+1}dx = \frac{2\pi i}{a}(H(a-1)\ln a - \ln(a+1))$.

6.4.9; $\int_{-\infty}^\infty \frac{e^{i\omega x}}{\cosh x}dx = \frac{\pi}{\cosh\frac{\omega\pi}{2}}$.

6.4.11; $\int_0^\infty \frac{dx}{x^2+x+2} = \frac{1}{4}\ln\sqrt{2} - \frac{3}{4\sqrt{7}}(\arctan\sqrt{7} - \pi)$.

6.4.12; $\int_0^{2\pi}\ln(a+b\cos\theta)d\theta = 2\pi\ln(\frac{a\pm\sqrt{a^2-b^2}}{2})$. Hint: Differentiate the integral with respect to b, and evaluate the derivative.)

6.4.13; $\int_0^\infty \frac{\sin\alpha x}{\sinh\pi x}dx = \frac{1}{2}\tanh\frac{\alpha}{2}$.

6.4.14; $\int_0^\pi \ln(\sin x)dx = -\pi\ln 2$.

6.4.15; $\int_{-\infty}^\infty \frac{e^{i\omega x}}{\cosh^3 x}dx = \frac{\pi(1+\omega^2)}{2\cosh\frac{\pi\omega}{2}}$.

6.4.16; $\int_{-\infty}^\infty \frac{e^{i\omega x}\sinh x}{\cosh^3 x}dx = \frac{i\pi\omega}{2\sinh\frac{\pi\omega}{2}}$.

6.4.17; $\int_0^\infty \frac{\ln(1+x^2)}{1+x^2}dx = \pi\ln 2$.

6.4.18; $\int_{-\infty}^\infty \frac{x^2}{\cosh^2 x}dx = \frac{\pi^2}{6}$.

6.4.20; Evaluate $\int_C \frac{dz}{P_n(z)}$ on some contour C that contains all the roots.

6.4.22; The change of variables $\rho^2 = r^2\cos^2\theta + z^2\sin^2\theta$ converts this to an integral for which complex variable techniques work nicely.

6.4.23; $\int_0^\infty t^{-1/2}e^{i\mu t}dt = \sqrt{\frac{\pi}{|\mu|}}\exp(i\frac{\pi}{4}\sqrt{\frac{\mu}{|\mu|}})$.

6.4.24; $\int_0^\infty t^{1/2}e^{i\mu t}dt = \frac{\sqrt{\pi}e^{3\pi i/4}}{2\mu^{3/2}}$.

6.4.27; $\int_{-\infty}^\infty \frac{e^{i\mu z}}{(z+i\alpha)^\beta}dz = 2e^{\alpha\mu}e^{-\pi i\beta/2}\sin\frac{\pi\beta}{2}|\mu|^{\beta-1}\Gamma(1-\beta)H(-\mu)$.

6.4.28; Use that $\frac{1}{\sqrt{z+i\alpha}+\sqrt{z+i\beta}} = \frac{\sqrt{z+i\alpha}-\sqrt{z+i\beta}}{i(\alpha-\beta)}$, but be careful to apply Jordan's lemma correctly. Then, $\int_{-\infty}^\infty \frac{e^{i\mu z}dz}{\sqrt{z+i\alpha}+\sqrt{z+i\beta}} = \frac{\sqrt{\pi}}{\mu^{3/2}}e^{-i\pi/4}\frac{e^{-\mu\alpha}-e^{-\mu\beta}}{\alpha-\beta}H(\mu)$.

6.5.4; Examine the expression $\int_0^\eta \frac{T(x)dx}{(\eta-x)^{1-\alpha}}$, make the change of variables $x = y\cos^2\theta + \eta\sin^2\theta$ and then use the beta function to show that $f(y) = \frac{\sin\alpha\pi}{\pi}\frac{d}{dy}\int_0^y \frac{T(x)}{(y-x)^{1-\alpha}}dx$. For another way to solve this problem, see Problem 7.3.4.

6.5.5; (a) $W(J_\nu, Y_\nu) = \frac{2}{\pi z}$.

(b) $W(J_\nu, H_\nu^{(1)}) = \frac{2i}{\pi z}$.

(c) $W(H_\nu^{(1)}, H_\nu^{(2)}) = -\frac{4i}{\pi z}$.

6.5.6; (b) $Y_n(z) = -\frac{2^n}{n\pi}\frac{\Gamma(n+1)}{z^n} +$ higher order terms.

6.5.8; Use that $\sum_{n=-\infty}^{\infty} J_n(z)t^n = e^{(t-1/t)z/2}$.

6.5.13; $\sum_{n=0}^{\infty} s^{2n} \int_{-1}^{1} P_n^2(t)dt = \int_{-1}^{1} \phi^2(z,s)dz = -\frac{1}{s}\ln(\frac{1-s}{1+s}) = 2\sum_{n=0}^{\infty} \frac{s^{2n}}{2n+1}$.

6.5.23; Period $= \sqrt{\frac{2l}{g}} B(\frac{1}{2}, \frac{1}{4}) = \sqrt{\frac{2l}{g}} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}$.

7.1.2; (a) λ with $|\lambda| < 1$ is residual spectrum, with $|\lambda| = 1$ is continuous spectrum, and with $|\lambda| > 1$ is resolvent spectrum.

(b) Notice that $L_2 = L_1^*$. Then, λ with $|\lambda| < 1$ is residual spectrum, with $|\lambda| = 1$ is continuous spectrum, and with $|\lambda| > 1$ is resolvent spectrum.

(c) $\lambda_n = \frac{1}{n}$ for positive integers n are point spectrum, there is no residual spectrum since L_3 is self adjoint, and $\lambda \neq \frac{1}{n}$ is resolvent spectrum, $\lambda = 0$ is continuous spectrum.

7.1.3; Use improper eigenfunctions to show $(L_4 - \lambda)^{-1}$ is unbounded. Show that $\{x_n\}$ with $x_n = \sin n\theta$ is an improper eigenfunction.

7.1.4; Show that $\phi(x) = \sin \mu x$ is an eigenfunction for all μ. Notice that the operator is not self-adjoint.

7.2.1; (a) $\delta(x-\xi) = 2\sum_{n=1}^{\infty} \sin(\frac{2n-1}{2}\pi x)\sin(\frac{2n-1}{2}\pi\xi)$.

(b) $\delta(x-\xi) = \frac{2}{\pi}\int_0^{\infty} \cos kx \cos k\xi dk$.

(c) $\delta(x-\xi) = \frac{2}{\pi}\int_0^{\infty} \sin k(x+\phi)\sin k(\xi+\phi)dk$ where $\tan\phi = \frac{k}{\alpha}$.

7.2.6; (a) $\frac{2a}{\mu^2+a^2}$.

(b) $\frac{1}{1-i\mu}$.

(c) $\sqrt{\frac{\pi}{a}} e^{\frac{-\mu^2}{4a}}$.

7.2.7; $H(\mu+\pi) - H(\mu-\pi)$.

7.2.8; $\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+i\mu}(i\mu F(\mu))e^{-i\mu x}d\mu = \int_x^{\infty} e^{x-s}f'(s)ds$.

7.2.9; Use the convolution theorem to find $u(x) = f(x) - \frac{4}{3}\int_{-\infty}^{\infty} f(t)e^{-3|x-t|}dt$.

7.2.10; $\int_{-\infty}^{\infty} f(\xi)\overline{f}(x-\xi)d\xi$.

7.2.11; (a) $E_x E_\mu = \frac{1}{4}$ which is optimal.

(b) $E_x E_\mu = \frac{\pi^2}{36} = 0.274$.

