Extortion Strategies & Zero Determinant (ZD) strategies

Here, we return to The iterated PD models with $\begin{pmatrix} R & S \\ T & P \end{pmatrix}$ as usual, e.g $\begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix}$

Press + Dyson show first, an interesting result. If a short memory player plays a long memory player, The short memory players score is The same as if The long memory guy played a short memory strategy. Because of This result, we can derive strategies for X (short mem) assuming each player has memory 1!

Let X, Y be random variables with values x, y That are The players respective moves on a given iteration. Suppose player X keeps history H_0 but Y keeps longer history H_1

We want to show That The joint prob distribution of (x, y) given history (H_0, H_1) averaged over (H_0, H_1) is same as joint prob of (x, y) averaged over The shorter history H_0

$$\langle P(x,y \mid H_0, H_1) \rangle_{H_0,H_1} \triangleq \sum_{H_0,H_1} P(x,y \mid H_0, H_1)\, P(H_0, H_1)$$

$$= \sum_{H_0,H_1} P(x \mid H_0)\, P(y \mid H_0, H_1)\, P(H_0, H_1) \quad \leftarrow \text{conditional independ ??}$$

Definition of conditional

$$= \sum_{H_0} P(x \mid H_0) \left[\sum_{H_1} P(y \mid H_0, H_1)\, P(H_1 \mid H_0)\, P(H_0) \right]$$

sum over H_1 gives marginal

$$= \sum_{H_0} P(x \mid H_0) \left[\sum_{H_1} P(y, H_1 \mid H_0) \right] P(H_0)$$

$$= \sum_{H_0} P(x \mid H_0)\, P(y \mid H_0)\, P(H_0)$$

$$= \langle P(x,y \mid H_0) \rangle_{H_0}$$

Intuitively - from X's point of view, X views Y's long strategy as a peculiar random number generator. Thus The player with The shortest memory sets the rules of The game

ZD strategies:

As usual let $xy \in (cc, cd, dc, dd)$, where c, d are cooperate, defect

X's strategy is $\vec{p} = (p_1, p_2, p_3, p_4)$ are probabilities to cooperate given the outcome + $\vec{q} = (q_1, q_2, q_3, q_4)$ are Y's strategies, seen from his perspective, ie $yx = \{cc, cd, dc, dd\}$ (Note this is a little diff then we used before. So payoff is

$$\begin{bmatrix} p_i q_j & p_i(1-q_j) \\ (1-p_i) q_j & (1-p_i)(1-q_j) \end{bmatrix}$$

As before we get a Markov transition matrix:

$$M = \begin{bmatrix} p_1 q_1 & p_1(1-q_1) & (1-p_1) q_1 & (1-p_1)(1-q_1) \\ p_2 q_3 & p_2(1-q_3) & (1-p_2) q_3 & (1-p_2)(1-q_3) \\ p_3 q_2 & p_3(1-q_2) & (1-p_3) q_2 & (1-p_3)(1-q_2) \\ p_4 q_4 & p_4(1-q_4) & (1-p_4) q_4 & (1-p_4)(1-q_4) \end{bmatrix}$$

$\vec{V}$ steady states left eigenvector: $\vec{V}^T M = \vec{V}^T \Rightarrow \vec{V}^T M' = 0$

$M' = M - I \Rightarrow \det M' = 0$

Recall Cramers Rule: $\underset{(4\times4)}{\mathrm{Adj}(M')} M' = \det(M') I = 0$

$\begin{pmatrix} a_{11} & \cdots & a_{14} \\ \vdots \end{pmatrix} M' = 0$

$\mathrm{Adj}(M')$ = matrix of minors (3x3) determinants, since $V^T M' = 0$

This means every row of $\mathrm{Adj}(M')$ is proportional to $\vec{V}$

$$\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \quad C_{ij} \text{ cofactor} \quad \mathrm{Adj}(M') = C^T$$

so, for example, up to sign, the components of $\vec{V}$ are the fourth row of $\mathrm{Adj}(M')$, which are the cofactors of the 4^{th} column of M' which are just the determinants of the first three columns of M' leaving out the relevant row. These determinants are unchanged by adding the first column of M' to the second + third columns

Let $\vec{f}$ be arbitrary vector. Then $\vec{V}\cdot\vec{f}$ is easy to compute using the above manipulations:

$$\vec{V}\cdot\vec{f} = \det\begin{bmatrix} -1+p_1q_1 & -1+p_1 & -1+q_1 & f_1 \\ p_2q_3 & -1+p_2 & q_3 & f_2 \\ p_3q_2 & p_3 & -1+q_2 & f_3 \\ p_4q_4 & p_4 & q_4 & f_4 \end{bmatrix} = D(p,q,f)$$

$$\qquad\qquad\qquad = \tilde{p} \qquad \tilde{q}$$

Note that two columns depend only on one player's strategy!

