Homework 4: Due March 19

1. Suppose that a system of equations on the one-dimensional line that is translation invariant in space and time and reflection invariant in space has a linearization with a nullspace proportional to

$$z_1 \exp(ix) + z_2 \exp(iqx) + c.c$$

where $q > 0$. (For example in the Lifshitz-Petrich equations.) Use symmetry to write down the bifurcation equations up to third order when (i) $q \neq 2$ and (ii) $q = 2$. The latter should give you additional terms when you apply translation invariance, so be careful!. Are there pure modes possible ($z_1 \neq 0$, $z_2 = 0$, etc?)

2. Consider a differential equation of the form:

$$\frac{du}{dt} = F(u, t, \lambda)$$

where $F(\cdot, t+1, \cdot) = F(\cdot, t, \cdot)$. This is invariant under time translations *which are integers* and these form a group under addition. Suppose that you linearize around some (possibly time-dependent) solution, and find a null-space (in the space of periodic functions) proportional to:

$$ze^{i\omega t} + c.c.$$

This is like a Hopf bifurcation. Derive the bifurcation equations up to order three for general $\omega \neq \pi, \pi/2, 2\pi/3$. Now, derive them for $\omega = \pi, \pi/2, 2\Pi/3$. These are called strong resonances. In the regular Hopf bifurcation, ω does not matter, but in this example, you have translation only with respect to the integers!

3. Suppose that we are lying in a square domain with periodic boundary conditions and the critical wave vector has length $\sqrt{5}$ so that the eigen space is formed as

$$u(x, y, \tau) = z_1 e^{i(2x+y)} + z_2 e^{i(-x+2y)} + z_3 e^{i(x+2y)} + z_4 e^{i(-2x+y)} + c.c$$

For the following, it may help to draw these vectors in the (x, y)-plane.

(a) How does translation $(x, y) \to (x + p_1, y + p_2)$ act on the nullspace?

(b) How does reflection about the $x - axis$ act on the nullspace?

(c) How does clockwise rotation by $\pi/2$ act on the nullspace?

Use these symmetries to write equations for $dz_1/d\tau, \ldots$ up to third order. Show that the coefficients are real. Show that solutions exist of the following form for (z_1, z_2, z_3, z_4): (a, a, a, a);$(a, 0, 0, 0)$; $(a, a, -a, -a)$; $(a, a, 0, 0)$; $(a, 0, 0, a)$; $(a, 0, a, 0)$. Use the compute to plot in gray scale (or color) on the square domain $x, y \in [0, 10\pi)$, the patterns formed when

$z_1 = z_2 = z_3 = z_4 \in R$ (supersquares) $z_1 = z_2 = -z_3 = -z_4 \in R$ (antisquares), $z_1 \in R, z_2 = z_3 = z_4 = 0$ (rolls), $z_1 = z_2 \in R, z_3 = z_4 = 0$ (simple squares), $z_1 = z_3 \in R, z_2 = z_4 = 0$ (rhombs 1), and $z_1 = z_4 \in R, z_2 = z_3 = 0$ (rhombs 2).

Here is Matlab code to plot 2D contours:

```
x=0:pi/20:10*pi;
y=0:pi/20:10*pi;
xl=repmat(x,201,1);
yl=repmat(x',1,201);
z=cos(2*xl+yl)+cos(yl-2*xl)+cos(-xl+2*yl)+cos(xl+2*yl);
pcolor(y,x,z)
shading interp
colorbar
```

and here is XPP code:

```
@ meth=discrete, total=200
u(x,y)=a*cos(2*x+y)+b*cos(-x+2*y)+c*cos(x+2*y)+d*cos(y-2*x)
par a=1,b=1,c=1,d=1
x[1..200]'=u(10*Pi*[j]/200,10*pi*t/200)
done
```

(To use the XPP code, click initconds go. Then View/array and fill in as:

```
column 1:x1
Ncols:200
NRows:200
RowSkip:1
Zmin:-4
ZMax:4
```

Resize to a square and redraw.) And here is gnuplot code:

```
set pm3d map
set xrange [0:10*pi];set yrange [0:10*pi];set samples 100;set isosamples 100
splot cos(2*x+y)+cos(-x+2*y)+cos(y-2*x)+cos(x+2*y)
# to get postscript hard copy
# set term post color
# set out "ss.eps"
#
# to get jpeg
# set term jpeg
# set out "ss.jpg"
# replot
```

4. Consider the two-dimensional Swift-Hohenberg equation:

$$\frac{\partial u}{\partial t} = \lambda u - u^3 + qu^2 - (1 + \nabla^2)^2 u$$

on a square of size $[0, 2\pi) \times [0, 2\pi)$ with periodic boundary conditions. Solve the linearized equation around $u = 0$ and show that when $\lambda = 0$, it has a two-dimensional nullspace spanned by

$$ze^{ix} + we^{iy} + c.c.$$

We derived equations in class for this situation (on a square lattice) having the form:

$$z_\tau = z(a\lambda + b|z|^2 + c|w|^2), \quad w_\tau = w(a\lambda + b|w|^2 + c|z^2|)$$

Compute the coefficients b, c in terms of the parameter q. When are stripes/spots stable?