(c) $E_x E_\mu = \frac{1}{2}$.

7.2.12; (a) $\int_{-\infty}^{\infty} e^{i\mu x} J_0(ax)dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{-\infty}^{\infty} e^{ix(\mu + a\sin\theta)}dx\right)d\theta$
$= \int_0^{2\pi}\delta(\mu + a\sin\theta)d\theta = \frac{2}{\sqrt{a^2-\mu^2}}$ if $|\mu| < |a|$ and $= 0$ otherwise.

(b) Take the inverse Fourier transform of the answer to part (a).

7.2.15; Show that $N_6(1) = 1/120$, $N_6(2) = 13/60$, and $N_6(3) = 11/20$. Then $E_3(z) = \frac{1}{120}z^2 + \frac{13}{60}z + \frac{11}{20} + \frac{13}{60z} + \frac{1}{120z^2}$.

7.3.2; (a) $s^{-(\alpha+1)}\Gamma(\alpha+1)$.

(b) $\frac{1}{a-s}$ provided Re $s > a$.

7.3.3; $u(x) = \int_0^x f(y)K(x-y)dy$ where $K = L^{-1}(\frac{1}{1+L(k(x))})$.

7.3.4; $f(t) = \frac{\sin\pi\alpha}{\pi}\frac{d}{dt}\int_0^t \tau^{\alpha-1}T(t-\tau)d\tau$.

7.3.6; (a) $M[H(x) - H(x-1)] = \frac{1}{s}$.

(b) $M[(1+x)^{-1}] = \frac{\pi}{\sin\pi s}$.

(c) $M[e^{-x}] = \Gamma(s)$.

(d) $M = \frac{1}{1-s}$.

(e) $M[e^{ix}] = i^s\Gamma(s)$.

(f) $M[\cos x] = \frac{1}{2}\cos\pi s\Gamma(s)$.

7.3.7; $F(\mu) = \int_0^\infty rf(r)\sin\mu r dr, rf(r) = \frac{2}{\pi}\int_0^\infty F(\mu)\sin\mu r d\mu$.

7.3.8; (a) Show that $\int_0^\infty J_0(x)dx = 1$, and then $G(\rho) = \frac{1}{\rho}$.

(b) $G(\rho) = \frac{1}{\rho}\sin a\rho$.

(c) $G(\rho) = \sinh^{-1}\frac{1}{\rho}$.

7.4.2; Let $u_n = \sum_j g_{nj}f_j$ where $g_{nj} = 0, n \le j, g_{nj} = \frac{\mu}{\mu^2-1}(\mu^{n-j} - \mu^{j-n})$ where $\mu^2 - \lambda\mu + 1 = 0$.

7.5.1; $u_1(x) = \begin{cases} \cos x, x > 0 \\ \cosh x, x < 0 \end{cases}$, $u_2(x) = \begin{cases} \sin x, x > 0 \\ \sinh x, x < 0 \end{cases}$.

7.5.3; (a) Eigenvalues are $\lambda = -\mu^2$ where $\tanh\mu = -\frac{\mu}{A+\mu}, \mu > 0$.

(b) $\lambda^2 = A - \mu$ where $\tan a\sqrt{A-\mu} = \frac{\mu}{\sqrt{A-\mu}}$, which has positive solutions if and only if $Aa^2 > \frac{\pi^2}{4}$.

(d) $c_{11} = 1 + \frac{2ik(\alpha+\beta)+\alpha\beta(e^{-2ika}-1)}{4k^2}$, $c_{12} = -e^{2ika}\frac{(2i\beta k-\beta\alpha)+(2i\alpha k+\beta\alpha)}{4k^2}$. $\lambda^2 = -\mu$ where $\tanh a\mu = -\frac{\mu(\alpha+\beta+2\mu)}{(\alpha+\mu)(\beta+\mu)+\mu^2}$.

7.5.5; (a) $R_L = R_R e^{-2ik\phi}$.

(b) $R_R^{(2)} = R_R^{(1)} e^{-2ik\phi}$.

7.5.6; $R = -e^{2ik_1 a}\frac{k_2\cos k_2 a + ik_1 \sin k_2 a}{k_2 \cos k_2 a - ik_1 \sin k_2 a}$ where $k_i = \frac{\omega}{c_i}$.

7.5.7; The general solution is $u(x) = \alpha e^{ikx}(k^2 + 1 + 3ik\tanh x - 3\tanh^2 x) + \beta e^{-ikx}(k^2 + 1 - 3ik\tanh x - 3\tanh^2 x)$. $T_r(k) = \frac{(ik-2)(ik-1)}{(ik+2)(ik+1)}$. The bound states are $\phi_1(x) = \tanh x \operatorname{sech} x$ for $k = i$, and $\phi_2(x) = \operatorname{sech}^2 x$ for $k = 2i$.

7.5.12; $R = -e^{-2ika}\frac{(ik+\tanh a)(ik-1)}{(ik-\tanh a)(ik+1)}$. There is one bound state having $\phi(x) = (\tanh a - \tanh x)e^{\tanh a(x-a)}$ at $k = -i\tanh a$ if $a < 0$.

7.5.13; Use (7.37) to show that $c_{11} = \frac{1}{2k(k-i)}$, $c_{12} = c_{21} = \frac{2k^2+1}{2k(k+i)}$, $c_{22} = -\frac{1}{2k(k+i)}$. There is a bound state having $\phi(x) = \begin{cases} e^{x/\sqrt{2}}, & x < 0 \\ e^{-x/\sqrt{2}}(1+\sqrt{2}\tanh x), & x > 0 \end{cases}$ at $k = \frac{i}{\sqrt{2}}$.

8.1.2; (a) $V_n(x) = a_n\frac{\sinh\frac{n\pi x}{b}}{\sinh\frac{n\pi a}{b}} + b_n\frac{\sinh\frac{n\pi(x-a)}{b}}{\sinh\frac{n\pi a}{b}}$, where $a_n = -\frac{2}{b}\int_0^b \sin\frac{n\pi y}{b} g(y)dy$, $b_n = \frac{2}{b}\int_0^b \sin\frac{n\pi y}{b} f(y)dy$.

(b) $U_n(y) = \frac{a\sinh(\frac{n\pi}{a}(y-b))}{n\pi\sinh\frac{n\pi b}{a}}\int_0^y \sinh\frac{n\pi\xi}{a} f_n(\xi)d\xi + \frac{a\sinh\frac{n\pi y}{a}}{n\pi\sinh\frac{n\pi b}{a}}\int_y^b \sinh(\frac{n\pi}{a}(\xi - b))F(\xi)d\xi$, where $F_n(\xi) = \frac{n\pi}{a}(f(\xi) - (-1)^n g(\xi))$. In the special case $f(y) = y(b-y)$, $g(y) = y^2(y-b)$, $a_n = -\frac{4b^3}{n^3\pi^3}(1 + 2(-1)^n))$, $b_n = \frac{4b^3}{n^3\pi^3}(1-(-1)^n)$. The first representation converges much faster than the second and gives a good representation of the solution with only a few terms.

8.1.3; (a) Require $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\phi_t + c\phi_x)dxdt = 0$ for all test functions $\phi(x,t)$.

(b) If $u = f(x - ct)$, make the change of variables $\xi = x + ct, \eta = x - ct$ to find $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\phi_t + c\phi_x)dxdt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\xi)\phi_\eta d\xi d\eta = 0$ since $\int_{-\infty}^{\infty}\phi_\eta d\eta = 0$.

8.1.4; $u(r,\theta) = \frac{3}{4}r\sin\theta - \frac{1}{4}r^3\sin 3\theta$.