$\vec{S}_X = [R, S, T, P]$, $\vec{S}_Y = [R, T, S, P]$

$$s_X = \frac{V\cdot\vec{S}_X}{V\cdot\mathbb{1}} = \frac{D(p,q,\vec{S}_X)}{D(p,q,\mathbb{1})}, \quad s_Y = \frac{V\cdot\vec{S}_Y}{V\cdot\mathbb{1}} = \frac{D(p,q,\vec{S}_Y)}{D(p,q,\mathbb{1})}$$

$$\alpha s_X + \beta s_Y + \gamma = \frac{D(p,q,\alpha\vec{S}_X+\beta\vec{S}_Y+\gamma\mathbb{1})}{D(p,q,\mathbb{1})} \longleftarrow \text{Really cool}$$

If I choose $\hat{p} = k[\alpha\vec{S}_X + \beta\vec{S}_Y + \gamma\mathbb{1}]$, then this determinant vanishes since two columns are proportional and this means I enforce a linear relationship between the payoffs no matter what q !!!

Of course, they may not be feasible since $\hat{p}_i \in [0,1]$

<u>EXAMPLE</u> X sets Y's score!

set $\alpha = 0$. Use $\tilde{p} = \beta\vec{S}_Y + \gamma\mathbb{1}$ & solve for p_2, p_3 in terms of p_1, p_4 eliminating β, γ. SHOW:

$$p_2 = \frac{p_1(T-P) - (1+p_4)(T-R)}{R-P}, \quad p_3 = \frac{(1-p_1)(P-S) + p_4(R-S)}{R-P}$$

To force $s_Y = \dfrac{(1-p_1)P + p_4R}{(1-p_1)+p_4}$ in PD $T > R > P > S$

so there is a feasible strategy when $p_1 \lessapprox 1$, $p_4 \gtrapprox 0 \Rightarrow p_2 \lessapprox 1$, $p_3 \approx 0$. Clearly s_Y is weighted average of $1-p_1$ & p_4 so all possible scores between P & R are possible for Y. X can completely ignore Y but set Y's score. X can spoof Y & then play a better strategy

X cannot really set his own score: $\tilde{p} = \alpha \vec{S}_X + \gamma \mathbb{1}$

$$p_2 = \frac{(1+p_4)(R-S) - p_1(P-S)}{R-P} \geq 1 \qquad p_3 = \frac{-(1-p_1)(T-P) - p_4(T-R)}{R-P} \leq 0$$

or never coop

Only one feasible point: $(1,1,0,0)$ which is always cooperate. So X cannot unilaterally set his score.

"Extortion" Suppose $\tilde{p} = \phi[(\vec{S}_X - P\mathbb{1}) - \chi(\vec{S}_Y - P\mathbb{1})]$. Then this means $\phi[(s_X-P) - \chi(s_Y-P)] = 0 \Rightarrow (s_X-P) = \chi(s_Y-P) \Rightarrow$

X can make his gain above mutual defection (P) χ times greater than Y's !!

Solving the equations for p_i:

$$p_1 = 1 - \phi(\chi-1)\frac{R-P}{P-S}, \quad p_2 = 1 - \phi\left(1+\chi\frac{T-P}{P-S}\right), \quad p_3 = \phi\left(\chi + \frac{T-P}{P-S}\right), \quad p_4 = 0$$

so feasible strategies exist for any χ + small ϕ, eg

$$0 \leq \phi \leq \frac{(P-S)}{(P-S)+\chi(T-P)}$$

Clearly X's score depends on Y's strategy + both are maximized when Y cooperates! $(1,1,1,1)$ in which case

$$s_X = \frac{P(T-R) + \chi[R(T-S) - P(T-R)]}{(T-R) + \chi(R-S)} \quad (\chi = 1 \Rightarrow s_X = s_Y = R)$$

E.G. $(5,3,1,0)$,

$\tilde{p} = [1-2\phi(\chi-1), 1-\phi(4\chi+1), \phi(\chi+4), 0]$, ok for $0 \leq \phi \leq (4\chi+1)^{-1}$

$s_X = \dfrac{2+13\chi}{2+3\chi} \quad s_Y = \dfrac{12+3\chi}{2+3\chi}$. Mutual coop $\Rightarrow 3$ $\quad s_X > 3$ if $\chi > 1$

$s_Y < 3$!!

