Homework 5: Due April ??

1. We have seen bifurcation of squares and rolls in a simple lattice and in a square periodic domain when there is a zero eigenvalue and we have seen bifurcation of waves and standing oscillations in a one-dimensional domain. So, now, suppose we are in a square periodic domain that has all the usual reflection and translation symmetries in space and translation in time but the nullspace is 8-dimensional and has the following form:

$$z_1 e^{i(t+x)} + z_2 e^{i(t+y)} + z_3 e^{i(t-x)} + z_4 e^{i(t-y)} + c.c$$

Derive the general bifurcation equations for this system. Here is a little help. Rotation of the square takes $z_1 \to z_2, z_2 \to z_3, z_3 \to z_4, z_4 \to z_1$ so all you need to compute is

$$z_1' = f_1(z_1, z_2, z_3, z_4)$$

and the rest will follow. Reflection across the x-axis leaves z_1 invariant but interchanges z_2, z_4, so this further restricts the number of different coefficients.

Determine algebraic conditions for the following types of solutions

   (a) Traveling rolls ($z_1 \neq 0$, $z_2 = z_3 = z_4 = 0$) and determine when they are stable
   (b) Standing rolls ($z_1 = z_3 \neq 0$, $z_2 = z_4 = 0$)
   (c) What does the pattern $z_1 = z_2 \neq 0$, $z_3 = z_4 = 0$ look like? and is it a solution?
   (d) What does $z_1 = z_2 = z_3 = z_4$ correspond to and is it a solution?
   (e) Alternating rolls - $z_1 = z_3 = -iz_2 = -iz_4$. Sketch these - they are really cool!

2. Consider the phase equation:

$$\theta_t = a(x) + \theta_x^2 + \theta_{xx}$$

with boundary conditions

$$\theta_x(0, t) = \theta_x(1, t) = 0.$$

This describes the evolution of the phase in the presence of spatial heterogeneities, $a(x)$. Assume $a(x)$ is continuous in $[0, 1]$. Let $\theta(x, t) = \omega t + \int u(y),\ dy$ so that

$$\omega = a(x) + u^2 + u_x.$$

Prove that there is some value of ω such that this equation has a solution satisfying $u(0) = u(1) = 0$. (There are several approaches to this; one is to make the Cole-Hopf transformation, $u = v_x/v$ and convert it to a Sturm-Liouville eigenvalue problem. But the most direct way is to use shooting.) Solve the BVP numerically for $a(x) = \exp(-4(x - 1/2)^2$ and estimate ω.

3. In class, we derived Burgers equation for the evolution of phase:

$$\theta_t = \alpha\theta_x^2 + \beta\theta_{xx}$$

where

$$\alpha =< U^*(t)DU''(t) >, \quad \beta < U^*(t)DU'(t) >$$

with U, U^* the oscillation and its adjoint and D the diffusion matrix. Consider the CGL in rectangular coordinates:

$$\begin{aligned} u_t &= u(1-u^2-v^2) - vq(u^2+v^2) + u_{xx} - dv_{xx} \\ v_t &= v(1-u^2-v^2) + uq(u^2+v^2) + v_{xx} + du_{xx} \end{aligned}$$

Given $U(t) = (\cos qt, \sin qt)$ and $U^*(t) = (\cos qt - (1/q)\sin qt, \sin qt + (1/q)\cos qt)$, compute α and β. Under what circumstances is diffusion, β positive?

4. Discretize the above equation with $dx = 1$ into 100 bins (total domain length is 100) with periodic boundary conditions. (In case you forget,

$$u_{xx} \approx \frac{u[j-1] - 2u[j] + u[j+1]}{dx^2}$$

and periodic boundary conditions mean $u[0] = u[100]$ and $u[101] = u[1]$. Solve the equations with random initial conditions and choose $d = 1$. Try $q = 2$ and $q = 0.5$. XPP code is included on the web page. You should integrate for a long periodi of time and see the result of the diffusive instability.