This is a long HW problem that involves a combination of simulations and also of analysis. We will look a model of all-to-all coupled quadratic integrate-and-fire neurons:

$$V_j' = V_j^2 + \mu + \Delta L_j + ks, \quad j = 1, \ldots, N$$

with the rule that when $V_j = +\infty$, it is reset to $-\infty$. We call the time at which V_j hits $+\infty$, t_j. The random numbers, L_j are taken from the Lorenz distribution (which we generate as)

$$L_j = \tan(\pi(R_j - 1/2))$$

where R_j are uniform random numbers on $(0, 1)$. The synaptic coupling obeys the ODE:

$$s' = -s/\tau + (1/N) \sum_{j=1}^{N} \delta(t - t_j)/\tau.$$

Note that the sum in this equation is the average firing rate of the network. **So, the first thing I want you to do is simulate the equations for $N = 400$ and "infinity" (the reset value) set to, say, 100. Set $\tau = 0.5$, $\mu = 5$, $\Delta = 0.25$. Use Euler to integrate with dt=0.005 and run for 40000 iterations (200 time units) with a variety of values of g, say $k = -3, -2, -1, 0, 1, 2, 3$ and plot s vs time; maybe zoom in to see if there is a rhythm at all. For k in the right range you should see some regular oscillations. I will post some XPP code if you don't want to try this using MatLab, Python, or something else (like Julia).**

So now we need to analyze this. Proceeding as we did in class with the Kuramoto, we will let $I_j = \mu + \Delta L_j$ and $N \to \infty$. We let $f(V, I, t)$ be the density of V, I at any point t and find that

$$\partial_t f + \partial_V [(V^2 + I + ks)f] = 0. \tag{1}$$

The firing rate is the flux at $V = +\infty$, that is

$$r(I, t) = \lim_{V \to \infty} (V^2 + I + ks) f(V, I, t)$$

Let $g(I)$ be the density for the applied currents. Then the average firing rate, $r_a(t)$ is

$$r_a(t) = \int_{-\infty}^{\infty} g(I) r(I, t) \, dI.$$

We also see that by this definition that s satisfies

$$\tau s' = -s + r_a(t)$$

which now closes the system.

Now it is time for an *ansatz*. We suppose that

$$f(V, I, t) = \frac{1}{\pi} \frac{\beta(I, t)}{(V - \alpha(V, t)) + \beta^2(V, t)}.$$

Using this *ansatz* derive equations for α, β. Plug the ansatz into Eq (1) and you will get something that looks like:

$$(AV^2 + BV + C)/M(V) = 0$$

This must be true for all V, so A, B, C must vanish. M is complicated, but we don't care since it is in the denominator and get rid of it. Solve A for β_t and you will get

$$\beta_t = 2\beta\alpha$$

and plug this into, say, C and you will find

$$\alpha_t = I + ks + \alpha^2 - \beta^2.$$

Finally, plug these into B to make sure it vanishes! (It will). Now we are cooking. Let's do one more thing. Let $w = \alpha + i\beta$. **Show that**

$$\frac{\partial w}{\partial t} = w^2 + I + ks$$

Now, lets close this sucker up. We note that $r(I, t) = \beta/\pi$ so that

$$\pi r_a(t) = \int_{-\infty}^{\infty} g(I)\beta(I, t)\ dI$$

so that we can now get an ODE for s. Also note that the average voltage is

$$V_a(t) = \int_{-\infty}^{\infty} g(I)\alpha(I, t)\ dI.$$

As with the Kuramoto done in class, we still have an infinite dimensional system. I will write it as

$$\frac{\partial w}{\partial t} = w^2 + \mu + \Delta L + ks$$

and parametrize it by L instead which is just taken from the Lorenzian in our simulations. That is, instead of $w(I, t)$, I have $w(L, t)$. We now see that

$$V_a(t) + i\pi r_a(t) = \int_{-\infty}^{\infty} g(L)w(L, t).$$

So, let's suppose that $g(L) = 1/(\pi[1 + L^2])$ and as in class use the residue theorem to evaluate the integral. We write

$$g(L) = \frac{1}{2\pi i}\left(\frac{1}{L - i} - \frac{1}{L + i}\right),$$

so that

$$V_a(t) + i\pi r_a(t) = w(\pm i, t)$$

depending on which of the two possible contours you take. (Note the $2\pi i$ from the residue theorem cancels with the $2\pi i$ in the denominator.) This yields:

$$\frac{dw(\pm i,t)}{dt} = w(\pm i,t)^2 + \mu + ks \pm i\Delta.$$

Justify why you must take the $+i$ root, recalling that β must be positive. This leads to the following ODEs that constitute the system to analyze:

$$\begin{aligned} a' &= \mu + ks + a^2 - b^2 \\ b' &= 2ab + \Delta \\ \tau s' &= -s + b/\pi. \end{aligned} \tag{2}$$

Here, $a = \alpha(+i,t), b = \beta(+i,t)$ as required. Notice that a is the average potential and b/π is the firing rate. **Do the following:**

1. Show that is $b(0) > 0$, then $b(t) > 0$ for all time as long as $\Delta > 0$.

2. Show that equilibria require that $a < 0$ since $b > 0$. Conclude that the average potential must be negative.

3. The uncoupled system has $k = 0$. Find the firing rate $F(\mu)b/\pi$ as a function of μ and sketch it for $\Delta = 0.1$ and $\mu \in (-2, 2)$. Show that $F(\mu)$ approaches $\sqrt{\mu}/\pi$ which is the single cell QIF firing rate curve.

4. For $k \neq 0$ explore the equilibria by setting $s = b/\pi$ and then exploring the (a, b) phase plane. Try to prove that there are at most 3 equilibria with $b > 0$ and always at least 1. Hint: Note the b nullcline asymptotes at the axes and that the a nullcline can be written as a hyperbola:

$$(b - K/2)^2 - a^2 = \mu + K^2/4$$

   where $K = k/\pi$. Recall from Calc I how to sketch hyperbolas!

5. Now, back to the full model (that is including s) This might be hard. Can you show that there are no Hopf bifurcations if $k > 0$? Show that there is only one $b > 0$ fixed point when $k < 0$

6. Sketch the bifurcation diagram for $k = 4, \tau = 0.5, \Delta = 0.1$ as the drive, μ to the network increases between -0.5 to 0.5.

7. Set $k = -3, \tau = 0.5, \Delta = 0.1$ and compute the bifurcation diagram as μ increases from -1 to 5. If there is a Hopf bifurcation, follow the periodic orbit. For $\mu = 5, k = -3$ does the behavior of the simple system appear to agree with the full blown coupled QIF you solved above?