- G. W. Stewart, *Introduction to Matrix Computations*, Academic Press, New York, 1980,
- G. E. Forsythe, M. A. Malcolm and C. B. Moler, *Computer Methods for Mathematical Computations*, Prentice-Hall, 1977,
- G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed., Johns Hopkins University Press, Baltimore, 1996,
- L. N. Trefethen and D. Bau, *Numerical Linear Algebra*, SIAM, Philadelphia, 1997,

and for least squares problems in

- C. L. Lawson and R. J. Hanson, *Solving Least Squares Problems*, SIAM, Philadelphia, 1995,

while many usable programs are described in

- W. H. Press, B. P. Flannery, S. A. Teukolsky, and W. T. Vetterling, *Numerical Recipes in Fortran 77: The Art of Computing*, Cambridge University Press, Cambridge, 1992.

This book (in its many editions and versions) is the best selling mathematics book of all time.

The best and easiest way to do numerical computations for matrices is with Matlab. Learn how to use Matlab! For example,

- D. Hanselman and B. Littlefield, *Mastering Matlab, Prentice-Hall*, Upper Saddle River, NJ, 1996.

While you are at it, you should also learn Maple or Mathematica.

- B. W. Char, K. O. Geddes, G. H. Gonnet, B. L. Leong, M. B. Monagan, and S. M. Watt, *Maple V Language Reference Manual*, Springer-Verlag, New York, 1991,
- A. Heck, *Introduction to Maple*, 2nd ed., Springer-Verlag, New York, 1996.
- S. Wolfram, *Mathematica*, 3rd ed., Addison-Wesley, Reading, MA, 1996.

Finally, the ranking of football teams using a variety of matrix algorithms is summarized in

- J. P. Keener, The Perron Frobenius Theorem and the ranking of football teams, SIAM Rev., 35, 80-93, 1993.

Problems for Chapter 1

Problem Section 1.1

1. Prove that every basis in a finite dimensional space has the same number of elements.

2. Show that in any inner product space

$$||x+y||^2 + ||x-y||^2 = 2||x||^2 + 2||y||^2.$$

Interpret this geometrically in $I\!R^2$.

3. (a) Verify that in an inner product space,

$$\text{Re } \langle x, y\rangle = \frac{1}{4}(||x+y||^2 - ||x-y||^2).$$

(b) Show that in any real inner product space there is at most one inner product which generates the same induced norm.

(c) In $I\!R^n$ with $n > 1$, show that $||x||_p = \left(\sum_{k=1}^n |x_k|^p\right)^{1/p}$ can be induced by an inner product if and only if $p = 2$.

4. Suppose $f(x)$ and $g(x)$ are continuous real valued function defined for $x \in [0,1]$. Define vectors in $I\!R^n$, $F = (f(x_1), f(x_2), \ldots, f(x_n))$ and $G = (g(x_1), g(x_2), \ldots, g(x_n))$, where $x_k = k/n$. Why is

$$\langle F, G\rangle_n = \frac{1}{n}\sum_{k=1}^n f(x_k)g(x_k)$$

with $x_k = \frac{k}{n}$, not an inner product for the space of continuous functions?

5. Show that

$$\langle f, g\rangle = \int_0^1 \left(f(x)\overline{g(x)} + f'(x)\overline{g'(x)}\right) dx.$$

is an inner product for continuously differentiable functions on the interval $[0,1]$.

6. Show that any set of mutually orthogonal vectors is linearly independent.

7. (a) Show that $I\!R^n$ with the supremum norm $||x||_\infty = \max_k\{|x_k|\}$ is a normed linear vector space.

(b) Show that $I\!R^n$ with norm $||x||_1 = \sum_{k=1}^n |x_k|$ is a normed linear vector.

8. Verify that the choice $\gamma = \frac{\langle x,y\rangle}{||y||^2}$ minimizes $||x - \gamma y||^2$. Show that $|\langle x, y\rangle|^2 = ||x||^2 \cdot ||y||^2$ if and only if x and y are linearly dependent.

9. Starting with the set $\{1, x, x^2, \ldots, x^k, \ldots\}$, use the Gram-Schmidt procedure and the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)\omega(x)dx, \qquad \omega(x) > 0$$

to find the first five orthogonal polynomials when

(a) $a = -1$, $b = 1$, $\omega(x) = 1$ (Legendre polynomials)

(b) $a = -1$, $b = 1$, $\omega(x) = (1 - x^2)^{-1/2}$ (Chebyshev polynomials)

(c) $a = 0$, $b = \infty$, $\omega(x) = e^{-x}$ (Laguerre polynomials)

(d) $a = -\infty$, $b = \infty$, $\omega(x) = e^{-x^2}$ (Hermite polynomials)

Remark: All of these polynomials are known by Maple.

10. Starting with the set $\{1, x, x^2, \ldots, x^n, \ldots\}$ use the Gram-Schmidt procedure and the inner product

$$\langle f, g \rangle = \int_{-1}^{1} (f(x)g(x) + f'(x)g'(x))dx$$

to find the first five orthogonal polynomials.

