Homework 3: Due Feb 26

1. This is a somewhat lengthy problem, but give it a whirl. We are going to study the following equation:

$$u_t(x,t) = -u(x,t) + \int_0^{2\pi} W(x-y) f(u(y,t))\ dy \tag{1}$$

where

$$f(u) = \nu u + \beta u^2 + \gamma u^3$$

and $W(x)$ is a 2π periodic function which is defined in the ensuing sections.

(a) Let $0 < a < 1$ and let

$$w_a(x) = \sum_{n=0}^{\infty} a^n \cos nx$$

Show that

$$w_a(x) = \frac{1 - a\cos x}{1 + a^2 - 2a\cos x}$$

Show that

$$\int_0^{2\pi} w_a(y) e^{-imy}\ dy = \pi a^m$$

for $m = \pm 1, \pm 2, \ldots$ and the integral is 2π for $m = 0$.

(b) Let $0 < b < a < 1$ and $0 < c$ consider

$$W(x) = w_a(x) - c w_b(x)$$

Plot $W(x)$ for, $a = .9, b = .1, c = .8$ on $[-\pi, \pi]$ and show that it has a Mexican hat like structure (positive near 0 and negative away from 0). More generally, compute

$$\hat{W}(m) := \int_0^{2\pi} W(x) e^{-imx}\ dx.$$

(c) $u(x,t) = 0$ is clearly a solution to eq (1). Determine the stability of $u = 0$ for $W(x)$ as defined above using ν as a a parameter. Under what conditions will there be the possibility for pattern formation?

(d) Compute the bifurcation equations for patterns that emerge as ν increases beyond the critical value. Let $n > 0$ be the critical value. (For notational simplicity, you can write $\hat{W}(m)$ instead of writing it out.) Proceed as follows:

i. Write $\tau = \epsilon^2 t$, $\nu = \nu^* + \epsilon^2 \alpha$,

$$u(x,t) = \epsilon z(\tau) e^{inx} + \epsilon^2 u_2 + \epsilon^3 u_3 + C.C$$

ii. You should get the following sequence of equations

$$\begin{aligned} 0 &= -u_2 + \nu^* W(x) * u_2 + \beta W(x) * [z^2 e^{2inx} + 2z\bar{z} + \bar{z}e^{-2inx}] \\ z_\tau e^{inx} + C.C. &= -u_3 + \nu^* W(x) * u_3 + \alpha W(x) * [ze^{inx} + C.C] + W(x) * N(x) \end{aligned}$$

where $N(x)$ is a bunch of nonlinear cubic terms in $ze^{inx}, \bar{z}e^{-inx}$ which you have to compute. Keep in mind that the linear operator

$$L_0 u(x) := -u(x) + \nu^* W(x) * u(x)$$

has a nontrivial nullspace and is self-adjoint (since W is symmetric).

iii. You should show that the terms in the u_2 equation are in the range so that you can solve for $u_2(x)$ as

$$u_2(x) = Ae^{2inx} + B + Ae^{-2inx}$$

Then you can get $N(x)$.

iv. If you correctly apply the Fredholm alternative, you will get something like

$$z_\tau = \alpha kz + dz^2\bar{z}$$

where k, d are constants depending on $\hat{W}, \beta, \gamma$. Of course, your main goal is to compute k, d.

2. Complete the multiplication table for the symmetry group, D_3.

3. Write down all the elements of the symmetry group of the square. (There are 8 of them) and describe how they transform the square by labeling the corners, A, B, C, D. You dont have to make a multiplication table.

4. Describe the symmetry group of a rectangle and show it is commutative.

5. We showed in class that the bifurcation equations for a Hopf bifurcation have the form:

$$z_\tau = z(a\lambda + b|z|^2)$$

where $a = a_1 + ia_2$ and $b = b_1 + ib_2$ ($a_1 > 0$). By writing $z = re^{i\theta}$, derive equations for r, θ and find solutions of the form $r = r_0$ and $\theta = \omega t$. Determine their stability. In particular show that $b_1 < 0$ is needed for stability.