Here is one more problem for your amusement. Consider the purely repulsive coupling case in which there is are two attractive sources that are separated by a repellor. Solve for ρ on the infinite domain where:

$$\int_{\alpha}^{\beta} e^{-|x-y|}\rho(y)\ dy = \lambda - F(x)$$

where $F(x) = bx^4 - ax^2$. Now before you start plugging and chugging, note that you *will not be able to solve for the width of the swarm analytically* because it requires solving a fifth order polynomial, $p(H) = M$. You should find this function $p(H)$. Henceforth, assume a, b positive. Show that $p(0) = 0$ and $p'(0) < 0$. Show that for h large enough, $p(H)$ is monotonically increasing. In fact, you should be able to use Descartes rule of signs to prove there is a single local positive minimum for p and thus conclude that $p(H)$ has a single positive root, H_1. Thus $p(H)$ is monotonically increasing and positive for $H > H_1$. How does H_1 depend on a? Show that the larger a is, the larger H_1 is. This means that for and $M > 0$ there exists a unique H solving $p(H) = M$. So this determines H. However, there is another wrinkle. Given $\rho(x; H)$ (which you had to already know since you computed $p(H) = \int_{-H}^{H} \rho(x)dx$), we also require that $\rho(x) \geq 0$. As you will have noted, $\rho(x)$ is symmetric about 0 and has a global minimum at $x = 0$. Thus, if $\rho(0) \geq 0$, then we have a valid solution. If $\rho(0) < 0$, then, the solution is invalid and our assumption of a single swarm is wrong. (In this case, the swarm bifurcates into two that are attracted to the two sources.) So, you now must analyze $q(H) = \rho(0; H)$ the value of ρ at $x = 0$. Show $q(0) < 0$ when $a > 0$, and thus conclude that if H is too small (which it will be for small M, as you proved above), then there cannot be a single swarm. Prove that for H large enough, $q(H) > 0$, so that massive swarms with two sources will form a single compact swarm. Thus, you have proven that as the mass of the swarm decreases, the single swarm must form two smaller swarms surrounding the source of the food. We can delve into this a bit more. As with $p(H)$, prove that $q'(H)$ has a single positive minimum (using Descartes rule of signs) and thus conclude that $q(H)$ has a single positive root, H_2. Show that the larger a is, the larger H_2 is. Now we can see another way to look at the bifurcation. Fix M large enough so that $q(H) > 0$ where $p(H) = M$. Now increase a (which is equivalent to separating the two attracting sources) you should be able to see that the swarm must split. Show this via an example: set $b = 0.25$ and choose $a = 1$ and $a = 3$. Graph p, q and show, e.g. that choosing $M = 13$ will produce a valid swarm with H close to 2 when $a = 1$ but an invalid swarm with H about 2.8 when $a = 3$.