General 2 unit problem

Let $X_j \in \mathbb{R}^n$ $\quad f_j : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$\dot{X}_1 = f_1(X_1, X_2) \qquad \dot{X}_2 = f_2(X_1, X_2)$

Want same eqns if $X_1 \mapsto X_2$

$\dot{X}_2 = f_1(X_2, X_1) \qquad \dot{X}_1 = f_2(X_2, X_1)$

$\Leftrightarrow f_1(X_2, X_1) = f_2(X_1, X_2)$

So:

$\dot{X}_1 = f(X_1, X_2)$, $\dot{X}_2 = f(X_2, X_1)$

Let $f(u,u) = 0$ + Let

$A = \partial_1 f(u,u)$, $B = \partial_2 f(u,u)$

$\dot{y}_1 = A y_1 + B y_2$, $\dot{y}_2 = A y_2 + B y_1$

To see symmetry:

Let $y_d = y_1 - y_2$, $y_s = y_1 + y_2$

$$\begin{aligned} \dot{y}_d = \cancel{A}\dot{y}_1 - \dot{y}_2 &= A y_1 + B y_2 - (A y_2 + B y_1) \\ &= A(y_1 - y_2) + B(y_2 - y_1) \\ &= (A - B) y_d \end{aligned}$$

$\dot{y}_s = (A+B) y_s$ $\qquad$ so reduce to 2 $n \times n$ instead of 1 $2n \times 2n$!

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Thus need only look at spectrum of smaller system.

Example: The Brusselator chemical reaction

This is the canonical reaction ~~equation~~ for oscillations + pattern formation.

$A \to X \qquad B + X \to Y \qquad 2X + Y \to 3X$

$X \to E$ all have rate 1 for simplicity

A B are fixed E doesnt matter

4 reactions $R_1 = A \qquad R_2 = BX \qquad R_3 = X^2 Y$

$R_4 = X$

$$\dot{X} = R_1 - R_2 - 2R_3 + 3R_3 - R_4$$

$$\dot{Y} = R_2 - R_3$$

So

$$\frac{dx}{dt} = A-(B+1)x + x^2 y \equiv f(x,y) \qquad \frac{dy}{dt} = Bx - x^2 y \equiv g(x,y)$$

$\boxed{X_1 Y_1} \leftrightarrow \boxed{\begin{matrix}X_2\\Y_2\end{matrix}}$ $\quad X_2 \xrightarrow{D_X} X_1$ +
$X_1 \xrightarrow{2 D_X} X_2$ sim for Y

$$\frac{dx_1}{dt} = f(x_1,y_1) + D_x(x_2 - x_1) \equiv F_1\left(\begin{bmatrix}x_1\\y_1\end{bmatrix},\begin{bmatrix}x_2\\y_2\end{bmatrix}\right)$$

$$\frac{dy_1}{dt} = g(x_1,y_1) + D_y(y_2 - y_1)$$

Note $F_1(\vec{x}_1, \vec{x}_2) = F_2(\vec{x}_2, \vec{x}_1)$ so symmetry holds.

Fixed points $x_1 = x_2 = u$ $\quad y_1 = y_2 = v$

$$A-(B+1)u + u^2 v = 0 \qquad Bu - u^2 v = 0$$

Add together to get $A-u=0 \Rightarrow u = A$ + then $v = B/A$.

With abuse of notation, let me call these a, b instead.

Matrix $A = \begin{bmatrix} \frac{\partial f}{\partial x} - D_x & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} - D_y \end{bmatrix}$ $\quad B = \begin{bmatrix} D_x & 0 \\ 0 & D_y \end{bmatrix}$

So... $\frac{\partial f}{\partial x} = -(b+1) + 2xy \quad \frac{\partial f}{\partial y} = x^2$

$\frac{\partial g}{\partial x} = b - 2xy \quad \frac{\partial g}{\partial y} = -x^2$

so

$$A = \begin{bmatrix} b-1-D_x & a^2 \\ -b & -a^2-D_y \end{bmatrix} \qquad B = \begin{bmatrix} D_x & 0 \\ 0 & D_y \end{bmatrix}$$

$$A+B = \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix} \equiv M_S \qquad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A-B = \begin{bmatrix} b-1-2D_x & a^2 \\ -b & -a^2-2D_y \end{bmatrix} \equiv M_D \qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\boxed{\mathrm{Tr}\, M_S = b-a^2-1 \qquad \mathrm{Det}\, M_S = a^2 > 0}$$

$\mathrm{Tr}\, M_D = b-a^2-1-2(D_x+D_y)$

$\mathrm{Det}\, M_D = a^2 - 2[D_y(b-1) - D_x a^2] + 4 D_x D_y$

$\mathrm{Tr}\, M_S < 0 \Rightarrow \mathrm{Tr}\, M_D < 0$

Can M_D ever be negative?

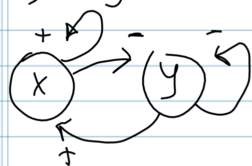
What does this mean?

