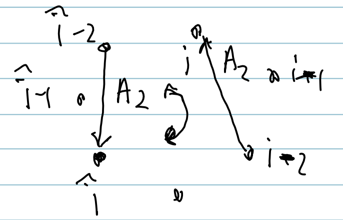
Some special cases:

Cyclic systems $\quad F: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$$\dot{x}_i = F\Big(X_i, \sum_{j=0}^{N-1} A_j X_{i-j}\Big) \qquad i = 0,\dots,N-1, \quad \underline{i-j \text{ mod } N}$$

A_j are $n \times n$ matrices

Note that you could regard the geometry as a ring and your interaction with unit k depends only on $k-j$, your "distance" from it!



These are often used in biology as simplification of the real geometry.

They are homogeneous in the sense that interactions depend on difference of location (index) & not on absolute location.

Let $\bar{X}$ solve $\quad 0 = F\Big(\bar{X}, \bar{X} \overbrace{\sum_{j=0}^{N-1} A_j}^{=\tilde{A}}\Big)$

Linearize (~~Let's just forget first company~~

$B = F_1(\bar{X}, \hat{A}\bar{X})$

$$\dot{Y}_i = BY_i + \sum_{j=0}^{N-1} C_j Y_{i-j} \qquad C_j = F_2(\bar{X}, \hat{A}\bar{X})A_j$$

Claim $\vec{Y}_i = e^{\lambda t} e^{\frac{2\pi\sqrt{-1}\, i l}{N-1}} \vec{V}$ is a solution

to this:

$$\lambda e^{\lambda t} e^{\frac{2\pi\sqrt{-1}\, i l}{N-1}} \vec{V} = B e^{\lambda t} e^{\frac{2\pi\sqrt{-1}\, i l}{N-1}} \vec{V} + \sum_{j=0}^{N-1} C_j e^{\frac{2\pi\sqrt{-1}\,(i-j)l}{N-1}} \cdot \vec{V}$$

$$\Rightarrow \lambda \vec{V} = B\vec{V} + \sum_{j=0}^{N-1} C_j e^{\frac{-2\pi\sqrt{-1}\, jl}{N}} \vec{V}$$

For each $l = 0, 1, \ldots, N-1$ this is an $n \times n$ system so get N $n \times n$ instead of 1 $(nN \times nN)$!

$$\boxed{\sum_{j=0}^{N-1} e^{\frac{-2\pi\sqrt{-1}\, jl}{N}} C_j \text{ is the discrete Fourier Transform of } C_j}$$

Key idea is as before: Need to use homogeneity + eigenproperties of the adjacency matrix.

Other examples 2D lattice with periodic B.C.s

Nearest Neighbor on a line

etc etc

Lets apply this to a ring of Brusselators with NN coupling.

$\dot{x}_i = f(x_i, y_i) + D_x(x_{i-1} - 2x_i + x_{i+1})$ $\quad i = 0, \dots, N-1$ all mod N

$\dot{y}_i = g(x_i, y_i) + D_y(y_{i-1} - 2y_i + y_{i+1})$

Linearization

$$\begin{pmatrix} \dot{u}_i \\ \dot{v}_i \end{pmatrix} = A \begin{pmatrix} u_i \\ v_i \end{pmatrix} + \begin{pmatrix} D_x & 0 \\ 0 & D_y \end{pmatrix} \begin{bmatrix} u_{i+1} - 2u_i + u_{i-1} \\ v_{i+1} - 2v_i + v_{i-1} \end{bmatrix}$$

$$e^{\frac{2\pi\sqrt{-1}\,l(i-1)}{N}} - 2 + e^{\frac{2\pi\sqrt{-1}\,l(i+1)}{N}} \leftarrow \text{cyclic sum part}$$

$$= 2\left[\cos\frac{2\pi l}{N} - 1\right] \triangleq \mu_l \quad (\mu_0 = 0)$$

(Note $N=2$ get $0, -4$ which is what we had before - well actually $0, -2$ but Here we counted twice since $i-1 = i+1$)

Letting $A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$

we have a series of matrices

$$M_l = \begin{bmatrix} \alpha + D_x \mu_l & \beta \\ \gamma & \delta + D_y \mu_l \end{bmatrix}$$

