The dipole distribution is frequently represented as $\Delta = -\delta'$ since $\phi'(0) = \lim_{\varepsilon\to 0} \frac{\phi(\varepsilon)-\phi(-\varepsilon)}{2\varepsilon} = \lim_{\varepsilon\to 0} \frac{1}{2\varepsilon}\left(\langle \delta_{-\varepsilon}, \phi\rangle - \langle \delta_{\varepsilon}, \phi\rangle\right)$ but since δ_ζ is not a function it is certainly not differentiable in the usual sense. We shall define the derivative of δ momentarily.

One often sees the notation $\int_{-\infty}^{\infty} \delta(x-t)\phi(t)dt = \phi(x)$. It should always be kept in mind that this is simply a notational device to represent the distribution $\langle \delta_x, \phi\rangle$ and is in no way meant to represent an actual integral or that $\delta(x-t)$ is an actual function. The notation $\delta(x-t)$ is a "SYMBOLIC FUNCTION" for the delta distribution.

The correct way to view δ_x is as an operator on the set of test functions. We should never refer to pointwise values of δ_x since it is not a function, but an operator on functions. The operation $\langle \delta_x, \phi\rangle = \phi(0)$ makes perfectly good sense and we have violated no rules of integration or function theory to make this definition.

The fact that some operators can be viewed as being generated by functions through normal integration should not confuse the issue. δ_x is not such an operator. Another operator that is operator valued but not pointwise valued is the operator (not a linear functional) $L = d/dx$. We know that d/dx cannot be evaluated at the point $x = 3$ for example, but d/dx can be evaluated pointwise only after it has first acted on a differentiable function $u(x)$. Thus, $du/dx = u'(x)$ can be evaluated at $x = 3$, only after the operand $u(x)$ is known. Similarly, $\langle \delta_x, \phi\rangle$ can be evaluated only after ϕ is known.

Although distributions are not always representable as integrals, their properties are nonetheless always defined to be consistent with the corresponding property of inner products. The following are some properties of distributions that result from this association.

1. If t is a distribution and $f \in C^\infty$ then ft is a distribution whose action is defined by $\langle ft, \phi\rangle = \langle t, f\phi\rangle$. For example, if f is continuous $f(x)\delta = f(0)\delta$. If f is continuously differentiable at 0, $f\delta' = -f'(0)\delta + f(0)\delta'$. This follows since

$$\langle f\delta', \phi\rangle = \langle \delta', f\phi\rangle = -(f\phi)'\,\big|_{x=0} = -f'(0)\phi(0) - f(0)\phi'(0).$$

2. Two distributions t_1 and t_2 are said to be equal on the interval $a < x < b$ if for all test functions ϕ with support in $[a,b]$, $\langle t_1, \phi\rangle = \langle t_2, \phi\rangle$. Therefore it is often said (and this is unfortunately misleading) that $\delta(x) = 0$ for $x \neq 0$.

3. The usual rules of integration are always assumed to hold. For example, by change of scale $t(\alpha x)$ we mean

$$\langle t(\alpha x), \phi\rangle \;=\; \frac{1}{|\alpha|}\left\langle t, \phi\left(\frac{x}{\alpha}\right)\right\rangle$$

and the shift of axes $t(x-\xi)$ is taken to mean

$$\langle t(x-\xi), \phi \rangle = \langle t, \phi(x+\xi) \rangle,$$

even though pointwise values of t may not have meaning. It follows for example that $\delta(x-\xi) = \delta_\xi$ and $\delta(ax) = \delta(x)/|a|$.

4. The derivative t' of a distribution t is defined by $\langle t', \phi \rangle = -\langle t, \phi' \rangle$ for all test functions $\phi \in D$. This definition is natural since, for differentiable functions

$$\langle f', \phi \rangle = \int_{-\infty}^{\infty} f'(x)\phi(x)dx = -\int_{-\infty}^{\infty} f(x)\phi'(x)dx = -\langle f, \phi' \rangle .$$

Since $\phi(x)$ has compact support, the integration by parts has no boundary contributions at $x = \pm\infty$.

