MATH BIOL 3380: Pattern formation

Introduction:

Many examples of PF in nature: fluids, animal coats, sensory maps, bodyplans, seashells, etc. But PF can be more general than this in the sense of what I will call symmetry breaking. Patterns can be more abstract and have aspects of time in them (as in the switch from synchrony to say anti-phase). This course will include some of these examples. I will use several references & papers, these will be introduced as needed. But here are two very good sources:

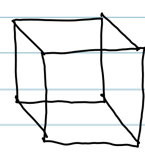
- ≈ 300 page review paper (Rev Mod Phys 1993) Cross & Hohenberg (link to PDF)
- J.D. Murray Math Biol Vol 2 (original book is also excellent & available in paperback)

Here are a few HW experiments to do:

1. Place a small amount of oil in a pan (enough to cover bottom with a few mm's oil). Place oil on burner & turn on. Look at the surface & report what you see

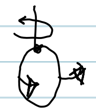
2. Press palms against eyes. At first you will see random sparks but after awhile, you will see more interesting structure

3. Look at time transitions!



 vs 

4. use hula-hoop or create a big hoop out of stiff wire. Tie a string to it & suspend from ceiling. Rotate at various speeds.



At sufficient speed, symmetry breaking instability & it will fail to rotate along vertical line

5. Come up with other examples of patterns arising from some local interactions?

Example 1: competitive network

$$(1) \quad \begin{aligned} \dot{x}_1 &= -x_1 + f(I - g x_2) \\ \dot{x}_2 &= -x_2 + f(I - g x_1) \end{aligned} \qquad f'(x) > 0, \; f > 0, \quad I, g > 0$$

This example will essentially show you the way to think of almost all pattern formation problems.

Note the symmetry. x_1 & x_2 can be exchanged with no change in the equations. This means that there will be a symmetric ~~solution~~ equilibrium to this equation, $x_1 = x_2 = u$ with

$$-u + f(I - gu) = 0 \quad (2).$$

$\boxed{\text{HW}}$ Given f is monotone increasing, $g > 0$, $I > 0$, $f > 0$ ~~suppose~~ prove there is a unique equilibrium point (Hint: Use intermediate value Thm)

Let u be the fixed point of the equation (2). Let's look at stability of this F.P.

Linearizing (1) + letting $\alpha = f'(I - gu)$ we get:

$$\dot{y}_1 = -y_1 - \alpha g y_2 \quad , \quad \dot{y}_2 = -y_2 - \alpha g y_1$$

The matrix is:
$$\begin{pmatrix} -1 & -\alpha g \\ -\alpha g & -1 \end{pmatrix} \qquad (3)$$

We could easily get eigenvalues and eigenvectors by direct computation. However, (3) has symmetry (It is an example of a circulant matrix; more on this later). Thus, the eigenvectors are straightforward:

$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ symmetric $\quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ antisymmetric

with corresponding eigenvalues

$(-1-\alpha g)$ + $(-1+\alpha g)$

If $-1+\alpha g > 0$ then $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ will start to grow exponentially in linear system. In non-linear system, we expect perturbations of symmetric state to grow. Since $(1,1)^T$ perturbations always decay while $(1,-1)^T$ can grow, This is the canonical example of a symmetry breaking instability.

KEY Feature: NEGATIVE - NEGATIVE interactions. (This is called "lateral inhibition")

We will examine the nonlinear aspects shortly

Example 2 PDE example.

$$\frac{\partial u}{\partial t} = -u + u_{xxxx} - \alpha u_{xx}, \quad \alpha \geq 0$$

$$u_x(0) = u_x(\pi) = u_{xxx}(0) = u_{xxx}(\pi) = 0$$

Clearly $u=0$ is always a solution. Since there is no time dependence in equation (it is time translation invariant), solutions to this linear PDE have the form