$\mathrm{Tr}\, M_0 \geq \mathrm{Tr}\, M_l \quad \forall l$ since $\mu_l \leq \mu_0$

$$\det M_l = \det M_0 + D_x D_y \mu_l^2 + \mu_l[\alpha D_y + \delta D_x]$$

For $N \gg 1$ and l small $\mu_l \simeq -\frac{4\pi^2 l^2}{N^2}$

Try some sims for this system!

ample 2 Global Inhibition

A classic example combines local interactions with global negative feedback:

$$\dot{x}_i = -x_i + f\Big(\alpha\underbrace{(x_{i+1} + x_i + x_{i-1}}_{\text{Averaging 3 neighbors}} - \beta\sum_{j=0}^{N-1} x_j\Big) \quad \alpha, \beta > 0$$

$f' > 0$

Linearize this about equilibrium $x_i = \hat{u}$

$$\dot{y}_i = -y_i + a(y_{i+1} + y_i + y_{i-1}) - b\sum_{j=0}^{N-1} y_j$$

$a, b > 0$

$$y_i = e^{\lambda_\ell t} e^{\frac{2\pi\sqrt{-1}\, i\ell}{N}}$$

$\lambda_0 = -1 + 3a - Nb \qquad \lambda_\ell = -1 + a\left[1 + 2\cos\frac{2\pi\ell}{N}\right]$

Note that $3a > a\left[1 + 2\cos\frac{2\pi\ell}{N}\right]$ $\ell > 0$

but $-Nb$ can dominate make $\lambda_0 < 0$

$\lambda_1 = \lambda_{N-1}$ is larger than all other eigenvalues.

Thus the most unstable mode will be $\lambda_1 = \lambda_{N-1}$ and the pattern will always be one full wavelength, no matter what the domain size (ring size, N)!

This is a mechanism to get size invariant patterns.

The point is that what sets the pattern is the interaction between negative & positive feedback. As an example you should consider instead:

$$-\beta \sum_{j=-m}^{m} x_{i+j}$$

The dominant eigenvalue depends on m.

Need to sum $\displaystyle\sum_{j=-m}^{m} e^{\frac{2\pi\sqrt{-1}\, j\ell}{N}}$

which you should be able to do..

We will get to ~~????~~ nonlinear systems & bifurcation shortly but want to turn to some ~~linear~~ continuum space examples. In the second half of the course, we will examine many other examples of pattern formation

Lets first consider a general reaction diffusion equation in $\mathbb{R}^n$ on a one dimensional domain of length L with different boundary conditions

We will just do the ~~scalar~~ planar RD equation

$$\frac{\partial u}{\partial t} = f(u,v) + D_u \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L$$
$$\frac{\partial v}{\partial t} = g(u,v) + D_v \frac{\partial^2 v}{\partial x^2}$$

BCS $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0$ at $x = 0, L$ $\boxed{\text{Neumann or No flux BCS}}$

$u(0,t) = u(L,t)$
$v(0,t) = v(L,t)$ $\boxed{\text{Periodic}}$

$u = \bar{u}$, $v = \bar{v}$ at $x = 0, L$ $\boxed{\text{Fixed or Dirichlet}}$

We assume that $f(\bar{u},\bar{v}) = g(\bar{u},\bar{v}) = 0$

Linear stability Theory.

$u(x,t) = \bar{u} + w(x,t)$ $v(x,t) = \bar{v} + z(x,t)$

Let $\vec{F} = \begin{pmatrix} f \\ g \end{pmatrix}$ $\partial F\big|_{\bar{u},\bar{v}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv M_0$

Assume $\det M_0 > 0$ $\operatorname{Tr} M_0 < 0$

$$\begin{pmatrix} w \\ z \end{pmatrix}_t = M_0 \begin{bmatrix} w \\ z \end{bmatrix} + \begin{bmatrix} D_u & 0 \\ 0 & D_v \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}_{xx} \qquad (\star)$$

Lets look at three different scalar eigenvalue problems

(D) $\lambda \phi = \phi_{xx}$ $\phi(0) = \phi(L) = 0$
(N) $\lambda \phi = \phi_{xx}$ $\phi_x(0) = \phi_x(L) = 0$
(P) $\lambda \phi = \phi_{xx}$ $\phi(0) = \phi(L)$, $\phi_x(0) = \phi_x(L)$

Standard Theory of ODEs

(D) $\phi_n(x) = \sin\frac{n\pi x}{L}$ $\quad \lambda_n = -\frac{n^2\pi^2}{L^2}$ $\qquad$ (N) $\phi_n(x) = \cos\frac{n\pi x}{L}$