Assume $\mathrm{Tr}\, M_S < 0 \Rightarrow a^2 > b-1$

Necessary condition for $\mathrm{Det}\ M_D < 0$ D

$D_y(b-1) - D_x a^2 > 0 \Rightarrow \boxed{b-1>0}$ +

$\Rightarrow D_y > D_x \frac{a^2}{b-1} > D_x$ since $a^2 > b-1$



Diffusion of X < Diff Y

Suppose $D_x = 0$ Then choose D_y large enough so that

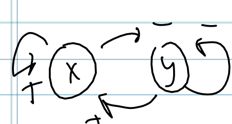
$$a^2 - 2D_y(b-1) < 0$$

Thus for small enough D_x can destabilize the uniform state!

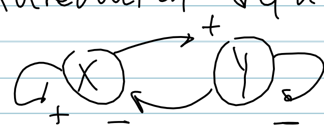
More generally for 2 dimensional coupled system with Diffusion:

$$A = \begin{bmatrix} \alpha - D_x & \beta \\ \gamma & \delta - D_y \end{bmatrix} \qquad B = \begin{bmatrix} D_x & 0 \\ 0 & D_y \end{bmatrix}$$

Two classes of interaction dynamics:



Positive feedback



Act-inhibitor

$\boxed{HW}$ Prove $\begin{bmatrix} - & + \\ - & + \end{bmatrix}$ can never have Diffusive patt formation!

(Note $\begin{pmatrix} - & - \\ + & + \end{pmatrix}$ is same since diagonal product is all that comes out)

(6)

HW Numerically explore Brusselator for:

HW Let $a = 0.6$, $b = 1.25$.

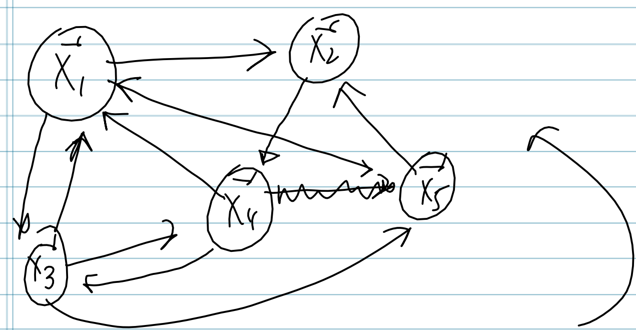
- Find a curve in (D_x, D_y) st $\det M_D = 0$

- Pick (D_x, D_y) st $\det M_D < 0$ & numerically solve ODE & show that there are equilibria st. $\vec{X}_1 \neq \vec{X}_2$

We have studied the case of a ~~single~~ pair of coupled cells.

How much can this be generalized?

Lets say we have N "cells"



and coupling between them (that is the same form of coupling but not necessarily to any element)

Furthermore, suppose that each cell has same number of arrows going into it

Since coupling is the same to everyone and have same # arrows, this means that there is symmetry in the sense that if we start with $\vec{X}_1 = \vec{X}_2 = \cdots = X_n$ then this is invariant!

The homogeneous solution is a solution

Lets look at the linearization of this system

Let A be linearizate matrix for self
& B " " " for coupling

In above picture

$$
\begin{array}{c c}
 & \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} &
\begin{bmatrix}
A & 0 & B & 0 & B \\
B & A & 0 & 0 & B \\
B & 0 & A & B & 0 \\
0 & B & B & A & 0 \\
B & 0 & B & B & A
\end{bmatrix}
\end{array}
= M
$$

Let $Q = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix}$

This is called $\boxed{\text{Adjacency matrix}}$

(can write Big matrix using Tensors
$A \otimes I + B \otimes Q$
but lets just keep it simple)

Let $\vec{e}$ be eigenvector of Q and λ be the corresponding eigenvalue

1. Note that $\lambda = 2$ and $\vec{e} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ 1)

The symmetric eigenvalue.

2. Note $A \to A - 2B$ for diffusive type coupling

Because of this symmetry, we can reduce this ~~nxn~~ $5n \times 5n$ system to 5 $n \times n$ systems.

Claim eigenvalues of M are eigenvalues of $A + \lambda_j B$ $j = 1, \ldots, 5$

Proof ~~Claim~~ Let $\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_5 \end{bmatrix}$ be eigen components of eigenvector with eigenvalue λ of Q

$Q\vec{e} = \lambda \vec{e}$.

Let $\vec{v} = \begin{bmatrix} \vec{u} e_1 \\ \vec{u} e_2 \\ \vdots \\ \vec{u} e_5 \end{bmatrix}$ $\vec{u} \in \mathbb{R}^n$

$M\vec{v} = \begin{bmatrix} A\vec{u}e_1 + B\vec{u}e_3 + B\vec{u}e_5 \\ B\vec{u}e_1 + A\vec{u}e_2 + B\vec{u}e_5 \\ \vdots \end{bmatrix}$

Note $e_3 + e_5 = \lambda e_1$
$e_1 + e_5 = \lambda e_2$ since eigenvector of Q
$\vdots$

Thus

$$\vec{M}\vec{V} = \begin{bmatrix} (A+\lambda B)\vec{u} \\ \vdots \\ (A+\lambda B)\vec{u} \end{bmatrix}$$