8.1.5; $G(z, z_0) = -\frac{1}{2\pi}\ln\left|\frac{z-z_0}{zz_0-1}\right|$.

8.1.6; $u(r,\theta) = \sum_{n\neq 0} a_n \frac{r^{|n|}}{|n|R^{|n|-1}}e^{in\theta}$, where $a_n = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)e^{-in\theta}d\theta$. Require $\int_0^{2\pi} g(\theta)d\theta = 0$.

8.1.8; $u(x,y) = \sum_{n=1}^{\infty}\left(a_n\frac{\cosh\frac{n\pi x}{b}}{\sinh\frac{n\pi a}{b}} + b_n\frac{\cosh\frac{n\pi(x-a)}{b}}{\sinh\frac{n\pi a}{b}}\right)\cos\frac{n\pi y}{b}$, where
$a_n = -\frac{10b^7}{n^7\pi^7}(120(-1)^n - 60n^2\pi^2 + n^4\pi^4 + 600)$,
$b_n = \frac{10b^7}{n^7\pi^7}(120 + 600(-1)^n - 60n^2\pi^2(-1)^n + n^4\pi^4)$.

8.1.9; $G(x, y; \xi, \eta) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin \frac{n\pi y}{a} \sin \frac{n\pi\eta}{a} e^{-\frac{n\pi}{a}|x-\xi|}$.

8.1.10; (a) $G(x, y, x_0, y_0) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ik(x - x_0))\frac{\cosh k(y_0-a)\sinh ky}{a\cosh ka}dk$ for $y > y_0$. (Use the Fourier transform in x.)

(b) Using Fourier series in y, $G(x, y, x_0, y_0) = \sum_{n=1}^{\infty} \frac{1}{a\lambda_n} \exp(-\lambda_n|x - x_0|)\sin\lambda_n y \sin\lambda_n y_0$ where $\lambda_n = \frac{2n+1}{2}\frac{\pi}{a}$.

8.1.11; $u(r, \theta) = \sum_{n=-\infty}^{\infty} \frac{a^n b^n}{b^{2n}-a^{2n}} a_n(\frac{r}{a})^n - (\frac{a}{r})^n e^{in\theta} - b_n(\frac{r}{b})^n - (\frac{b}{r})^n e^{in\theta}$, where $a_n = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)d\theta$, $b_n = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta$.

8.1.12; If $f(\theta) = \sum_{n=0}^{\infty} a_n \cos n(\theta - \phi_n)$, then $u(r, \theta) = \sum_{n=0}^{\infty} a_n(\frac{a}{r})^n \cos n(\theta - \phi_n) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)d\theta + \frac{1}{\pi}\sum_{n=1}^{\infty}(\frac{a}{r})^n \int_0^{2\pi} f(\phi)\cos n(\theta - \phi)d\phi$. This infinite sum can be summed by converting the cosine to complex exponentials and using geometric series.

8.1.13; $u(r, \theta) = -\frac{2}{\pi}\int_0^{\infty} F(\mu)\frac{\sinh\mu(\theta-\alpha)}{\sinh\mu\alpha}\sin(\mu \ln \frac{r}{R})d\mu$, where $F(\mu) = \int_0^{\infty} f(Re^{-t})\sin\mu t dt$.

8.1.14; $u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n \left(\frac{r}{R}\right)^{\frac{n\pi}{\alpha}} \sin \frac{n\pi\theta}{\alpha}$, where $a_n = \frac{2}{\alpha}\int_0^{\alpha} f(\theta)\sin\frac{n\pi\theta}{\alpha}d\theta$.

8.1.16; Hint: Evaluate $\sum_{j=1}^{\infty} r^j \cos j\phi$ using Euler's formula $\cos j\phi = \frac{1}{2}(e^{ij\phi} + e^{-ij\phi})$ and geometric series.

8.1.17; Using Mellin transforms,
$u(r, \theta) = \frac{1}{2\pi}\int_0^{\infty} e^{-ik\ln r}\frac{1}{k\cosh k\pi}(G(k)\sinh k\theta - kF(k)\cosh k(\theta - \pi))\,dk$
where $F(k) = \int_0^{\infty} f(r)e^{ik\ln r}\frac{dr}{r}$, $G(k) = \int_0^{\infty} g(-r)e^{ik\ln r}dr$.

8.1.18; $u(r, \theta) = \sum_{n=-\infty}^{\infty} \frac{I_n(\alpha r)}{I_n(\alpha R)}e^{in\theta}$, where $a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta}f(\theta)d\theta$, and $I_n(x)$ is the modified Bessel function of first kind.

8.1.20; $u = \frac{1}{\alpha^2+\beta^2}(\alpha f - \beta H(f))$.

8.1.21; $u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n \frac{|n|}{\alpha+|n|\beta} r^{|n|}e^{in\theta}$, where $a_n = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)e^{-in\theta}d\theta$.

8.1.22; $-v(\xi) = \int_\Omega \mathbf{n}\cdot\nabla_\xi G(\mathbf{x}, \xi)f(\mathbf{x})dV_\mathbf{x} - \int_{\partial\Omega} \mathbf{n}\cdot\nabla_\xi G(\mathbf{x}, \xi)v(\mathbf{x})dS_\mathbf{x}$, where $v = \frac{\partial u}{\partial n}$.

8.1.24; Eigenfunctions are $\phi_{nm}(x, y) = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$, with eigenvalues $\lambda_{nm} = -\pi^2(\frac{n^2}{a^2} + \frac{m^2}{b^2})$.

8.1.25; Eigenfunctions are $J_n(\mu_{nk}\frac{r}{R})\sin n\theta$ for $n > 0$. Thus, eigenvalues are the same as for the full circle, with $n = 0$ excluded.

8.1.26; Eigenfunctions are $\phi_{nmk} = J_n(\mu_{nk}\frac{r}{R})e^{in\theta}\sin\frac{m\pi z}{a}$ with eigenvalues $\lambda = -(\frac{\mu_{nk}^2}{R^2} + \frac{m^2}{a^2})$.

8.1.27; Eigenfunctions are $\phi_{nm}(\phi, \theta) = P_m^n(\cos\theta)\sin n\phi$ (or $\cos n\phi$) with $\lambda_{nm} = \frac{1}{R^2}m(m + 1)$.

8.1.29; Eigenfunctions are $u(r,\theta,\phi) = \frac{1}{\sqrt{r}} J_{m+1/2}(\mu_{mk}\frac{r}{R}) P_m^n(\cos\theta)\cos n\phi$ with eigenvalues $\lambda_{mk} = (\frac{\mu_{mk}}{R})^2$, where $J_{m+1/2}(\mu_{mk}) = 0$ for $m \geq n$. Note that $\mu_{01} = \pi, \mu_{11} = 4.493, \mu_{21} = 5.763, \mu_{02} = 2\pi, \mu_{31} = 6.988$, etc.

8.2.1; Require $h'(x) - \alpha h(x) = f'(x) + \alpha f(x)$.

8.2.2; $p_{tt} + (\lambda^+ + \lambda^-)p_t = c(\lambda^+ - \lambda^-)p_x + c^2 p_{xx}$.

8.2.4; $G = \frac{1}{2}H(t - \tau - |x - \xi|) + \frac{1}{2}H(t - \tau - |x + \xi|)$.

8.2.7; With appropriately scaled space and time variables, $\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = \sin\phi$.