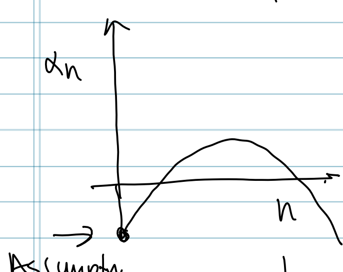
(P) $\phi_n(x) = \sin\frac{2\pi n x}{L}$, $\cos\frac{2\pi n x}{L}$ $\qquad \lambda_n = -\frac{4n^2\pi^2}{L^2}$

Thus we can use this idea to solve the linear equation (★) under the three assumptions on BCs.

$$\begin{pmatrix}W\\Z\end{pmatrix} = e^{\nu t}\phi_n(x)\begin{bmatrix}p\\q\end{bmatrix} \quad \text{with}$$

$$\nu\begin{bmatrix}p\\q\end{bmatrix} = M_0\begin{bmatrix}p\\q\end{bmatrix} + \lambda_n\begin{bmatrix}D_x & 0\\0 & D_y\end{bmatrix}\begin{bmatrix}p\\q\end{bmatrix}$$

Thus we get eigenvalues ν_n and plot these as a function of n. If write $\mathrm{Re}\,\nu_n = \alpha_n$



Called "Dispersion relation" often

Want some maximum at $n \neq 0$. Note $n \to -n$ symmetry due to isotropic diffusion (Diffusion is same in both directions)

These ideas work for any domain — we need to know the eigenfunctions for the particular domain.

For any kind of system of equations with a nice linearization and regular enough domain, we can find eigenvalues for the linear equations & explore pattern formation

one class of models we will look at later are integral operators.

~~One~~ Crucial property we will want is some sort of homogeneity. Otherwise cannot get eigenvalues analytically

Let's take as an example the convolution equation:

$$\lambda u(x) = \int_0^L k(x-y)u(y)dy \qquad x \in [0,L]$$

where we suppose that $k(x+L) = k(x)$ so k is a periodic "kernel"

The eigenvalues of this operator are easy to find. Try

$$u_n(x) = e^{\frac{2\pi i n x}{L}}$$

$$\lambda u_n(x) = \lambda e^{\frac{2\pi i n x}{L}} = \int_0^L k(x-y) e^{\frac{2\pi i n y}{L}} dy$$

Let $x' = x-y$ so $dx' = -dy$ and we get

$$\lambda_n e^{\frac{2\pi i n x}{L}} = \int_{x-L}^{x} k(x') e^{\frac{2\pi i n x}{L}} e^{\frac{-2\pi i n x'}{L}} dx'$$

$$\Rightarrow \boxed{\lambda_n = \int_0^L k(x') e^{-\frac{2\pi i n x}{L}} dx'}$$

For this reason we can study many nonlocal problems. In 2 dimensions typically consider ~~2~~ periodic domain or infinite domains.

Infinite domains have some mathematical problems, due to a continuum of solutions.

Consider, say

$$\Delta u = \lambda u \qquad u : \mathbb{R}^{2} \to \mathbb{R}$$

Bounded function are $u(\vec{x}) = e^{i\vec{k}\cdot\vec{x}}$

and $\lambda = -|k|^2$

There are infinitely many functions for each λ.

One simplification that is sometimes done is restrict there to a periodic lattice such as square or hexagonal lattice.

We will look at this type of system later on.

Consider a square domain in $\mathbb{R}^2$ with length + width L

$$\Delta u = \lambda u \qquad u : [0,L]\times[0,L] \to \mathbb{R}$$

and assume periodic boundary condition.

Then $u(x,y) = e^{\frac{2\pi i (nx+my)}{L}}$ + $\lambda = -\frac{4\pi^2}{L^2}(n^2+m^2)$

So for a given λ, there could be ~~an~~ as little as four dimensional ~~null space~~ eigenspace

(n_0, m_0) $(-n_0, m_0)$ $(n_0, -m_0)$ $(-n_0, -m_0)$

or much larger. Say $\lambda = -\frac{4\pi^2}{L^2} \cdot 25$

$n^2 + m^2 = 25$ $(\pm 5, 0)$ $(0, \pm 5)$, $(\pm 3, 4)$ $(\pm 4, 3)$ $(\pm 3, -4)$ $(\pm 4, -3)$ 12 dimensions!!