So eigenvalues of $\vec{M}$ are found by solving $(A+\lambda_j B)\vec{u}$ eigenvalue problem

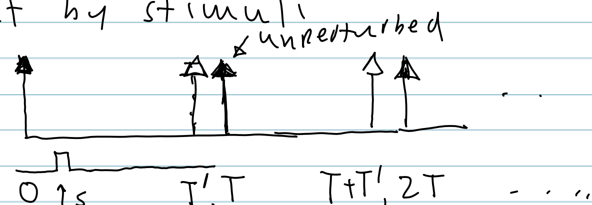
$j = 1, \dots, S$

Nothing special about all the "1's" in Q could be arbitrary

Application: pulse coupled oscillators

Consider a firefly. He flashes periodically with frequency ω. With his flash, he is able to communicate to other fireflies

<u>Aside</u> Limit cycle oscillators can be reset by stimuli



Define

$$\Delta(s) := T - T'(s)$$

called phase resetting curve.

~~Let p(θ)~~ Let θ_j define the phase of jth insect (defined mod T)

Let $P(\theta)$ be the pulse of light emitted by the firefly.

$$\frac{d\theta_j}{dt} = \omega_j + \left[\sum_{j=1}^{N} \cancel{P_k}\, g_{jk} P(\theta_k)\right] \Delta(\theta_j)$$

Firefly frequency — input from other FF's — response function

Simplification: $\omega_j = \omega \ \forall j$ (identical)

$\sum_{j=1}^{N} g_{jk} = g \quad \forall j \qquad$ Let $T = 2\pi$

Synchrony: $\theta_j(t) = \theta(t)$ h

$$\frac{d\theta}{dt} = \omega + g\, P(\theta)\, \Delta(\theta)$$

Assume $\omega + g \cancel{*} P(t) \Delta(\theta) > 0$ for all θ

This means $\dot{\theta} > 0$ and period is:

$$\int_0^{2\pi} \frac{d\theta}{\omega + g P(\theta) \Delta(\theta)} = P_0 = \text{period}$$

Is synchrony stable?

Lets look at the linearization around $\theta_j(t) = \Theta(t)$

$$\dot{Y}_j = \Delta(\Theta(t)) \sum_{k=1}^{N} g_{jk}\, P'(\Theta(t))\, Y_k(t) + g\,\Delta'(\Theta(t))\, P(\Theta(t))\, Y_j(t)$$

Using the theory above, let λ_ℓ be an eigenvalue of the matrix G. Note $\lambda_1 = g$.

$\vec{Y} = u(t)\,\vec{e}$ +

$$\dot{u} = \left[\Delta(\Theta(t))\,\lambda_\ell\, P'(\theta(t)) + g\,\Delta'(\theta(t))\,P(\theta(t))\right] u$$

$$\equiv \alpha(t)\, u(t)$$

If $\int_0^{P_0} \alpha(s)\,ds > 0$ then unstable since

$$u(t) = u(0)\, e^{\int_0^t \alpha(s)\,ds}$$

Let us do a little calculation.

$$\dot{\Theta} = \omega + g\, P(\theta)\,\Delta(\theta)$$

$$\ddot{\theta} = \left[g\, P'(\theta)\,\Delta(\theta) + g\, P(\theta)\,\Delta'(\theta)\right]\dot{\theta}$$

Since $\dot{\theta}$ is periodic, we have

$$g\int_0^{P_0} P'(\theta(t))\,\Delta(\theta(t)) + P(\theta(t))\,\Delta'(\theta(t))\,dt = 0\,!$$

So this means

$$\int_0^{P_0} P'(\theta(t))\,\Delta(\theta(t))\,dt = -\int_0^{P_0} P(\theta(t))\,\Delta'(\theta(t))\,dt$$

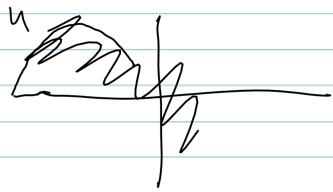
Thus

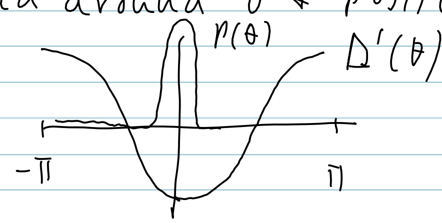
$$\boxed{\int_0^{p_0} \alpha(s)\,ds = (g-\lambda_\ell)\int_0^{p_0} p(\theta(t))\,\Delta'(\theta(t))\,dt}$$

Suppose all entries of G are positive. Then F.P. $\Rightarrow$ $\mathrm{Re}[g-\lambda_\ell] > 0$ since $|g| > |\lambda_\ell|$ $(\ell \neq 1)$ & $g > 0$ is real.

If $\int_0^{p_0} p(\theta(t))\,\Delta'(\theta(t))\,dt < 0$ then synch is stable. For Fireflies $\Delta(\theta) \sim -K \sin\theta$ so $\Delta'(\theta) = -K\cos\theta$

If $p(\theta)$ is centered around 0 & positive then





integral will be negative!