8.2.8; $u(x,t) = \sum_{n=0}^{\infty}(a_n \sin\frac{cn\pi t}{L} + b_n \cos\frac{cn\pi t}{L})\sin\frac{n\pi x}{L}$ where

(a) $b_n = \frac{2L}{n^2\pi^2}\sin\frac{n\pi}{2}$, $a_n = 0$,

(b) $a_n = \frac{2L}{n^3\pi^3}\sin\frac{n\pi}{2}$, $b_n = 0$,

(c) $b_n = \frac{200}{9n^2\pi^2}\sin\frac{n\pi}{10}$, $a_n = 0$,

(d) $a_n = -\frac{20L}{cn^2\pi^2}(\cos\frac{n\pi}{5} - \cos\frac{n\pi}{10})$, $b_n = 0$.

8.2.9; For a rectangle with sides a and b, $\omega = \frac{\lambda}{2\pi} = \frac{1}{2}\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}$. Set $A = ab$, and find that the minimum is at $a = \sqrt{A}$.

8.2.10; For a square of side L, the fundamental eigenvalue is $\lambda = \sqrt{2}\frac{\pi}{L}$, whereas for a circle of radius R the fundamental eigenvalue is $\lambda = \frac{2.40482}{R}$. Take $\pi R^2 = L^2$ and use that $\frac{2\pi\omega}{c} = \lambda$. The fundamental frequency for the circle is smaller than for the square, $\lambda_{11}^{circle} = 0.959\lambda_{11}^{square}$.

8.2.12; $G(r) = -\frac{i}{4}H_0^{(1)}(r)$ is outgoing as $r \to \infty$.

8.2.13; Construct the Green's function from $H_0^{(1)}(\lambda|r - \xi|)$ with $\lambda = \frac{\omega}{c}$, and then the solution is proportional to (up to a scalar constant) $\psi(r,\theta) = \frac{1}{\sqrt{r}}e^{i\lambda r}\frac{\sin(\lambda a\sin\theta)}{\lambda\sin\theta}$ for large r.

8.3.1; $C\frac{\partial V}{\partial t} = \frac{1}{R_1}\frac{\partial^2 V}{\partial x^2} - \frac{V}{R_2}$.

8.3.3; $G(x,t;\xi,\tau) = 2\sum_{n=1}^{\infty}(1 - e^{-n^2\pi^2(t-\tau)})\cos n\pi x\cos n\pi\xi$, for $t > \tau$.

8.3.4; $G(x,t) = \frac{1}{2\sqrt{\pi t}}e^{-\frac{x^2}{4t} - \alpha t}$.

8.3.5; $G(r,t) = (4\pi t)^{-n/2}e^{-r^2/4t}$, where $r = |x|$.

8.3.7; $T(x,t) = T_0 + \alpha x + (T_1 - T_0)\frac{e^{-a}}{\pi}\int_{-\infty}^{\infty}\frac{e^{-a\eta^2}}{\eta^2+1}d\eta$, where $a = \frac{x^2}{4Dt}$.

8.3.8; $x = \ln 2\sqrt{\frac{2D}{\omega}} = 0.82\text{m}$, $t = \frac{\ln 2}{\omega} = 3.47 \times 10^6\text{s} = 40$ days.

8.3.9; $t = \frac{4L^2}{D\pi^2} \ln \frac{40}{\pi^2} = 5.5$ hours. At this time the temperature at the bottom of the cup is 116^0 F. This seems like a long time. What might be wrong with the model?

8.3.10; (a) Require $u_t = D_i \nabla^2 u$ on regions $i = 1, 2$ subject to the conditions that u and $\mathbf{n} \cdot k_i \nabla u$ be continuous at the interface.

(b) In a spherical domain, eigenfunctions are

$$\phi(r) = \begin{cases} \frac{1}{r} \sin \frac{\mu r}{\sqrt{D_1}}, & 0 < r < r_p \\ \frac{1}{r} \sin \frac{\mu(r-R)}{\sqrt{D_2}}, & r_p < r < R \end{cases}$$

The temperal behavior is $\phi(r)e^{-\mu^2 t}$ and the values μ must satisfy the transcendental equation $\frac{k_1}{\sqrt{D_1}} \tan \frac{\mu r_p}{\sqrt{D_1}} = -\frac{k_2}{\sqrt{D_2}} \tan \frac{\mu(R-r_p)}{\sqrt{D_2}}$.

(c) $\mu^2 \approx 6.6 \times 10^{-4}$/s.

8.3.11; Solve the transcendental equation $\frac{1}{\sqrt{2\pi t}} e^{-\frac{h^2}{Dt} - \alpha t} = \theta$ for t. The velocity is $\frac{h}{t}$.

8.3.12; Propagation occurs provided $h^3\theta < f(\frac{1}{6}) = 0.076$, where $f(y) = (4\pi y)^{-3/2} e^{-1/4y}$. The propagation velocity is v where $f(\frac{hv}{D}) = h^3\theta$, and the minimal velocity is $\frac{1}{6}\frac{D}{h}$.

8.3.13;

$$u(x,t) = e^{i\omega t} \frac{(1 + i\omega + i\alpha\gamma)e^{i\alpha(L-x)} - (1 + i\omega - i\alpha\gamma)e^{i\alpha(x-L)}}{(1 + i\omega + i\alpha\gamma)e^{i\alpha L} - (1 + i\omega - i\alpha\gamma)e^{-i\alpha L}},$$

where $\alpha^2 = 1 + i\omega$.

8.3.14; $p(x,t) = e^{i\omega t}P(x)$, where $P(x) = \exp(-\eta(\omega)x)$, $\eta^2(\omega) = \frac{i\omega}{k + ik\mu\omega}$.

8.4.1; $u_n(t) = \exp(-(\frac{2\sin(\pi/k)}{h})^2)\sin(\frac{2n\pi}{k})$. If we set $n = \frac{kx}{L}$, and $h = \frac{1}{k}$, we have in the limit $k \to \infty$, $u(x,t) = \exp(-\frac{4\pi^2}{l^2})\sin(\frac{2\pi x}{L})$, which is the correct solution of the continuous heat equation with periodic initial data.

8.4.2; $u_n(t) = J_n(-\frac{t}{h})$.

8.4.5; $u_n(t) = J_{2n}(\frac{2t}{h})$.

8.4.6; $k(\omega) = \cos^{-1}(1 - \frac{\omega^2 h^2}{2})$.

8.4.7; For stability, require $-2 < \delta t \lambda < 0$, for all eigenvalues λ of A.

8.4.8; For stability, require that all eigenvalues of A be negative.

9.1.3; $q(x) = -2H(x)\text{sech}^2 x$. (See Problem 7.5.13.)

9.2.2; (a) $\frac{da_n}{dt} = \frac{1}{2}a_n(a_{n+1}^2 - a_n^2)$.

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$r < R$

alues μ must satisfy the
$\frac{2}{D_2} \tan \frac{\mu(R-r_p)}{\sqrt{D_2}}$.

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where
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$\frac{i\omega - i\alpha\gamma)e^{i\alpha(x-L)}}{\omega - i\alpha\gamma)e^{-i\alpha L}}$,

$^2(\omega) = \frac{i\omega}{k+ik\mu\omega}$.

$\imath = \frac{kx}{L}$, and $h = \frac{1}{k}$, we
$) \sin(\frac{2\pi x}{L})$, which is the
›ith periodic initial data.

›alues λ of A.

negative.

(b) $\frac{du_n}{dt} = e^{-u_{n-1}} - e^{-u_{n+1}}$.

9.2.3; $q_{xt} = \sinh q$.

9.3.2; The bound state has $\mu = -\frac{A}{2}$. There is a single soliton with amplitude $\frac{1}{2}A^2$ and speed $c = 4\mu^2 = A^2$.