Summary For continuum & discrete spatial problems that have some homogeneity or symmetry properties, we can usually find eigenvalues & ~~as long~~ use this to decompose problem into simpler ones.

Two more examples:

(1) $u_{1t} = -u_1 + F_1(k_{11} * u_1 - k_{12} * u_2)$ $\quad k * u = \int_{\Omega} k(x-y) u(y)\,dy$

$u_{2t} = -u_2 + F_2(k_{21} * u_1 - k_{22} * u_2)$

Assume $k_{ij} * \mathbb{1} = \alpha_{ij}$ is ind of x. (or $F_j(0) = 0$ so $\vec{u} \equiv 0$)

Linearize about homogeneous state

$u_{1t} = -u_1 + \beta_1 [k_{11} * u_1 - k_{12} * u_2]$
$u_{2t} = -u_2 + \beta_2 [k_{21} * u_1 - k_{22} * u_2]$

Problem is that $k_{ij} * u = \lambda u$ may NOT have same eigenfunctions, so No simplification is possible!. Need ~~single~~ same eigenfunction for all!

Example 2

$\vec{u}_t = f(\vec{u}) + \vec{D}\,\Delta\vec{u}$

with Boundary conditions

Must choose BC's so that eigenfunctions of $\Delta w = \lambda w$ are same!

Then can simplify. (For example could not have $\nabla u_1 \cdot \vec{n} = 0$ and $u_2 = 0$ on $\partial\Omega$ for $\vec{u} = (u_1, u_2)$)

In practice, this is not a bad restriction. But you should be aware of it.

Now what can we do with all this!

It is time to turn to nonlinear equations and bifurcation theory, which I will review for you.

Most ~~of~~ of the ideas we will use will be applied to spatial problems, but it is easier to start with ODEs & then assert that this ~~happens~~ holds in infinite dimensions (It usually does!)

$\frac{dx}{dt} = f(x)$ $\quad x \in \mathbb{R}^n \quad$ ☆

$\bar{x}$ is a fixed point or equilibrium to ☆ if
$f(\bar{x}) = 0$.

Example $\dot{x} = x - x^3$, $\quad x = 0, \pm 1$

$\dot{x} = x(1 - x^2 - y^2) - y \quad \dot{y} = y(1 - x^2 - y^2) + x$

has $(0,0)$ as fixed point

Let $A = Df = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}}$ be the linearization

If eigenvalues of A have all negative real parts, then $\bar{x}$ is asympt stable

If at least one eigenvalue has a positive real part, $\bar{x}$ is unstable

If no eigenvalues of A have zero real parts, we say that ~~...~~ $\bar{x}$ is a hyperbolic equilibrium + the behavior near $\bar{x}$ is the same as the linear equation.

Center manifold Theorem provides a way to study nonlinear systems ~~near the~~ at non-hyperbolic equilibria

We write $\frac{dx}{dt} = f(x, \mu) \quad (x \in \mathbb{R}^n, \mu \in \mathbb{R}^m)$

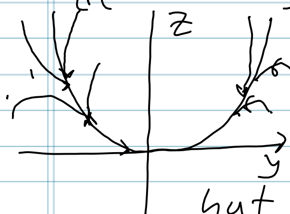
near non hyperbolic equilibrium as

$$\frac{dx}{dt} = Ax + g(x)$$

* separate out stable + zero (lets forget unstable part) $x = y + z$

zero ↗ ↑ negative real parts

$\frac{dy}{dt} = By + g_y(y,z)$ $\frac{dz}{dt} = Cz + g_z(y,z)$



Cz makes z decay but it will eventually be balanced by $g_z(y,z)$ so z does not $\to 0$

but lands on some manifold $z = h(y)$

+ $|h(y)| = O(|y|^2)$ as $y \to 0$

Center manifold theorem guarantees existence of this manifold tangent at $z=0$ at $y=0$

It is invariant (start there, stay there)

Locally $z = h(y)$ so that

$$\boxed{\frac{dy}{dt} = By + g_y(y, h(y))}$$

Note dim y is typically much less than dim $x = n$ so this is much simpler.