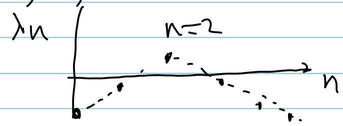
$$u(x,t) = e^{\lambda t}\varphi(x)$$

with

$$\lambda\varphi = -\varphi + \varphi_{xxxx} - \alpha\varphi_{xx}$$

Claim $\varphi(x) = \psi_n(x) = \cos nx$ will satisfy ODE and BCs when $n = 0, 1, 2, \ldots$

Proof substitute.



$$\lambda_n = -1 - n^4 + \alpha n^2$$

If $\lambda_n > 0$ for some n, then solutions will grow like $e^{\lambda_n t}\cos nx$ for that n. Idea is that the biggest λ_n will grow fastest.

☆ HW Treating n as a continuous variable, find maximum of λ_n for given value of α & determine the minimum value of α such that λ_n^{max} is non-negative.

Return to Equation (1). Lets treat g as a parameter and suppose at g^*,

$$-1 + g^* f'(I - g^* u^*) = 0$$

(Note this is not easy to find analytically but we will assume that the point exists)

Amplitude expansion In order to get to the simplest type of equation, it is necessary to introduce amplitude equations (mathematically, these are called Normal forms)

Write $\cancel{(x_1, x_2)^T = X}$ $u^* = (u,u)^T$ as a vector. Let us write:

$$\tau = \varepsilon^2 t, \qquad \cancel{X = u^* + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots}$$

$$x_1 = u + \varepsilon y_1 + \varepsilon^2 z_1 + \varepsilon^3 w_1 + \cdots$$

$$g = g^* + \varepsilon^2 \hat{g} \qquad x_2 = u + \varepsilon y_2 + \varepsilon^2 z_2 + \varepsilon^3 w_2 + \cdots$$

y_j, z_j etc function of τ

(How did I know ε^2 for τ, g? → I did not have to know this, but used ~~a~~ prior knowledge to make it simpler.)

<u>Aside</u>: The second most important theorem in applied mathematics is the Fredholm alternative which we now review.

Let A be a linear operator (it could be matrix multiplication or differentiation or even integration.)

Let $\langle , \rangle$ be an inner product.

The adjoint is A^* : $\langle x, Ay \rangle = \langle A^* x, y \rangle$

Example (1) A is a matrix Then $A^* = A^T$

(2) $Lu = \frac{du}{dt} + Au \qquad u(0) = u(1) \leftarrow$ B.C.s

$$\langle u, v \rangle = \int_0^1 u \cdot v \, dt$$

$$\langle v, Lu \rangle = \int_0^1 v \cdot \left(\frac{du}{dt} + Au\right) dt = \int_0^1 v \cdot \frac{du}{dt} + \int_0^1 v \cdot Au \, dt$$

$$= \cancel{uv\Big|_0^1}^{\,0} - \int_0^1 \frac{dv}{dt} \cdot u \, dt + \int_0^1 A^T v \cdot u \, dt$$

$$= \int_0^1 \left(-\frac{dv}{dt} + A^T v\right) \cdot u \, dt \qquad \text{so } L^* v = -\frac{dv}{dt} + A^T v$$

☆ $\boxed{HW}$ (1) $Lu = \int_0^1 k(x,y)\, u(y)\, dy \qquad \langle u, v \rangle = \int_0^1 u \cdot v \, dt$

Find L^*

(2) $Lu = D\frac{d^2u}{dx^2} + Au \qquad A$ is $n \times n \quad D$ is $n \times n$

$\frac{du}{dx}(0) = \frac{du}{dx}(1) = 0 \quad$ BC's

Find L^*

$\boxed{\text{Fredholm ALTERNATIVE}}$ $\qquad R_L = N_{L^*}^T$ or

$Lx = b$ has a soln iff $\langle v^*, b \rangle = 0 \quad L^* v^* = 0$

All of perturbation & bifurcation Theory depends on This notion!