Problem Section 1.2

1. (a) Represent the transformation whose matrix representation with respect to the natural basis is

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}$$

relative to the basis $\{(1,1,0)^T, (0,1,1)^T, (1,0,1)^T\}$.

(b) The representation of a transformation with respect to the basis $\{(1,1,2)^T, (1,2,3)^T, (3,4,1)^T\}$ is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 1 & 0 & 1 \end{pmatrix}.$$

Find the representation of this transformation with respect to the basis $\{(1,0,0)^T, (0,1,-1)^T, (0,1,1)^T\}$.

2. (a) Prove that two symmetric matrices are equivalent if and only if they have the same eigenvalues (with the same multiplicities).

(b) Show that if A and B are equivalent, then

$$\det A = \det B.$$

(c) Is the converse true?

3. (a) Show that if A is an $n \times m$ matrix and B is an $m \times n$ matrix, then AB and BA have the same nonzero eigenvalues.

(b) Show that the eigenvalues of AA^* are real and non-negative.

4. Show that the eigenvalues of a real skew-symmetric ($A = -A^T$) matrix are imaginary.

5. Find a basis for the range and null space of the following matrices:

(a)

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 5 \end{pmatrix},$$

(b)

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 1 & 1 & 1 \end{pmatrix}.$$

6. Find an invertible matrix T and a diagonal matrix Λ so that $A = T\Lambda T^{-1}$ for each of the following matrices A:

(a)

$$\begin{pmatrix} 1 & 0 & 0 \\ 1/4 & 1/4 & 1/2 \\ 0 & 0 & 1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(c)

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 3 \end{pmatrix}$$

(d)

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

(e)

$$\begin{pmatrix} 1/2 & 1/2 & \sqrt{3}/6 \\ 1/2 & 1/2 & \sqrt{3}/6 \\ \sqrt{3}/6 & \sqrt{3}/6 & 5/6 \end{pmatrix}.$$

7. Find the spectral representation of the matrix

$$A = \begin{pmatrix} 7 & 2 \\ -2 & 2 \end{pmatrix}.$$

Illustrate how $Ax = b$ can be solved geometrically using the appropriately chosen coordinate system on a piece of graph paper.

8. Suppose P is the matrix that projects (orthogonally) any vector onto a manifold M. Find all eigenvalues and eigenvectors of P.

9. The sets of vectors $\{\phi_i\}_{i=1}^n$, $\{\psi_i\}_{i=1}^n$ are said to be **biorthogonal** if $\langle \phi_i, \psi_j \rangle = \delta_{ij}$. Suppose $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ are biorthogonal.
   (a) Show that $\{\phi_i\}_{i=1}^n$ and $\{\psi_i\}_{i=1}^n$ each form a linearly independent set.
   (b) Show that any vector in $\mathbb{R}^n$ can be written as a linear combination of $\{\phi_i\}$ as
   $$x = \sum_{i=1}^n \alpha_i \phi_i$$
   where $\alpha_i = \langle x, \psi_i \rangle$.
   (c) Express (b) in matrix form; that is, show that
   $$x = \sum_{i=1}^n P_i x$$
   where P_i are projection matrices with the properties that $P_i^2 = P_i$ and $P_i P_j = 0$ for $i \neq j$. Express the matrix P_i in terms of the vectors ϕ_i and ψ_i.

10. (a) Suppose the eigenvalues of A all have algebraic multiplicity one. Show that the eigenvectors of A and the eigenvectors of A^* form a biorthogonal set.
   (b) Suppose $A\phi_i = \lambda_i \phi_i$ and $A^*\psi_i = \bar{\lambda}_i \psi_i$, $i = 1, 2, \ldots, n$ and that $\lambda_i \neq \lambda_j$ for $i \neq j$. Prove that $A = \sum_{i=1}^n \lambda_i P_i$ where $P_i = \phi_i \psi_i^*$ is a projection matrix. Remark: This is an alternate way to express the spectral decomposition theorem for a matrix A.
   (c) Express the matrices C and C^{-1}, where $A = C\Lambda C^{-1}$, in terms of ϕ_i and ψ_i.
   (d) Suppose $A\phi = \lambda\phi$ and $A\psi = \bar{\lambda}\psi$ and the geometric multiplicity of λ is one. Show that it is not necessary that $\langle \phi, \psi \rangle \neq 0$.

Problem Section 1.3

1. Use the minimax principle to show that the matrix
$$\begin{pmatrix} 2 & 4 & 5 & 1 \\ 4 & 2 & 1 & 3 \\ 5 & 1 & 60 & 12 \\ 1 & 3 & 12 & 48 \end{pmatrix}$$
has an eigenvalue $\lambda_4 < -2.1$ and an eigenvalue $\lambda_1 > 67.4$.