If there are parameters, μ we can write

Lots of zero eigenvalues → $\frac{d\mu}{dt} = 0$ $\frac{dy}{dt} = B'y + g_y'(y,z,\mu)$

$\frac{dz}{dt} = Cz + g_z'(y,z,\mu)$

$\Rightarrow \boxed{\frac{dy}{dt} = By + g_y(y, h(y,\mu))}$

You will learn all of this in Rubin's class. But lets do an example.

$\frac{dy}{dt} = By + g_y(y,z) \qquad \frac{dz}{dt} = Cz + g_z(y,z)$

write $z = h(y)$

$\frac{dz}{dt} = Dh \frac{dy}{dt} = \boxed{Dh\left[By + g_y(y, h(y))\right] = Ch(y) + g_z(y, h(y))}$

Need to solve $\boxed{}$ for $h(y)$. Ugly PDE. Only need it near $y=0$ so usually expand in polynomial & match coefficients.

$\frac{dx}{dt} = \mu x - xy \qquad \frac{dy}{dt} = -y + x^2 \qquad \frac{d\mu}{dt} = 0$

$(x=y=0)$ is equilibrium point $\lambda = -1, \mu$

For $\mu < 0$ stable, $\mu > 0$ unstable $\mu = 0$ nonhyperbolic.

so $y = h(x,\mu) \qquad \frac{dy}{dt} = \frac{\partial h}{\partial x}\frac{dx}{dt} + \frac{\partial h}{\partial \mu}\frac{d\mu}{dt}$

$\frac{dy}{dt} = -h(x,\mu) + x^2 = \frac{\partial h}{\partial x}\left[\underset{\text{quad}}{\mu x} - \underset{\text{quad}}{x\,h(x,\mu)}\right]$

Write $h(x,\mu)$ as polynomial.

$h = ax^2 + bx\mu + c\mu^2 + \dots$

$x^2 - [ax^2 + bx\mu + c\mu^2] = (2ax + b\mu)\, x\, (\mu - ax^2 - bx\mu - c\mu^2 \dots)$

$a - 1 = 0$ "x^2" "$x\mu$" $b = 0$, "μ^2" $c = 0$

so $y = x^2 + \dots$ + $\boxed{\frac{dx}{dt} = \mu x - x^3}$!!

This method is great for simple examples but really sucks for realistic problems so we will use the method of multiple scales + perturbation theory later on!

PF $\frac{dx}{dt} = \mu x \pm x^3$ $\frac{dx}{dt} = \mu + x^2$ F

TC $\frac{dx}{dt} = \mu x \pm x^2$

These are the three types of steady state bifurcations you get in simple 0 eigenvalues. The "problem" with symmetry is that there are usually multiple eigenvalues as we saw in the last few pages.

Before continuing I want to introduce another way to compute the nonlinear portions of a bifurcation calculation.

It is called the Lyapunov - Schmidt technique and it applies with many functional equations as well. The advantage of LS is that there is no need to first transform the problem into Jordan form (for the linear part)

Let $F: B \times \mathbb{R} \to B$ be a nonlinear mapping of some function space (or could be $\mathbb{R}^n$) but more generally, it is infinite dimensional (like a PDE or something)

Want solutions $F(u,\lambda) = 0$ ($\lambda \in \mathbb{R}$ is parameter $u \in B$). Suppose $u=0$ is a solution for $\lambda = \lambda_0$

Linearize $L(\lambda_0) = D_u F(0,\lambda_0)$

If $L(\lambda_0)$ is invertible then from IFT $\exists\, u(\lambda) \in B$ st

$$F(u(\lambda),\lambda) = 0 \quad + \quad u(\lambda_0) = 0.$$

This is the unique solution near $u=0$. Since $F(0,\lambda)=0$ for λ in some nbhd of λ_0, $u=0$ is only small solution.

So if we want to find branches of nontrivial solutions near $u=0$, we had better not have $L(\lambda_0)$ invertible. So we assume that $L_0 \equiv L(\lambda_0)$ has a finite dimensional null space.

Assume Null_{L_0} is finite dimensional closed subspace of B, $\mathrm{Ran}_{L_0} \equiv \{f \in B \mid \exists u,\ L_0 u = f\}$

has finite codimension ($\mathrm{Ran}_{L_0}^{\perp}$ is finite dimensional)

and $\dim N_{L_0} = \dim R_{L_0}^{\perp}$. (This means Fredholm alternative holds: $Lu = f$ iff $f \in N_{L^*}^{\perp}$)

Define projections
$$P: B \to R_{L_0}$$
$$Q: B \to N_{L_0}$$

Write $u \in B$ as $u = v + w$ where $v \in N_{L_0}$ & $w \in N_{L_0}^{\perp}$, That is $v = Qu$, $w = (I-Q)u$.

