Homework #1
Due January 23, 2019

1. (a) Do the oil in the frying pan experiment. See page 6 in Hoyle. Place a pan on the stove and add 1-2 mm of oil. Heat it up and look at the surface. Sprinkle uniformly with celery seed or black pepper to help visualize it. Report the patterns you see. If you want, take a picture. The best picture will get posted on the web. (b) Press on your eyeballs until you start to see patterns. What do you see?

2. Suppose $f(u)$ is monotone increasing and positive. Suppose $f(-\infty) = 0$ and $f(+\infty) = A < \infty$. prove there is a unique equilibrium point to $-u + f(I - gu)$ for all I and $g > 0$.

3. Given $\lambda_n = -1 - \beta n^4 + \alpha n^2$ find conditions on α, β such that λ_n has a positive maximum viewed as a function of n. (Pretend n is real instead of an integer.)

4. (a) Let f be as in problem 2. Let β, g, I, τ be positive parameters. The following model was suggested by Rinzel, Wilson, and others as a mechanism for binocular rivalry in which a different pattern is presented to the left and right eyes. The perceived pattern alternates with a characteristic frequency. Here are the equations:

$$\begin{aligned} x_1' &= -x_1 + f(I - gx_2 - \beta z_1) \\ z_1' &= (-z_1 + x_1)/\tau \\ x_2' &= -x_2 + f(I - gx_1 - \beta z_2) \\ z_2' &= (-z_2 + x_2)/\tau \end{aligned}$$

   Using the methods studied in class (as well as exercise 2), prove that there is a unique symmetric equilibrium, $x_1 = x_2 = u$, $z_1 = z_2 = u$ and study its stability by decomposing the 4×4 system into two 2×2 systems. Show that solutions restricted to the symmetric subspace are always stable but that two different ways to lose stability can occur in the anti-symmetric subspace. (b) Let $f(x) = 1/(1+e^{-x})$, $I = 4, \tau = 10, \beta = 2$ and $g = 2, 5, 8$. Simulate the ODEs by solving them numerically starting from non-symmetric initial conditions (e.g. $x_1(0) = 1, x_2(0) = 0$) and describe the behavior. (c) Starting with $g = 0$ compute the bifurcation diagram as g increases to 10.

5. (a) Let $a = 0.6, b = 1.25$ in the Brusselator. Find a curve in (D_x, D_y) parameters such that $\det M_D = 0$. Recall

$$M_D := \begin{pmatrix} b - 2D_x & a^2 \\ -b & -a^2 - 2D_y \end{pmatrix}.$$

   (b) Pick (D_x, D_y) such that $\det M_D < 0$. Simulate the coupled Brusselator pair (start with small positive non-symmetric initial data.) Show that there are non-symmetric equilibria.

6. The Gierer-Meinhardt model kinetics has the form:

$$\begin{aligned} x' &= x^2/y - ax \\ y' &= x^2 - by \end{aligned}$$

Using the same techniques as applied to the Brusselator to find conditions on a, b and D_x, D_y for loss of stability of the symmetric state for a diffusively coupled pair:

$$X_j' = F(X_j) + \begin{pmatrix} D_x & 0 \\ 0 & D_y \end{pmatrix} [X_k - X_j], \qquad j = 1, 2, \qquad k = 2, 1$$

where F is the GM kinetics and $X = (x, y)$. If you assume that the symmetric solution is stable in the symmetric subspace, what do your conditions say about D_x and D_y ?