Homework 2: Due Feb 12

1. I may have done this in class, but it doesn't hurt for you to rederive it in a different context. Consider the general planar reaction-diffusion equation

$$\begin{aligned} u_t &= f(u,v) + D_u u_{xx} \\ v_t &= g(u,v) + D_v v_{xx} \end{aligned}$$

with periodic boundary conditions on a large domain $0 < x < L$. $D_u, D_v > 0$ Let (u_0, v_0) be an equilibrium point ($f(u_0, v_0) = g(u_0, v_0) = 0$).

   (a) Let $k = 2\pi n/L$ where n runs through the integers. Show that the equilibrium is asymptotically stable if the eigenvalues of the matrices

$$M_k := \begin{pmatrix} a - k^2 D_u & b \\ c & d - k^2 D_v \end{pmatrix}$$

   Have negative real parts for all k. Note $a = f_u(u_0, v_0), b = f_v(u_0, v_0)$, $c = g_u(u_0, v_0), d = g_v(u_0, v_0)$. We say that a system can undergo a pattern forming instability if M_0 has eigenvalues with negative real parts, but M_k has some eigenvalue(s) with positive real parts for some values of k.

   (b) Prove that the eigenvalues of M_k has begative real parts if k is sufficiently large.

   (c) Find conditions on M_0 so as to guarantee there will never be a pattern forming instability for any k. (That is, find conditions on a, b, c, d such that M_k has negative eigenvalues for any choice of D_u, D_v and all k.)

2. Consider the Brusselator model on the interval $[0, 100]$ (treat this as esentially infinite for now). Recall:

$$\begin{aligned} u_t &= a - (b+1)u + u^2 v + D_u u_{xx} \\ v_t &= bu - u^2 v + D_v v_{xx} \end{aligned}$$

Choose periodic boundary conditions. Let $k = 2\pi n/100$ and treat k as a continuous parameter. Pick $a = 4.5, D_u = 2, D_v = 16$. Find the linear stability of the equilibrium point $u = a, v = b/a$ and determine for what value of b and k the system first loses stability. Use this critical value of k to get n. Based on this linear stability analysis, one predicts n stripes in the domain. Suppose that $D_u = 8$ and $D_v = 64$. How many stripes do you predict? Can you suggest a scaling law? That is suppose that $D_u = 2M$ and $D_v = 16M$ with $M > 0$ a parameter. How many stripes do you expect as a function of M?

3. In the above problem, with $M = 1$, simulate the Brusselator for b less than and greater than the value required for stability of the homogeneous state. Here is some XPP code

```
!dx=1
u0=u100
u101=u1
v0=v100
v101=v1
f(u,v)=a-(b+1)*u+u^2*v
g(u,v)=b*u-u^2*v
par b=5 a=4.5,du=2,dv=16
u[1..100]'=f(u[j],v[j])+du*(u[j+1]-2*u[j]+u[j-1])/dx^2
v[1..100]'=g(u[j],v[j])+dv*(v[j+1]-2*v[j]+v[j-1])/dx^2
u[1..100](0)=a+ran(1)*.02
v[1..100](0)=b/a+ran(1)*.02
@ meth=euler,dt=.01,maxstor=100000
@ total=500,nout=5
done
```

Note that I made a really crude discretization of the domain, but that doesnt seem to make much difference here. You should be able to see the right number of stripes

4. The Gray-Scott chemical system is commonly used for pattern formation and has the form

$$\begin{aligned} u_t &= -uv^2 + f(1-u) + D_u u_{xx} \\ v_t &= uv^2 - fv + D_v v_{xx} \end{aligned}$$

Here f, D_u, D_v are parameters. Find all the spatially independent equilibria and their stability. Are any of these starting points for pattern formation? That is, can you find values of f and equilibria which have the form needed as per exercise 1? Let's say that you said yes to the above. Choose $f = 0.15$ and find a pair D_u, D_v such that M_k has a positive eigenvalue for some $k > 0$.

5. Analyze the stability of the zero solution to the linear integral operator on the space of $2\pi-$periodic functions in x:

$$u_t = -u + \int_0^{2\pi} [a + b\cos(x-y)]u(y,t)\ dy$$

as a function of a, b. (Hint: Reduce it to a finite dimensional problem as follows. Since $u(x,t)$ is periodic in x, write

$$u(x,t) = c_0(t) + \sum_{j=1}^{\infty}[c_j(t)\cos nx + s_j(t)\sin nx]$$

and derive equations for c_j, s_j using the mutual orthogonality of the trig functions.)