$F(u,\lambda) = 0 \Rightarrow$ (a) $PF(v+w,\lambda) = 0$, $(I-P)F(v+w,\lambda) = 0$ (b)

Regard (a) as a map from $(I-Q)B \to R_{L_0}$ with v, λ fixed. Clearly $PF(0,\lambda) = 0$ and $PL_0 = D_u PF(0,\lambda_0)$ is <u>invertible</u> as a map from $(I-Q)B \to R_{L_0}$ since we have projected out the nullspace! Thus from IFT we can uniquely solve for $w = \tilde{w}(v,\lambda)$ when $|v|$ & $|\lambda - \lambda_0|$ are small. st

$$\tilde{w}(0,\lambda_0) = 0$$

It remains to solve ☆ $f(v,\lambda) \equiv (I-P)F(\tilde{w}(v,\lambda)+v,\lambda) = 0$

But now f is a map from a finite dim space N_{L_0} to another FDS $(I-P)B$. If we write $v = \sum_{i=1}^{n} z_i \phi_i$ $z_i \in \mathbb{C}$ where $\{\phi_i\}$ is the basis for N_{L_0} & we let $\{\psi_j\}_{j=1}^{n}$ span $(I-P)B$

Then solving (☆) is equivalent to solving

$$f_1(z_1,\ldots,z_n;\lambda) = 0 \quad f_2(z_1,\ldots,z_n;\lambda) = 0 \ldots f_n(z_1,\ldots,z_n;\lambda) = 0$$

where $f\left(\sum_{i=1}^{n} z_i \phi_i;\lambda\right) = \sum_{j=1}^{n} f_j(z_1,\ldots,z_n;\lambda)\psi_j$

How does this help? . For those of you who have done any perturbation theory, this is all just tantamount to the application of the Fredholm Alternative Theorem.

We now apply this to ~~an~~ ~~example~~ example problem.

We now apply this to an example from the previous lectures, the symmetric coupled path. Let $F: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and consider:

$$\dot X_1 = F(X_1, X_2, \lambda), \quad \dot X_2 = F(X_2, X_1, \lambda)$$ with λ a parameter.

We assume $\exists!\, u(\lambda)$ for $|\lambda - \lambda_0|$ small st

$$F(u(\lambda), u(\lambda), \lambda) = 0.$$ WLOG let $\lambda_0 = 0$.

Lets write $X_j(\lambda) = u(\lambda) + Y_j(\lambda)$ so that we get

$$\dot Y_1 = F(Y_1, Y_2, \lambda), \quad \dot Y_2 = F(Y_2, Y_1, \lambda)$$ and $Y_j = 0$ is a solution for $|\lambda|$ small. Lets linearize around $Y_j = 0$. Let $A = \partial_1 F_2(0,0,\lambda)$, $B = \partial_2 F_2(0,0,\lambda)$

be $n \times n$ matrices. So full linearization is
$$\begin{bmatrix} A & B \\ B & A \end{bmatrix}$$ + eigenvectors have form $\begin{bmatrix} v_s \\ v_s \end{bmatrix}$ + $\begin{bmatrix} v_d \\ -v_d \end{bmatrix}$ with eigenvalues λ_s, λ_d of $A+B$ + $A-B$ respectively. ~~By hypothesis ($\exists\, u(\lambda)$) there are no zero eigenvalues of $A+B$ (since then IFT)~~

Assume (1) $A+B$ has all eigenvalues with negative real parts near $\lambda = 0$ (2) $A-B$ has zero eigenvalue at $\lambda = 0$ + all other eigenvalues have negative real part (3) Let $\mu(\lambda)$ be the eigenvalue st $\mu(0) = 0$. Then $d\mu/d\lambda|_{\lambda=0} > 0$.

We will now derive the equations for the dynamics near $\lambda = 0$, using perturbation theory.

Aside: Fredholm Alternative. $Lu = f$ has a soln iff $\langle v^*, f \rangle = 0$ for all v^* st $L^* v^* = 0$.

