1. Consider the following rule for pursuit in three-dimensions:

$$\frac{A-P}{|A-P|} \cdot \frac{\dot{P}}{|\dot{P}|} = 1$$

along with the normalization that $|\dot{P}| = \sigma$ to derive the following equations for $P = (x, y, z)$ given $A = (a, b, c)$ as parameterized functions of time. Let $(u, v, w) = (a - x, b - y, c - z)$ and $d = |(u, v, w)|$. Show

$$\begin{aligned} x' &= \sigma u/d \\ y' &= \sigma v/d \\ z' &= \sigma w/d \end{aligned}$$

Use your equations to chase (a) the helix $(r\cos t, r\sin t, Kt)$ where $r = .8, K = .6.$ and choose a speed a little under 1, say 0.8. What is your predator's strategy? (b) the trefoil knot, $[(2+\cos 3t)\cos 2t, (2+\cos 3t)\sin 2t, \sin 3t]$ (note that this does not have unit speed, so you should increase the speed accordingly) Plot some projections in two d to see it work. I will put an XPP file on the web site so you can play with it. (chase3d.ode)

2. In class, we derived equations for the pursuer in two-dimensions chasing an agent who also has a strategy which is to move at an angle ϕ away from the pursuer at speed σ. We were able to reduce the system to equations for the differences, $u = a - x, v = b - y$ and wrote:

$$\begin{aligned} u' &= \frac{1}{d}(-u - (\cos\phi u + \sin\phi v)\sigma) \\ v' &= \frac{1}{d}(-v + (\sin\phi u - \cos\phi v)\sigma) \end{aligned}$$

Here $d = \sqrt{u^2 + v^2}$ makes this a nonlinear equation. Now, find conditions of σ, ϕ such that (u, v) go to zero (that is , the prey is caught) and such that (u, v) grow (that is, the prey escapes). Using this analysis, fix ϕ such that for σ large enough the prey gets away. For σ small enough, the prey is always caught. Now consider $\sigma = f(d)$ where $f(0) = \sigma_{max}$ is large enough for escape and $f(d)$ gets small for d large. Say, $f(d) = C\exp(-kd)$ where $-C\cos\phi > 1$ and $k > 0$. Prove that there is a limit cycle (by explicitly constructing it; it will be of the form $r\cos\omega t, r\sin\omega t$) Find r, ω as a function of ϕ, C, k.

3. Suppose we go to three-dimensions and assume that the predator obeys the chase dynamics in problem 1, but the prey attempts to move away with a distance dependent velocity and at "strategy" (multiplication by a matrix that preserves distance, e.g. has determinant ± 1) relative to the predator. Say, we use a matrix R. Then our dynamics are

$$\begin{aligned} P' &= (A-P)/|A-P| \\ A' &= \sigma R(P-A)/|A-P| \end{aligned}$$

Letting $U = (P - A)$, we see that

$$U' = \frac{1}{|U|}(-I - \sigma R)U$$

The question is, what kind of dynamics does this system have, say, where the speed is a function of distance and decreases for large distances. My conjecture is that the only dynamics are limit cycles and fixed points when $\sigma = f(|U|)$. To see why this is, let λ be an eigenvalue of R and study the linear dynamics

$$V' = (-I + \sigma R)V$$

as a function of σ. What can happen as σ increases from 0? (Recall that U and its surrogate, V describe the distance between the predator and the prey.)

4. Let $f_a(d) = -c_a \exp(-d/l_a)/d$ and $f_r(d) = c_r \exp(-d/l_r)/d$ and assume $0 < c_a < c_r$ and $c_a l_a > c_r l_r$. Consider two animals in the plane with dynamics

$$\begin{aligned} \frac{d\vec{x}_1}{dt} &= f_a(d)(\vec{x}_1 - \vec{x}_2) \\ \frac{d\vec{x}_2}{dt} &= f_r(d)(\vec{x}_2 - \vec{x}_1) \end{aligned}$$

with $d = |\vec{x}_1 - \vec{x}_2|$. Find conditions where the distance between the two animals tends to a constant; what is the speed of their motion? Hint, first reduce it to a planar equation,letting $\vec{u} = \vec{x}_1 - \vec{x}_2$. This will let you find d. Then assume a constant speed and try to find this unknown from the above equations.