9.3.3; Use that $r(x) = 6e^{-x+8t} + 12e^{-2x+64t}$ and find a solution of the form (9.18) with $\xi = x - 4t$, $\eta = 2x - 32t$.

9.3.4; One soliton will emerge.

9.3.5; $\frac{dc_{11}}{dt} = -\frac{i}{2k}c_{11}$, $\frac{dc_{12}}{dt} = 0$.

9.4.3; Verify (9.20) and (9.21).

9.4.4; (a) $\alpha = \frac{Bz}{1-z^2}z^{-2j}$, $\beta = 1 - \alpha$, $R = \frac{\beta}{\alpha}$.

(b) $\alpha = -\frac{1}{4}\frac{(4a_0^2-1)(4a_0^2z^2-1)}{a_0^2z^2(z^2-1)}$, $\beta = \frac{1}{4}\frac{z^2(16a_0^4-4a_0^2+1)-4a_0^2}{a_0^2(z^2-1)}$. There are two roots of $\beta(z) = 0$ at $z^2 = \frac{4a_0^2}{16a_0^4-4a_0^2+1}$.

(c) $\alpha = -\frac{2B}{z(z^2-1)}$, $\beta = \frac{z^2+2Bz-1}{z^2-1}$.

9.4.5; $a_nW_n = \frac{\alpha(z)}{2}(z - \frac{1}{z})$.

9.4.6; With $q_0 > 0$, set $a_0 = \frac{1}{2}e^{-q_0/2}$, $a_1 = \frac{1}{2}e^{q_0/2}$, and then use Problem 9.4.4b to show that two solitons moving in opposite directions emerge.

9.4.7; Set $b_1 = -\frac{dq_0}{dt} \neq 0$ and then use Problem 9.4.4c to show that one soliton emerges.

9.4.8; Choose spring constants k_n with $4k_n^2 = 1 + \sinh^2\omega\,\mathrm{sech}^2 n\omega$.

10.2.1; $E_n(x) = \frac{e^{-x}}{\Gamma(n)}\sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(n+k)}{x^{k+1}}$.

10.2.2; $\int_0^1(\cos\pi t + t^2)e^{ixt}dt = (-\frac{i}{x} + \frac{2}{x^2} + \frac{2i}{x^3})e^{ix} - \frac{2i}{x^3} + \frac{i}{x}(1 + e^{ix})\sum_{k=0}^{\infty}(\frac{\pi}{x})^{2k}$.

10.2.3; $\int_0^1 e^{ixt}t^{-1/2}dt = \int_0^\infty e^{ixt}t^{-1/2}dt - \int_1^\infty e^{ixt}t^{-1/2}dt$
$= \sqrt{\frac{\pi i}{x}} + \frac{e^{ix}}{\sqrt{\pi}}\sum_{k=0}^{\infty}(-1)^{k-1}\frac{\Gamma(k+1/2)}{x^{k+1}}$.

10.2.4; $C(x) = \frac{1}{2}\sqrt{\frac{\pi}{2}} - P(x)\cos x^2 + Q(x)\sin x^2$, $S(x) = \frac{1}{2}\sqrt{\frac{\pi}{2}} - P(x)\sin x^2 - Q(x)\cos x^2$, where $P(x) = \frac{1}{2}\left(\frac{1}{2x^3} - \frac{15}{8x^7} + \cdots\right)$,
$Q(x) = \frac{1}{2}\left(\frac{1}{x} - \frac{3}{4x^5} + \frac{105}{16x^9} + \cdots\right)$.

10.3.1; $E_1(x) = e^{-x}\sum_{k=0}^{\infty}(-1)^k\frac{k!}{x^{k+1}}$.

10.3.2; $\int_0^\infty \frac{e^{-zt}}{1+t^4}dt = \sum_{k=0}^{\infty}(-1)^k\frac{(4k)!}{z^{4k+1}}$.

10.3.5; $\int_0^1 e^{-xt}t^{-1/2}dt = \sqrt{\frac{\pi}{x}} - e^{-x}\sqrt{\pi}\sum_{k=0}^{\infty}\frac{1}{x^{k+1}\Gamma(1/2-k)}$.

10.3.6; $\int_0^\infty e^{xt}t^{-t}dt = \sqrt{2\pi y}e^y(1-\frac{1}{24y}-\frac{23}{576y^2}+\cdots)$, where $y=e^{x-1}$.

10.3.7; $\int_0^1 t^x\sin^2\pi t\,dt = \frac{2\pi^2}{x^3}-\frac{12\pi^2}{x^4}+(50-8\pi^2)\frac{\pi^2}{x^5}+\cdots$.

10.3.8; $\int_0^\pi e^{xt^2}t^{-1/3}\cos t\,dt = \frac12\sum_{k=0}^\infty \frac{(-1)^k}{(2k)!}\frac{\Gamma(k+1/3)}{x^{k+1/3}}$.

10.3.11; (a) $\sum_{k=0}^n \binom{n}{k} k!\,n^{-k} = n\int_0^\infty e^{n(\ln(1+x)-x)}dx \sim \sqrt{\frac{\pi n}{2}}$ for large n.

(b) $\sum_{k=0}^n\binom{n}{k} k!\,\lambda^k = \int_0^\infty e^{-x}(1+\lambda x)^n dx \sim (n\lambda)^n e^{-n+\frac{1}{\lambda}}(\frac{\lambda}{\lambda-1})\sqrt{\frac{2\pi}{n}}$.

10.3.13; $\int_0^\infty t^x e^{-t}\ln t\,dt = \sqrt{2\pi}x^{x+1}e^{-x}\ln x(\frac{1}{2x^{3/2}}-\frac{1}{24x^{5/2}}+\cdots)$.

10.3.16; $\binom{100}{10}\sim 1.82\times10^{13}$.

10.4.1; $\int_C \frac{e^{k(z^2-1)}}{z-1/2}dz = 2\pi i e^{-3k/4} - 2ie^{-k}\sum_{j=0}^\infty(-4)^j\frac{\Gamma(j+1/2)}{k^{j+1/2}}$.

10.4.3; $I(x)=\int_\infty^\infty \exp[ix(t+t^3/3)]dt = \sqrt{\frac{\pi}{x}}e^{-2x/3}(1-\frac{5}{48x}+\frac{385}{4608x^2}+O(x^{-3}))$.

10.4.4; $J_n(z)=\sqrt{\frac{2}{z\pi}}\cos(z-\frac{n\pi}{2}-\frac{\pi}{4}) - \frac{4n^2-1}{8}\sqrt{\frac{2}{z^3\pi}}\sin(z-\frac{n\pi}{2}-\frac{\pi}{4})+O(z^{-5/2})$.

10.4.5; $\mu_{nk}=(2n+4k-1)\frac{\pi}{4}-\frac{1}{2\pi}\frac{1-2n^2}{2n+4k-1}+O(\frac{1}{k^2})$.

10.4.6; $\int_0^1\cos(xt^p)dt=\frac{1}{p}(\frac{i}{x})^{1/p}\Gamma\frac{1}{p})-\frac{ie^{ix}}{px}$.

10.4.7; $\int_0^{\pi/2}\left(1-\frac{2\theta}{\pi}\right)^{1/2}\cos(x\cos\theta)d\theta$
$= \mathrm{Re}\left\{e^{ix}\left(\sqrt{\frac{\pi}{2ix}}-\frac{1}{x^{3/2}}(\frac{1}{(2i\pi)^{3/2}}+\frac{i}{8}\sqrt{\frac{\pi}{2i}})+\cdots\right)\right\}-\frac{\sqrt{2}}{2}\frac{1}{x^{3/2}}+\cdots$.