As $\chi \to \infty$, $s_X \to \frac{13}{3}$ + $s_Y \to 1$ so Y has no reason to cooperate so X should not get too greedy!

Say $\chi = 3$ + ϕ is midpoint of feasibility $\vec{p} = (\frac{11}{13}, \frac{1}{2}, \frac{7}{26}, 0)$

$s_X \simeq 3.73$, $s_Y \simeq 1.91$. Note $\chi = 1$, $\phi = 1/5 \to$ TFT $(1,0,1,0)$

Spatial games in a continuum

This is my version of a spatial game. We will later look at some others. Lets consider a two strategy game with a 2x2 payoff matrix $\begin{pmatrix} R & S \\ T & P \end{pmatrix}$ (not necessarily PD)

Let $u(x,t)$ be density of players using strategy C so that $v(x,t) = 1 - u(x,t)$ is density of players using strategy D. Now here is how I do the spatial game. Let me look at a weighted nbhd around x & compute the fitness if I played C:

$$f_C(x) = R \int w(y)\, u(x-y,t)\, dy + S \int w(y)\, v(x-y,t)\, dy$$

and fitness if I played D:

$$f_D(x) = T \int w(y)\, u(x-y,t)\, dy + P \int w(y)\, v(x-y,t)\, dy$$

Here $w(y)$ is a symmetric weighting function, say exponential or Gaussian.

I will go from $C \to D$ at a rate β & from $D \to C$ at a rate α where

$$\alpha = H(f_C - f_D), \quad \beta = H(f_D - f_C)$$ where H is some monotonic non negative function, eg $1/(1+e^{-\gamma x})$

so, eg if $f_C > f_D$, then α will be bigger than β & there will be a loss of D & gain of C at x. $\therefore \frac{\partial u}{\partial t} = \alpha v - \beta u = \alpha(1-u) - \beta u$

$$\alpha = H(f_C(x) - f_D(x)), \quad \beta = H(f_D(x) - f_C(x))$$

It is now clear how to do this for m strategy games

Let $A = (a_{ij})$ be payoff matrix. Let $u_i(x,t)$ be density playing strategy i. Then

$$f_i(x,t) = \int w(y) \sum_{j=1}^{n} a_{ij} u_j(x-y,t)\, dy$$

$$\alpha_{ij} = H(f_i - f_j) = \text{rate } j \text{ to } i$$

$$\frac{\partial u_i}{\partial t} = \sum_{j=1}^{n} \alpha_{ij} u_j \qquad \alpha_{ij} = H(f_i(x,t) - f_j(x,t))$$

This is not like replicator dynamics, which we will look at later. It is a <u>new</u> model. Lets look at equilibria for $n=2$

$$\frac{du}{dt} = \alpha(1-u) - \beta u = 0 \Rightarrow u = \frac{\alpha}{\alpha+\beta}$$

$$\alpha = H(a_{11}u + a_{12}(1-u) - a_{21}u - a_{22}(1-u))$$

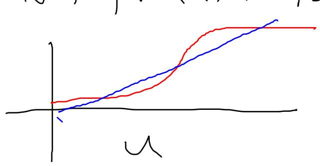
$$= H(a_{12} - a_{22} + (a_{11} - a_{21} + a_{22} - a_{12})u]$$

$$= H(c - (d-c)u)$$

$\beta = H(-c + (d-c)u)$ so as expected all that matters is the off diagonal terms, So WLOG we can Assume $A = \begin{bmatrix} 0 & c \\ d & 0 \end{bmatrix}$

For PD $d > 0$, $c < 0$. For bistable game: $c, d < 0$

So lets try that: Take $c = -1$, $d = -4$ for example + $\gamma = 10$



3 roots bistable