5. This may be too hard, but I think it will be interesting. In class, we will analyze a compact one-dimensional swarm. Suppose that in addition to the swarming animals, there is a predator who is attracted to them and they are repelled by him (see previous problem for a swarm of 1).Try simulating this with a swarm of 100. (see XPP code.) Fix the rate of repelling (as above) and vary the rate of attraction. I found a very interesting set of bifurcations as the rate of attraction of the predator to the herd is increased. I don't know if it can be analyzed, but there is an oscillatory solution I think. I will post the ODE on line. Try to find values of the parameter `ca` which have oscillatory dynamics. I do not see this in two space dimensions (see next problem). It would be interesting to see if it can occur in a low dimesional model such as the one that I looked at in class with 2+1. (Update - it is found for 4 sheep and 1 wolf). As I read this, I note it is fairly vague. So, using the ODE l1d50.ode (which I have set up so that the initial values are linearly increasing and the prey is on the outside), start with c_a small, say $c_a = -.02$ and show that the predator remains on the outside and pushes the herd away. Make c_a more negative

and note that he gets between the sheep but herds from within since the guys on the opposite side want to be with their sheep buddies. Continue to strengthen the attraction and note that at some range of parameters, the whole herd freezes up with the wolf in the middle. Explain intuitively what is going on. Finally make the attraction even stronger and observe limit cycles. (I varied c_a between $-.02$ and $-.2$.

6. This one is also challenging. Let's consider the two spatial dimension analogue of a case where there are two sheep and one wolf. The sheep have an attraction-repulsion kernel,

$$q(x) = K(|x|)x$$

where $K(r) = [G\exp(-r/l) - \exp(-r)]/r$ as usual and we assume that the wolf is attracted to the sheep with kernel, $g_a(x) = f_a(x)x$ where $f_a(x) = c_a \exp(-|x|/l_a)/|x|$ and the sheep are repelled with kernel $g_r(x) = f_r(x)x$ where $f_r(x) = c_r \exp(-|x|/l_r)/|x|$ where in all cases x is a vector in R^2. Now, I will abuse notation and let the positions of the sheep be (x_j, y_j) and that of the wolf (u, v). Let ρ be the distance between the two sheep and r_j be the distance between the sheep and the wolf. The dynamics are

$$\begin{aligned} x_j' &= K(\rho)(x_k - x_j) + f_r(r_j)(x_j - u) \\ y_j' &= K(\rho)(y_k - y_j) + f_r(r_j)(y_j - v) \\ u' &= f_a(r_1)(u - x_1) + f_a(r_2)(u - x_2) \\ v' &= f_a(r_1)(v - y_1) + f_a(r_2)(v - y_2) \end{aligned}$$

where $j = 1, 2$ and $k = 2, 1$. This is a 6 dimensional ODE. However, it can be reduced to a four dimensional ODE by replacing x_j by $x_j - u$ and y_j by $y_j - v$. Then we get an ode in the 4 variables which are the sheep coordinates relative to the wolf. Letting $z_j = x_j - u$ and $w_j = y_j - v$, write down the 4D system. I have simulated this and find two different types of stable solutions. One in which the wolf is symmetrically placed between the sheep, so his attraction cancels and he stays exactly between them. This, of course is a one-dimensional situation. The other solution I have found is one in which the two sheep are paired and the wolf lies between them but offset along the line perpendicular to the sheep axis. This is a two-dimensional configuation. In the real coordinates (not relative), this represents a scenario where the wolf "pushes" the sheep orthogonally while the latter solution is a frozen one. All of these relative and absolute configurations have a great deal of symmetry, so even in the relative coordinates, there will be zero eigenvalues (corresponding to rotation in the plane). Presumably you could get rid of the rotations and be left with a three-d system. We will do something simpler! Let the relative coordinates be $(z_1, w_1), (z_2, w_2)$ Both these solutions have a symmetry that $z_1 = -z_2$ and $w_1 = w_2$. Show that the full 4D system is invariant under this symmetry. This allows us to write a planar system for just (w_1, z_1) and thus we can study some bifurcations of this!. I have posted the symmetric ODE (3-2-sym.ode) on line and you should vary the prarmeter, c_a which is the wolf's attraction to the sheep. Show that there is a bifurcation as c_a changes, say from -.1 to -.03. You could do this all numerically by looking at the nullclines.