If t is a distribution, then t' is also a distribution. If $\{\phi_n\}$ is a zero sequence in D, then $\{\phi_n'\}$ is also a zero sequence, so that

$$\langle t', \phi_n \rangle = -\langle t, \phi_n' \rangle \to 0 \quad \text{as} \quad n \to \infty.$$

It follows that for any distribution t, the nth distributional derivative $t^{(n)}$ exists and its action is

$$\left\langle t^{(n)}, \phi \right\rangle = (-1)^n \left\langle t, \phi^{(n)} \right\rangle .$$

Thus any L^2 function has distributional derivatives of all orders.

Examples

1. The Heaviside distribution $\langle H, \phi \rangle = \int_0^\infty \phi(x)dx$ has derivative

$$\langle H', \phi \rangle = -\int_0^\infty \phi'(x)dx = \phi(0)$$

since ϕ has compact support, so that $H' = \delta_0$.

2. The derivative of the δ-distribution is

$$\langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0)$$

which is the negative of the dipole distribution.

3. Suppose f is continuously differentiable except at the points $x_1, x_2, \ldots, x_n$ at which f has jump discontinuities $\Delta f_1, \Delta f_2, \ldots, \Delta f_n$, respectively. Its distribution has derivative given by

$$\langle f', \phi \rangle = -\langle f, \phi' \rangle = -\int_{-\infty}^{\infty} f(x)\phi'(x)dx$$

$$
\begin{aligned}
&= \int_{-\infty}^{x_1} f(x)\phi'(x)dx \\
&+ \int_{x_1}^{x_2} f(x)\phi'(x)dx + \cdots + \int_{x_n}^{\infty} f(x)\phi'(x)dx \\
&= \int_{-\infty}^{\infty} \frac{df}{dx}\phi(x)dx + \sum_{k=1}^{n} \Delta f_k \phi(x_k).
\end{aligned}
$$

It follows that the distributional derivative of f is

$$f' = \frac{df}{dx} + \sum_{k=1}^{n} \Delta f_k \delta_{x_k},$$

where df/dx is the usual calculus derivative of f, wherever it exists.

4. For $f(x) = |x|$, the distributional derivative of f has action

$$
\begin{aligned}
\langle f', \phi\rangle &= -\langle f, \phi'\rangle = -\int_{-\infty}^{\infty} |x|\phi'(x)dx \\
&= \int_{-\infty}^{0} x\phi'(x) - \int_{0}^{\infty} x\phi'(x)dx \\
&= -\int_{-\infty}^{0} \phi(x)dx + \int_{0}^{\infty} \phi(x)dx \\
&= -\int_{-\infty}^{\infty} \phi(x)dx + 2\int_{0}^{\infty} \phi(x)dx
\end{aligned}
$$

so that $f' = -1 + 2H(x)$, and $f'' = 2\delta_0$.

Definition

A sequence of distributions $\{t_n\}$ is said to converge to the distribution t if their actions converge in $\mathbb{R}$, that is, if

$$\langle t_n, \phi\rangle \to \langle t, \phi\rangle \quad \text{for all} \quad \phi \quad \text{in} \quad D.$$

This convergence is called convergence in the sense of distribution or WEAK CONVERGENCE.

If the sequence of distributions t_n converges to t then the sequence of derivatives t_n' converges to t'. This follows since

$$\langle t_n', \phi\rangle = -\langle t_n, \phi'\rangle \to -\langle t, \phi'\rangle = \langle t', \phi\rangle$$

for all ϕ in D.

Example

The sequence $\{t_n\}$ =
sequence of distributi
(pointwise) and as a
to the zero distributio

Using distributions, w
derivative to many objects
nitions. It is also possible

Definition

The differential equati
of distribution (i.e., in
all derivatives are inte
ential equation is calle
equation.

Examples

1. To solve the equati
a distribution u for
This rather munda
its solutions. Althou
usual. We want to
that the action of u
function. Said anot
set of test functions
functions. We first
$\int_{-\infty}^{\infty} \psi\, dx = 0$. Cert
compact support. I
$\phi = \int_{-\infty}^{x} \psi\, dx$ is a t
support.
Now comes the trick
$\int_{-\infty}^{\infty} \phi_0(x)dx = 1$. A
combination of ϕ_0 a

$\phi(x) = \phi_0(x)$

$= \phi_0(x)$