10.4.8; If $x>0$, the change of variables $s=\sqrt{x}t$ converts this to $\frac12\sqrt{x}I(x^{3/2})$, where $I(x)$ is defined in Problem 10.4.3. If $x<0$, $\int_0^\infty\cos(sx+s^3/3)ds\sim\frac{\sqrt{\pi}}{2(-x)^{1/4}}\cos(\frac{2x}{3}+\frac{\pi}{4})$.

10.4.9; Use the answer to Problem 10.4.8 to show that the kth zero of the Airy function is at $x_k\sim\frac38(-4k+1)\pi$.

10.4.10; (a) $\int_{-\infty}^\infty\frac{e^{-a\eta^2}}{1+\eta^2}d\eta\sim\sqrt{\frac{\pi}{a}}(1-\frac{1}{2a}+\frac{3}{4a^2}\cdots)$, for large a.

10.5.1; $\int_a^b f(x)e^{ikg(x)}dx = f(\alpha)\Gamma(\frac43)(\frac{kg^{(3)}(\alpha)}{6})^{2/3}e^{ikg(\alpha)}$.

10.5.4; Set $u(x,t)=e^{-\epsilon t/2}w(x,t)$ so that $w_{tt}=w_{xx}+\frac{\epsilon^2}{4}w$.

11.1.1; $x_1=-0.010101$, $x_{2,3}=-0.49495\pm0.86315i$.

11.1.2; $x_1=0.98979$. Since $x=0$ is a double root of $x^3-x^2=0$, the implicit function theorem does not apply for roots near $x=0$.

11.1.3; $x_1 = 1 - \epsilon + 3\epsilon^2 - 12\epsilon^3 + O(\epsilon^4)$, $x_{2,3} = \pm\frac{i}{\epsilon} - \frac{1}{2} \pm \frac{3}{8}i\epsilon + \frac{1}{2}\epsilon^2 + O(\epsilon^3)$.

11.1.4; $\phi + i\psi = U(z - \frac{a^2}{z}) + i\epsilon a(\frac{a}{z})^{20} + O(\epsilon^2)$.

11.1.5; $\phi + i\psi = z + i\epsilon e^{iz} + O(\epsilon^2)$.

11.1.6; $u(x,y) = a(1-y) + by + \epsilon(a + (b-a)y)\frac{x^3}{6} - \epsilon v(x,y) + O(\epsilon^2)$, where $\nabla^2 v = 0$ and $v_x(0,y) = 0, v_x(1,y) = a/2 + (b-a)y/2, v(x,0) = \frac{ax^2}{2}, v(x,1) = \frac{bx^2}{2}$. Use separation of variables to find v.

11.1.8; $u(r,\theta) = r\cos\theta + \epsilon(\frac{3}{2}(1 - r^4) + \frac{1}{24}(r^2 - r^4)\cos 2\theta) + O(\epsilon^2)$.

11.1.9; $u(r,\theta) = r\cos\theta - \frac{1}{2}\epsilon r^2 \sin 2\theta + O(\epsilon^2)$.

11.1.11; $s/v = \epsilon^2 k^2/2 + \epsilon^4 k^4/4 + O(\epsilon^6)$.

11.1.12; For a channel of width $2l$, $s = \epsilon^2 \frac{\omega k}{2} \frac{\sinh^2 2kl + 2k^2l^2}{\sinh^2 2kl - 2k^2 l^2} + \cdots$.

11.2.1; (a) Eigenvalues are 1, 2, 3, independent of ϵ. Eigenvectors are $(-2, 0, 1-\epsilon)^T, (0,1,0)^T$, and $(0,0,1)^T$.

(b) The eigenvalue $\lambda = 1$ has algebraic multiplicity 2, but geometric multiplicity 1, so the "standard" perturbation method fails. The eigenvalues are $\lambda = \pm\sqrt{\epsilon - \epsilon^2} = 1 \pm (\epsilon^{1/2} - \frac{1}{2}\epsilon^{3/2} - \frac{1}{8}\epsilon^{5/2} + O(\epsilon^{7/2}))$, with corresponding eigenvectors $(1, x_2)^T$ where $x_2 = \pm\sqrt{\frac{\epsilon}{1-\epsilon}} = \pm(\epsilon^{1/2} + 1/2\epsilon^{3/2} + 3/8\epsilon^{5/2} + O(\epsilon^{7/2}))$.

11.2.2; $y(x) = \sin nx + \frac{\epsilon}{4n^2}(-x\sin nx - \pi n x\cos nx + nx^2\cos nx)$, $\lambda = n^2 - \epsilon\frac{\pi}{2} + O(\epsilon^2)$.

11.2.3; (b) For x_1 sufficiently large (since $f(x)$ has compact support), $\lambda \sim \lambda_0 + e^{-x_1}\frac{\int f(x)\phi_0(x)dx}{\int \phi_0^2(x)dx}$.

11.3.1; Steady solutions are at $\theta = 0$ and at $\cos\theta = \frac{g}{l\Omega^2}$ provided $\Omega^2 > \frac{g}{l}$.

11.3.2; $y(x) = \epsilon\sin n\pi x$

$$+ \epsilon^2\frac{(1-\cos n\pi x)^2 + 2x\cos n\pi x}{3n^2\pi^2} - \epsilon^2(\frac{2}{3n^3\pi^3} - 1)\sin n\pi x + 2\epsilon^2\frac{\sin n\pi x - n\pi x\cos n\pi x}{3n^3\pi^3\cos n\pi},$$

$$\lambda = n^2\pi^2 + \frac{4\epsilon}{3n\pi}(1 - \cos n\pi) + O(\epsilon^2).$$

11.3.3; Bifurcation points occur at solutions of $\frac{L^2P}{EI}(1 - \frac{P}{EA}) = n^2\pi^2$ and there are a finite number of such solutions. The nth solution exists if $\frac{L^2A}{4I} > n^2\pi^2$.

11.3.4; $u(x,y) = \epsilon\sin n\pi x\sin\frac{m\pi y}{2} + O(\epsilon^2)$, $\lambda = -(n^2 + \frac{1}{4}m^2)\pi^2 - \frac{9}{16}\epsilon^2\alpha + O(\epsilon^4)$.

11.3.5; $u = \epsilon\phi + O(\epsilon^4)$, $\lambda = \lambda_0 - \epsilon^3 a_4\frac{\int\phi^5 dx}{\int\phi^2 dx}$ if $a_4 \neq 0$ whereas $u = \epsilon\phi + O(\epsilon^5)$, $\lambda = \lambda_0 - \epsilon^4 a_5\frac{\int\phi^6 dx}{\int\phi^2 dx}$ if $a_4 = 0$, $a_5 \neq 0$.

11.3.6; The exact solution is $\phi = 1 + \lambda\alpha$ where $\alpha = (1 + \lambda\alpha)^2$. A plot of the amplitude α as a function of λ is shown in Fig. A.5.

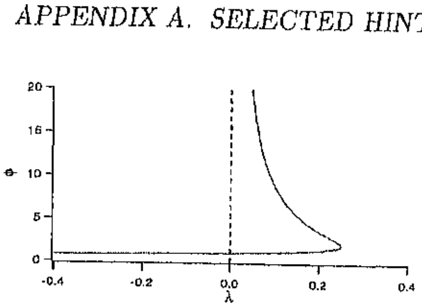

Figure A.5: Plot of ϕ as a function of λ for Problem 11.3.6.

11.3.7; (a) $u(x) = A\alpha(x)$, where A satisfies the quadratic equation $A^2\int_0^1 \alpha^3(y)dy - \frac{A}{\lambda} + \frac{1}{2}\int_0^1 \alpha(y)dy = 0$.