2. (a) Prove an inequality relating the eigenvalues of a symmetric matrix before and after one of its diagonal elements is increased.
   (b) Use this inequality and the minimax principle to show that the smallest eigenvalue of
   $$A = \begin{pmatrix} 8 & 4 & 4 \\ 4 & 8 & -4 \\ 4 & -4 & 3 \end{pmatrix}$$
   is smaller than $-1/3$.

3. Use the minimax principle to show that the intermediate eigenvalue λ_2 of
$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{pmatrix}$$
is not positive.

4. The moment of inertia of any solid object about an axis along the unit vector x is defined by
$$I(x) = \int_R d_x^2(y) \rho dV,$$
where $d_x(y)$ is the perpendicular distance from the point y to the axis along x, ρ is the density of the material, and R is the region occupied by the object. Show that $I(x)$ is a quadratic function of x, $I(x) = x^T A x$ where A is a symmetric 3×3 matrix.

5. Suppose A is a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 > \lambda_3 \geq \ldots$. Show that
$$\max_{\langle u,v \rangle = 0} \langle Au, u \rangle + \langle Av, v \rangle = \lambda_1 + \lambda_2$$
where $\|u\| = \|v\| = 1$.

Problem Section 1.4

1. Under what conditions do the matrices of Problem 1.2.5 have solutions $Ax = b$? Are they unique?

2. Suppose P projects vectors in $\mathbb{R}^n$ (orthogonally) onto a linear manifold M. What is the solvability condition for the equation $Px = b$?

3. Show that the matrix $A = (a_{ij})$ where $a_{ij} = \langle \phi_i, \phi_j \rangle$ is invertible if and only if the vectors ϕ_i are linearly independent.

4. A square matrix A (with real entries) is positive-definite if $\langle Ax, x \rangle > 0$ for all $x \neq 0$. Use the Fredholm alternative to prove that a positive definite matrix is invertible.

Problem Section 1.5

1. Use any of the algorithms in the text to find the least squares pseudo-inverse for the following matrices:

   (a)
   $$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

   (b)
   $$\begin{pmatrix} 3 & -1 \\ 1 & 1 \\ -1 & 3 \end{pmatrix}$$

   (c)
   $$\begin{pmatrix} 3 & 1 & 2 \\ -1 & 1 & -2 \end{pmatrix}$$

   (d)
   $$\begin{pmatrix} -1 & 0 & 1 \\ -1 & 2/3 & 2/3 \\ -1 & -2/3 & 7/3 \end{pmatrix}.$$

   (e) The linear algebra package of Maple has procedures to calculate the range, null space, etc. of a matrix and to augment a matrix with another. Use these features of Maple to develop a program that uses exact computations to find A' using the Gaussian elimination method (Method 1), and use this program to find A' for each of the above matrices.

2. Verify that the least squares pseudo-inverse of an $m \times n$ diagonal matrix D with $d_{ij} = \sigma_i \delta_{ij}$ is the $n \times m$ diagonal matrix D' with $d'_{ij} = \frac{1}{\sigma_i} \delta_{ij}$ whenever $\sigma_i \neq 0$ and $d'_{ij} = 0$ otherwise.

3. (a) For any two vectors $x, y \in \mathbb{R}^n$ with $||x|| = ||y||$ find the Householder (orthogonal) transformation U that satisfies $Ux = y$.

   (b) Verify that a Householder transformation U satisfies $U^*U = I$.

4. Use the Gram-Schmidt procedure (even though Householder transformations are generally preferable) to find the QR representation of the matrix
$$A = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 2 & 1 \\ 9 & 1 & 2 \end{pmatrix}.$$

5. For the matrix
$$A = \begin{pmatrix} 5 & -3 \\ 0 & 4 \end{pmatrix},$$
illustrate on a piece of graph paper how the singular value decomposition $A = U\Sigma V^*$ transforms a vector x onto Ax. Compare this with how $A = T\Lambda T^{-1}$ transforms a vector x onto Ax.

6. For the matrix
$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix},$$
illustrate on a piece of graph paper how the least squares pseudo-inverse $A' = Q^{-1}\Lambda' Q$ transforms a vector b into the least squares solution of $Ax = b$.

7. For each of the matrices in Problem 1.2.6, use the QR algorithm to form the iterates $A_{n+1} = Q_n^{-1} A_n Q_n$, where $A_n = Q_n R_n$. Examine a few of the iterates to determine why the iteration works and what it converges to.

8. For the matrices
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$
and
$$A = \begin{pmatrix} 1.002 & 0.998 \\ 1.999 & 2.001 \end{pmatrix},$$
illustrate on a piece of graph paper how the least squares pseudo-inverse $A' = V\Sigma' U^*$ transforms a vector b onto the least squares solution of $Ax = b$. For the second of these matrices, show how setting the smallest singular value to zero stabilizes the inversion process.

9. For a nonsymmetric matrix $A = T^{-1}\Lambda T$, with Λ a diagonal matrix, it is not true in general that $A' = T^{-1}\Lambda' T$ is the pseudo-inverse. Find a 2×2 example which illustrates geometrically what goes wrong.