$\langle L^* v, u \rangle = \langle v, Lu \rangle$ defines L^*

- method of multiple scales. Assume some function depends on s, $\underset{\tau_1}{\varepsilon t}$, $\underset{\tau_2}{\varepsilon^2 t}$, ... so $\frac{d}{dt} = \frac{\partial}{\partial s} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + \cdots$

Let us look for solution to $s = t$ is fast time

No depon Fast time

$$\dot{Y}_1 = F(y_1, y_2, \lambda), \quad \dot{Y}_2 = F(y_2, y_1, \lambda)$$

Preliminaries..

General Taylor series.

$$F(y_1, y_2) = Ay_1 + By_2 + Q_1(y_1, y_1) + Q_2(y_1, y_2) + Q_3(y_2, y_2) + C_1(y_1, y_1, y_1) + C_2(y_1, y_1, y_2) + C_3(y_1, y_2, y_2) + C_4(y_2, y_2, y_2) + \cdots$$

$Q(w, z) = Q(z, w)$ is bilinear form:

$Q(aw_1 + bw_2, z) = a\,Q(w_1, z) + b\,Q(w_2, z)$ $\quad Q: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

$C(w, z, p)$ is trilinear form. $C: \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$

~~Example Brusselator~~

Write $Y = (Y_1, Y_2)^T$ and assume Y is small, $0 < \varepsilon \ll 1$ is small parameter. We use this as parametric equations $Y = Y(\varepsilon)$, $\lambda = \lambda(\varepsilon)$ to express Y in terms of λ.

Y may also depend on time, and we use above timescales.

Write $Y = \varepsilon P_1 + \varepsilon^2 P_2 + \varepsilon^3 P_3 + \cdots$

$\lambda = \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + \cdots$

Let $(A_0, B_0) = A(0) + B(0)$. Let

$$(A(\lambda), B(\lambda)) = A_0 + B_0 + \lambda(A_1 + B_1) + \lambda^2(A_2 + B_2), \ldots$$

Let $(A_0 + B_0)V = 0$ and $(A_0^T - B_0^T)V^* = 0$ with $V^* \cdot V = 1$ (WLOG)

$$\frac{dY}{dt} = \varepsilon \frac{dP_1}{ds} + \varepsilon^2 \frac{dP_1}{d\tau_1} + \varepsilon^3 \frac{dP_1}{d\tau_2} + \varepsilon^2 \frac{dP_2}{ds} + \varepsilon^3 \frac{dP_2}{d\tau_1} + \varepsilon^3 \frac{dP_3}{ds} + \cdots$$

lowest order "ε"

$$\frac{dP_1}{ds} = \begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix} P_1$$

only nonzero solution is $P_1 = r\begin{bmatrix} V \\ -V \end{bmatrix}$ where r is scalar and function of τ_1, τ_2 only . .

Order ε^2

$$\frac{dr}{d\tau_1}\begin{bmatrix} V \\ -V \end{bmatrix} + \frac{dP_2}{ds} = \begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix} P_2 + \lambda_1 r \begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix}\begin{bmatrix} V \\ -V \end{bmatrix} + r^2 \begin{bmatrix} Q_1(V,V) - Q_2(V,V) + Q_3(V,V) \\ Q_1(V,V) - Q_2(V,V) + Q_3(V,V) \end{bmatrix}$$

write this as

$$-\begin{bmatrix} A_0 & B_0 \\ B_0 & A_0 \end{bmatrix} P_2 = -\begin{pmatrix} V \\ -V \end{pmatrix}\frac{dr}{d\tau_1} + \lambda_1 r \begin{pmatrix} A_1 & B_1 \\ B_1 & A_1 \end{pmatrix}\begin{bmatrix} V \\ -V \end{bmatrix} + r^2 \begin{bmatrix} \tilde{Q} \\ \tilde{Q} \end{bmatrix}$$

$\begin{bmatrix} A_0 & B_0 \\ B_0 & A_0 \end{bmatrix}$ has 1 dim null space spanned by $\begin{bmatrix} V \\ -V \end{bmatrix}$

and $\begin{pmatrix} A_0^T & B_0^T \\ B_0^T & A_0^T \end{pmatrix}$ has 1 dim null space $\begin{bmatrix} V^* \\ -V^* \end{bmatrix}$

so from F.A. no solution P_2 unless RHS is orthogonal to $\begin{bmatrix} V^* \\ -V^* \end{bmatrix} \Rightarrow$

$$-\frac{dr}{d\tau_1} + \lambda_1 r \underbrace{\begin{pmatrix} V^* \\ -V^* \end{pmatrix} \cdot \begin{pmatrix} A_1 & B_1 \\ B_1 & A_1 \end{pmatrix} \begin{pmatrix} V \\ -V \end{pmatrix}}_{} = 0$$