(b) $u(x) = A\alpha(x)$, where $1 = \lambda\sin(A)\int_0^1 \alpha^2(y)dy$.

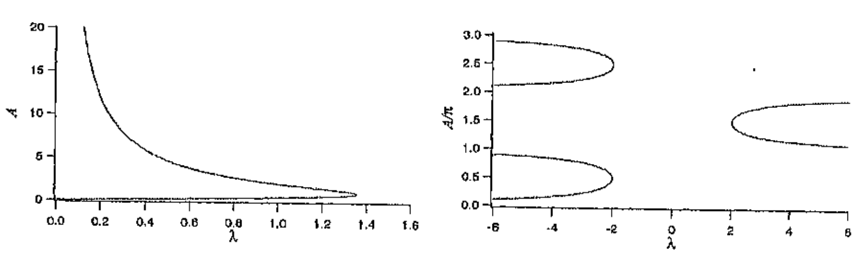

Figure A.6: Left: Plot of A as a function of λ for Problem 11.3.7a with $\alpha(x) = \sin(x)$; Right: Plot of A as a function of λ for Problem 11.3.7 with $\alpha(x) = \sin(x)$.

11.3.9; $u(x) = \exp(A\cos x)$, where $A = \lambda\int_{-\pi}^{\pi} e^{A\cos y}\cos y dy$.

11.3.10; Solve the system of equations $v'' + n^2\pi^2 v = \epsilon(\phi+v)^2 - \mu\phi - \mu v$, $\int_0^1 v\phi dx = 0$, $\mu = 2\epsilon\int_0^1(\phi+v)^2\phi dx$, where $\phi = \sin n\pi x$.

11.4.3; (a) $a = 1 - \frac{1}{12}\epsilon^2 + O(\epsilon^3)$, $u(t) = \sin(t) + \frac{1}{3}\epsilon\cos t\sin t + O(\epsilon^2)$, or $a = 1 + \frac{5}{12}\epsilon^2 + O(\epsilon^3)$, $u(t) = \cos(t) + \frac{1}{3}\epsilon(\cos^2 t - 2) + O(\epsilon^2)$.

(b) The two curves are $a = 4 - \frac{1}{30}\epsilon^2 - \frac{317}{21600}\epsilon^4 + O(\epsilon^5)$, and $a = 4 - \frac{1}{30}\epsilon^2 + \frac{433}{21600}\epsilon^4 + O(\epsilon^5)$.

11.4.5; Set $u = Ae^{i\omega t}$ and find that $A^2\left((b^2A^2 + 1 - \omega^2)^2 + a^2\omega^2\right) = \frac{1}{4}F^2$.

11.4.6; Set $U = a(x)u_0'(x)$ and show that for $a(x)$ to be bounded, it must be that $\int_{-\infty}^{\infty} g(x)u_0'(x)dx = 0$.

11.4.7; (a) Use the implicit function theorem to show that $u = -\frac{\epsilon}{1+\omega^2}\cos\omega t + O(\epsilon^2)$.

(b) Require $a = \Phi(\phi) = \frac{3\pi\omega}{4}\text{sech}\,\frac{\omega\pi}{2}\sin\phi$.

'S AND SOLUTIONS

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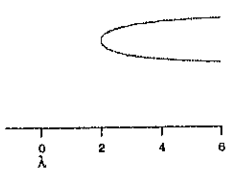

m 11.3.7a with $\alpha(x) =$
.3.7 with $\alpha(x) = \sin(x)$.

dy.

$)^2 - \mu\phi - \mu v$, $\int_0^1 v\phi dx =$

$3t\sin t + O(\epsilon^2)$, or $a =$
$2) + O(\epsilon^2)$.
(ϵ^5), and $a = 4 - \frac{1}{30}\epsilon^2 +$

$+ a^2\omega^2) = \frac{1}{4}F^2$.

ounded, it must be that

hat $u = -\frac{\epsilon}{1+\omega^2}\cos\omega t +$

11.4.8; Require $a = \Phi(\phi) = \frac{3}{2}\frac{\pi\omega}{\sinh\frac{\pi\omega}{\sqrt{2}}}\cos(\phi)$.

11.5.1; There are two Hopf bifurcation points, at $\lambda = \frac{1}{2} \pm \frac{\sqrt{3}}{6}$.

11.5.3; Steady state solutions have $s = \lambda$, $x = (\lambda + a)(1 - \lambda)$. A Hopf bifurcation occurs at $\lambda = \frac{1}{2}(a + 1)$.

11.5.4; Suppose ψ_0 satisfies $A^T\psi_0 = 0$, and ψ_1 satisfies $A^T\psi_1 = -i\lambda\psi_1$. Then a small periodic solution takes the form $x = \epsilon(a\phi_0 + be^{i\lambda t}\phi_1 + \bar{b}e^{-i\lambda t}\bar{\phi}_1) + O(\epsilon^2)$ provided $2|b|^2\langle Q(\phi_1, \bar{\phi}_1), \psi_0\rangle + a\langle B\phi_0, \psi_0\rangle + a^2\langle Q(\phi_0, \phi_0), \psi_0\rangle = 0$, where $2a\langle Q(\phi_0, \phi_1), \psi_1\rangle = -\langle B\phi_1, \psi_1\rangle$.

12.1.1; $u(t, \epsilon) = a(1 + \frac{\epsilon^2}{2})\cos((1 - \frac{\epsilon^2}{2}t - \epsilon) + O(\epsilon^3)$.

12.1.2; $u(t) = A(\epsilon t)\sin((1 + \epsilon^2\omega_2)t) + \epsilon B(\epsilon t)\cos((1 + \epsilon^2\omega_2)t) + O(\epsilon^2)$ where $A_t = \frac{1}{2}A(1 - \frac{1}{4}A^2)$, $B_t = \frac{1}{2}B(1 - \frac{A^2}{4}) + \frac{A}{16}(A^2 - 4)(6A^2 - 1)$, and $\omega_2 = -\frac{1}{16}$.

12.1.3; The solution is $u(t) = \epsilon A(\tau)\sin(t) + O(\epsilon^2)$ where $A_\tau = \frac{1}{2}A(\eta(\tau) - \frac{1}{4}A^2)$, and $\mu = \epsilon^2\eta$, $\tau = \epsilon t$.

12.1.4; (a) The Landau equation is $A_\tau = -\frac{3}{8}A^3$, with solution $A(\tau) = \frac{2A_0}{\sqrt{3A_0\tau + 4}}$, where $A(0) = A_0$, so that $u(t) = \frac{A_0}{\sqrt{1 + \frac{3}{4}A_0\epsilon t}}\sin(t + \phi_0) + O(\epsilon)$.

(b) The Landau equation is $A_\tau = -\frac{4}{3\pi}A^2$, with solution $A(\tau) = \frac{3\pi A_0}{4A_0\tau + 3\pi}$ where $A(0) = A_0$.

12.1.5; Take $a = 1 + \epsilon^2 a_2$, take the slow time to be $\tau = \epsilon^2 t$ and then the solution is $u(t) = A(\tau)\sin(t + \phi(\tau) + O(\epsilon)$ where $\phi_\tau = \frac{1}{2}(a_2 - \frac{5}{12} + \frac{1}{2}\cos^2\phi)$, $A_\tau = \frac{1}{4}\sin\phi\cos\phi$.

12.1.6; The leading order solution is $u(t) = A(\tau)\sin(t + \phi(\tau))$ where $\tau = \epsilon t$ and $A_\tau = \frac{3}{8}A^2\cos(\phi)$, $\phi_\tau = \frac{1}{8}A\sin\phi$.