Let's get back to our equation (1). We assume $f(x)$ can be expanded around $I - g^* u^*$. Let

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{6}(x-x_0)^3$$

<u>Define</u> $u^* = f(I - g^* u^*)$ $\quad \alpha := f'(I - g^* u^*)$

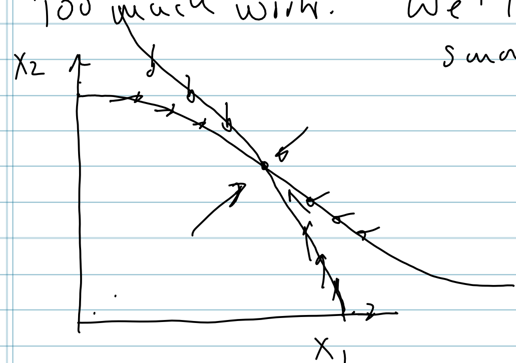
$\alpha_2 := f''(I - g^* u^*)/2$, $\quad \alpha_3 := f'''(I - g^* u^*)/6$

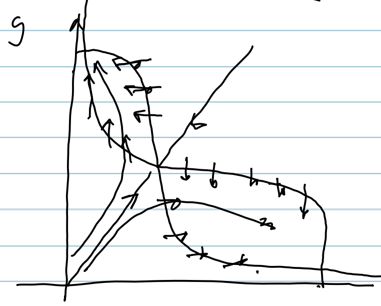
$$f(I - g x_1) = f(I - [g^* + \varepsilon^2 \tilde{g}][u + \varepsilon y_1 + \varepsilon^2 z_1 + \varepsilon^3 \cdot W_1 + \cdots])$$

$$= u^* - \varepsilon \alpha g^* y_1 - \varepsilon^2 \alpha g^* z_1 - \varepsilon^3 g^* W_1 + \varepsilon^2 \alpha_2 g^{*2} y_1^2$$

Too much work. We'll us phase plane

small g





(HW) analyze the following model:

$$\dot{x}_1 = -x_1 + f(I - g x_2 - z_1)$$

$$\dot{x}_2 = -x_2 + f(I - g x_1 - z_2)$$

$$\tau \frac{dz_1}{dt} = -z_1 + \beta x_1 \qquad \tau \frac{dz_2}{dt} = -z_2 + \beta x_2$$

HINT:

STEP 1: Show there is a symmetric equilibrium $x_1 = x_2 = u$, $z_1 = z_2 = \beta u$

STEP 2: Linearize about this F.P. & show matrix can be written as

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ w_1 \\ w_2 \end{pmatrix} = \begin{bmatrix} -1 & -\alpha g & -\alpha & 0 \\ -\alpha g & -1 & 0 & -\alpha \\ \beta/\tau & 0 & -\frac{1}{\tau} & 0 \\ 0 & \beta/\tau & 0 & -\frac{1}{\tau} \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ w_1 \\ w_2 \end{pmatrix}$$

STEP 3 Use symmetry arguments to reduce this to 2 2x2 matrices

(Meta hint:
suppose $y_1 = y_2$ & $w_1 = w_2$
or suppose $y_1 = -y_2$ & $w_1 = -w_2$
to get 2 redundant ~~set~~ pairs)

STEP 4 Recall for 2x2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that eigenvalues have negative real parts if & only if $a+d<0$ & $ad-bc>0$

STEP 5 Show the $y_1 = y_2$, $w_1 = w_2$ system is always stable. Show that $y_1 = -y_2$, $w_1 = -w_2$ can be unstable in 2 different ways (Trace >0 or det <0)

Finally solve this for following:

$\tau = 10$, $I = 4$, $\beta = 2$, $f(x) = 1/(1+\exp(-x))$

$g = 2$, $g = 5$ $g = 8$

Can you compute bifurcation diagram as a function of g? (Hard, ~~on~~ or easy dep on what you know!)