η ||

(~~term~~ $\eta = 2V^*(A_1 - B_1)V$. Let's suppose $\eta > 0$

Then $\frac{dr}{d\tau_1} = \lambda_1 \eta_1 r$ solutions are $r = e^{\lambda \eta t} r(0)$

decay/grow exponentially unless $\lambda_1 = 0$ (either $r \to 0$ or $r \to \infty$, neither is any good!) so we pick $\lambda_1 = 0$ and $\frac{dr}{d\tau_1} = 0$ so r is ind of τ_1.

Need to prove $\eta \neq 0$.

Let $\xi(\lambda)$ be eigenvector corresponding to $\mu(\lambda)$

with $[A(\lambda) - B(\lambda)]\xi(\lambda) = \mu(\lambda)\xi(\lambda)$

& $\mu(0) = 0$. differentiate this wrt λ & set $\lambda = 0$

$(A_1 - B_1)V + (A_0 - B_0)\xi_1 = \mu'(0) V$ where

$A_j = \frac{dA}{d\lambda}\Big|_{\lambda=0}$ by def and $\xi(0) = V$ & $\xi_1 \equiv \frac{d\xi}{d\lambda}\Big|_{\lambda=0}$

Multiply both sides by V^* & take inner product:

$V^* \cdot (A_1 - B_1)V = \frac{1}{2}\mu'(0)$ since $V^* \cdot V = \frac{1}{2}$

$\Rightarrow 2V^* \cdot (A_1 - B_1)V = \mu'(0) > 0$ by Hypothesis !!

(Note $V^* \cdot (A_0 - B_0)\xi_1 = (A_0^T - B_0^T)V^* \cdot \xi_1 = 0 \cdot \xi_1 = 0$)

So we find r ind of $\varepsilon t \equiv \tau_1$ & $\lambda_1 = 0$

Thus:

$$-\begin{bmatrix} A_0 & B_0 \\ B_0 & A_0 \end{bmatrix} P_2 = r^2 \begin{bmatrix} \tilde{Q} \\ \tilde{Q} \end{bmatrix}$$

and from this we see that $P_2 = r^2 \begin{bmatrix} q \\ q \end{bmatrix}$

where $-(A_0 + B_0) q = \tilde{Q}$. Since $A_0 + B_0$ is invertible we can find

$$q = -(A_0 + B_0)^{-1} \tilde{Q}$$

Summarizing, so far:

$$\begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = \varepsilon r \begin{bmatrix} v \\ -v \end{bmatrix} + \varepsilon^2 r^2 \begin{bmatrix} q \\ q \end{bmatrix} \quad \bigstar$$

Now we go to cubic order ε^3

$$-\begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix} P_3 = \left(-\begin{bmatrix} v \\ -v \end{bmatrix} \frac{dr}{d\tau_2} + \lambda_2 r \begin{bmatrix} A_1 & B_1 \\ B_1 & A_1 \end{bmatrix} \begin{bmatrix} v \\ -v \end{bmatrix} + \begin{bmatrix} 2Q_1(v,q) - 2Q_3(v,q) \\ -2Q_1(v,q) + 2Q_3(v,q) \end{bmatrix} r^3 + \begin{bmatrix} C_1(v,v,v) - C_2(v,v,v) + C_3(v,v,v) - C_4(v,v,v) \\ -C_1(v,v,v) + C_2(v,v,v) - C_3(v,v,v) + C_4(v,v,v) \end{bmatrix} r^3 \right) \equiv r^3 \begin{bmatrix} z \\ -z \end{bmatrix}$$

Apply FA to this & we get

$$\frac{dr}{d\tau_2} = \lambda_2 r \mu'(0) + \gamma_3 r^3$$

$$\gamma_3 = 2 v^* \cdot z = 2 \left(v^* \cdot \left[2Q_1(v,q) - 2Q_3(v,q) + C_1 - C_2 + C_3 - C_4 \right] \right)$$

where $C_i = C_i(v,v,v)$.

Last hypothesis $\gamma_3 \neq 0$.