12.1.7; (a) The Landau equation is $A_\tau = -\frac{3}{8}A^3\cos(\tau)$, with solution $A(\tau) = \frac{2A_0}{\sqrt{3A_0^2\sin\tau + 4}}$, where $A(0) = A_0$. This solution exists for all time provided $A_0^2 < \frac{4}{3}$.

(b) The Landau equation is $A_\tau = -\frac{3}{4}A^3 q(\tau)$.

12.1.8; Let $\tau = \epsilon^2 t$ and then $u(t) = A(\tau)\sin(t + \phi(\tau))$ where $A_\tau = -\frac{1}{2}A$, and $\phi_\tau = \frac{1}{6}A^2$.

12.1.9; With $u = R\cos\theta$, then (from higher order averaging) $R_t = -\frac{3\epsilon}{8}R^3 - \frac{\epsilon}{2}R + O(\epsilon^3)$. The solution is $R^2(t) = 16\epsilon R_0^2 e^{-\epsilon^2 t}(16\epsilon + 3R_0^2(1 - e^{-\epsilon^2 t}))^{-1}$. The behavior predicted by the leading order equation $R_t = -\frac{3\epsilon}{8}R^3$ is incorrect, because it predicts algebraic rather than exponential, decay.

2.1.11; The exact solution is $u(x) = aJ_0(\frac{A}{a}e^{ax}) + bY_0(\frac{A}{a}e^{ax})$. The approximate solution is (set $\epsilon = \frac{a}{A}$ and employ adiabatic invariance, or approximate the Bessel functions for large arguments) is $u(x) \sim e^{-ax/2}\sin(\frac{A}{a}e^{ax} + \phi)$.

12.1.12; Write the equation (12.16) as the system $u_x = r(\frac{x}{\epsilon})v$, $v_x = f(x, \frac{x}{\epsilon})$, and then make the exact change of variables $u = U_0 + \epsilon W(\frac{x}{\epsilon})z$, $v = z + \epsilon Z$, where $\frac{dW}{d\sigma} = r(\sigma) - \bar{r}$, and $Z = \int_0^\sigma f(y,\sigma)d\sigma - \sigma\int_0^1 f(y,\sigma)d\sigma$, with $\sigma = \frac{x}{\epsilon}$.

12.1.13; Set $v = z + \epsilon h(\frac{t}{\epsilon})u$ where $h' = \bar{g} - g$, and then $\frac{du}{dt} = z + \epsilon h(\frac{t}{\epsilon})u$, $\frac{dz}{dt} = -\bar{g}u - \epsilon h(\frac{t}{\epsilon})z - \epsilon^2 h(\frac{t}{\epsilon})^2 u$.

12.1.15; In dimensionless time, r_c has units of length^{-2}. A good dimensionless parameter is $\epsilon = \frac{h}{\sqrt{\bar{r}}}$, $\bar{r} = r_c + r_m/h$. The function W is a piecewise linear "sawtooth" function with slope $-\frac{r_m}{r_c h + r_m}$.

12.1.16; Write the equation as the system $u' = v, v' = u - (1 + g'(\frac{t}{\epsilon}))u^3 + \epsilon a v$. Then the transformation $v = z + \epsilon g(\frac{t}{\epsilon})$ transforms this into a system to which Melnikov's method can be applied. The requirement is that $a\int_{-\infty}^{\infty} U_0'^2(t)dt = \int_{-\infty}^{\infty} g'(\frac{t+\phi}{\epsilon})U_0^3(t)U_0'(t)dt$, where $U_0(t) = \sqrt{2}\text{sech}\ t$.

12.2.2; The device is a hysteretic switch. There are two stable steady outputs, $v_0 = v_{R_+}$ if $v_< v_{R_+}$ and $v_0 = v_{R_-}$ if $v_- > v_{R_-}$.

12.2.3; (a) $i = \frac{v - v_0}{R_3}$ where $v_0 = \begin{cases} v_{R+} & \text{for } v > Av_{R+} \\ \frac{v}{A} & \text{for } Av_{R-} < v < Av_{R+} \\ v_{R-} & \text{for } v < Av_{R-} \end{cases}$, where $A = \frac{R_1}{R_1 + R_2}$.

(b) The current in the left branch of the circuit satisfies $CR_pR_s\frac{di_2}{dt} + R_s i_2 = v - v_g$, thus the effective inductance is $L = CR_pR_s$.

12.2.5; $u(t) = \frac{t}{1+t} + e^{-t/\epsilon} + O(\epsilon)$.

12.2.6; $u(t) = \frac{1}{1+t} + O(\epsilon), v(t) = \frac{-1}{(1+t)^2} + e^{-t/\epsilon} + O(\epsilon)$.

12.3.2; To leading order in ϵ, $u(t) = -\ln(\frac{1+t}{2}) - \ln(2)e^{-t/2\epsilon^{1/3}}(\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}t}{2\epsilon^{1/3}} + \cos\frac{\sqrt{3}t}{2\epsilon^{1/3}})$.

12.3.3; $u(t) = -\tan^{-1}(t) + \epsilon^{1/3}\exp(\frac{-t}{2\epsilon^{1/3}})\left(\frac{1}{3}\sin(\frac{\sqrt{3}t}{2\epsilon^{1/3}}) - \cos(\frac{\sqrt{3}t}{2\epsilon^{1/3}})\right)$
$+ \frac{1}{4}\epsilon^{1/3}2^{2/3}\exp(\frac{2^{1/3}(t-1)}{\epsilon^{1/3}})$

12.3.4; For all β there is a solution with $u(x) \sim -1+$ a boundary layer correction at $x = 1$. For $\beta > 0.2753$ (the real root of $3x^3 + 12x^2 + 11x - 4 = 0$), there is a solution with $u(x) \sim 2+$ a boundary layer correction at $x = 1$.

12.3.5; The solutions are $u(x) = \frac{x}{x+1} - \tanh(\frac{x - \eta_1}{2\epsilon}) + O(\epsilon)$ and $u(x) = \frac{x-1}{2-x} - \tanh(\frac{x-\eta_2}{2\epsilon}) + O(\epsilon)$ with η_1 and η_2 appropriately chosen.

12.3.6; The two solutions are $u(x) = \frac{x-1}{x+1} + 2e^{-x/\epsilon} + O(\epsilon)$, and $u(x) = \frac{x-9}{x-7} - \frac{2}{7}e^{-x/\epsilon} + O(\epsilon)$.

12.3.7; (a) $u(x) = \frac{x}{x+1} - \tanh(\frac{x}{2\epsilon} - \tanh^{-1}(\frac{2}{3}))$.

(b) $u(x) = 4\frac{x-1}{2x-3} - 2\tanh(\frac{x-1}{\epsilon} + \tanh^{-1}(\frac{1}{4}))$

(c) $u(x) = H(x-\frac{1}{4})\frac{4x-1}{5(x+1)} + \frac{2}{5}(1-H(x-\frac{1}{4}))\frac{11-4x}{2x-3} - \frac{4}{5}\tanh(\frac{2}{5\epsilon}(x-\frac{1}{4}))$ where $H(x)$ is the usual Heaviside function.

12.3.8; $u(x) = \frac{\alpha+\beta-1}{2} + \epsilon\ln(\cosh(\frac{2x-\alpha+\beta-1}{2\epsilon})) + O(\epsilon)$.

12.3.11; (a) $T \sim 2\ln\frac{2-\alpha}{\alpha}$.

(b) $T(v) = \frac{2}{\gamma+1}\ln\frac{1+(1+\gamma)v}{1-(1+\gamma)v}$, where $c(v) = \frac{2v}{\sqrt{1-v^2}}$.