Example calculation.
We'd like to do more than scalar problems, but the algebra is so tedious that it could take a week just to do it

Thus I will do a simple biharmonic problem on a 1D periodic domain:

$$u_t = \lambda u + \beta u^2 + \gamma u^3 - 2u_{xx} - u_{xxxx}$$

$u(0)=u(2\pi)$, $u_x(0)=u_x(2\pi)$, $u_{xx}(0)=u_{xx}(2\pi)$, $u_{xxx}(0)=u_{xxx}(2\pi)$

One solution is $u=0$.
Linearized $v_t = \lambda v + 2v_{xx} - v_{xxxx}$, $v = e^{\nu t} e^{inx}$

$\nu_0 = \lambda$, $\nu_1 = \lambda+1$, $\nu_2 = \lambda - 8$, so if $\lambda < -1$ AS when $\lambda > -1$, $n=\pm 1$ pattern is unstable. $\lambda_0 = -1$
write $\lambda = -1 + \varepsilon^2 \hat{\lambda}$ $\tau = \varepsilon^2 t$

$u = \varepsilon z(\tau) e^{ix} + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots + cc$

$$0 = \beta \underbrace{[z^2 e^{2ix} + 2z\bar{z} + \bar{z}^2 e^{-2ix}]}_{**} \underbrace{- u_2 - 2u_2'' - u_2''''}_{\mathcal{L}} \quad \mathcal{L} e^{\pm ix} = 0$$

This is self adjoint. But $(**)$ is in range since orthog to $e^{\pm ix}$ so we guess $u_2 = Az^2 e^{2ix} + Bz\bar{z} + \bar{A}\bar{z}^2 e^{-2ix}$
$Az^2(1-8+16) = \beta \Rightarrow A = \beta/9$ $B = 2\beta$ so at this point $u = \varepsilon z e^{ix} + \varepsilon \bar{z} e^{-ix} + \varepsilon^2 \left[\frac{\beta}{9} z^2 e^{2ix} + 2\beta z\bar{z} + \frac{\beta}{9}\bar{z}^2 e^{-2ix}\right]$

so $\beta u^2 = \varepsilon^3 \left[\frac{2\beta}{9} + 4\beta\right] z^2\bar{z} e^{ix} + cc +$ other terms

$\gamma u^3 = 3\gamma z^2\bar{z} e^{ix} + cc +$ other terms

$$\boxed{z_\tau = \hat{\lambda} z + \left(\frac{38}{9}\beta^2 + 3\gamma\right) z^2\bar{z}} \quad *$$

Note if $\gamma = 0$ always subcritical. Need $\gamma < 0$ + large enough.

Reaction diffusion calculation

$$\frac{\partial u}{\partial t} = F(u;\lambda) + D\frac{\partial^2 u}{\partial x^2} \qquad 0 < x \le 2\pi$$

and lets try periodic BC's for simplicity

Preliminaries:

Let u_0, λ_0 be such that

(a) $F(u_0, \lambda_0) = 0$

(b) $A(\lambda) = D_u F(u,\lambda)\big|_{u=u(\lambda)}$

(c) Let n_0 be such that $\quad A_0 \equiv A(\lambda_0)$

$A(\lambda_0) - n_0^2 D$ has 1 dim nullspace spanned by φ. Let $(A^T(\lambda_0) - n_0^2 D^T)\psi = 0$ with $\varphi \cdot \psi = 1$

(d) $A_0 - n^2 D$ is invertible for all $n \neq \pm n_0$

Write $F(u_0 + w, \lambda_0) = A_0 w + Q(w,w) + C(w,w,w) + \cdots$

By symmetry we know it will end up being a pitchfork so write $\lambda = \lambda_0 + \varepsilon^2 \nu$, $\tau = \varepsilon^2 t$

$$u(x,t) = u_0 + \underbrace{\varepsilon\varphi[z e^{in_0x} + \bar z e^{-in_0x}]}_{u_1} + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots$$

$z = z(\tau)$

$$\frac{\partial}{\partial t} = \varepsilon^2 \frac{\partial}{\partial \tau} \qquad A(\lambda_0) = A_0 + (\lambda - \lambda_0)B$$

So:

"0" order $\quad 0 = F(u_0, \lambda_0)$ ✓

"1" order $\quad 0 = [(A_0 - n_0^2 D)\varphi] z e^{in_0x} + ((A_0 - n_0^2 D)\varphi)\bar z e^{-in_0x}$ ✓

"2" order

$$0 = A_0 u_2 + D u_{2xx} + z^2 e^{2in_0x} Q(\varphi,\varphi) + 2z\bar z\, Q(\varphi,\bar\varphi) + \bar z^2 e^{-2in_0x} Q(\varphi,\varphi)$$

Guess a solution

$$u_2(x) = z^2 e^{2in_0x}\zeta_2 + \bar z^2 e^{-2in_0x}\zeta_2 + 2z\bar z\,\zeta_0$$

$(A_0 - 4n_0^2 D)\zeta_2 + Q(\varphi,\varphi) = 0$

Since $(A_0 - 4n_0^2 D)$ is invertible, we get

$\boxed{\zeta_2 = -(A_0 - 4n_0^2 D)^{-1} Q(\varphi,\varphi)}$ & similarly

$\boxed{\zeta_0 = -A_0^{-1} Q(\varphi,\varphi)}$

Thus $u_2(x) = z^2 e^{2in_0x}\zeta_2 + \bar{z}^2 e^{-2in_0x}\zeta_2 + 2z\bar{z}\zeta_0$

$Q(\varepsilon[ze^{in_0x} + \bar{z}e^{-in_0x}]\varphi + \varepsilon^2[z^2 e^{2in_0x}\zeta_2 + \bar{z}^2 e^{-2in_0x}\zeta_2 + 2z\bar{z}\zeta_0],$
$\varepsilon\bigcirc + \varepsilon^2\bigcirc) =$

Cubic terms that matter:

$2Q(\varphi,\zeta_2) z^2\bar{z} e^{in_0x} + 4Q(\varphi,\zeta_0)\bar{z}^2 z e^{in_0x} +$ ~~other~~ cc

Lets write as:

$\mathcal{L}u_3 =$
$[A_0 u_3 + D u_{3xx}] = \frac{\partial z}{\partial \tau}\varphi e^{in_0x} - \{\nu z B\varphi e^{in_0x} + 3C(\varphi,\varphi,\varphi) z^2\bar{z} e^{in_0x}$
$+ 2Q(\varphi,\zeta_2) z^2\bar{z} e^{in_0x} + 4Q(\varphi,\zeta_0) z^2\bar{z} e^{in_0x}\} + cc +$
other terms in $e^{\pm 3in_0x}, e^{0n_0x}, \ldots$

$\mathcal{L}u = \square e^{in_0x}$ is solvable iff $\square e^{in_0x}$ is orthogonal to the adjoint nullspace, spanned by $\psi e^{\pm in_0x}$.

Thus Applying This to ☆, we get:

$0 = \frac{\partial z}{\partial \tau} - \{\nu z b + c z^2\bar{z}\}$
$b = \psi\cdot B\varphi, \quad c = \psi\cdot[3C(\varphi,\varphi,\varphi) + 2Q(\varphi,\zeta_2) + 4Q(\varphi,\zeta_0)]$

Our bifurcation equations are:

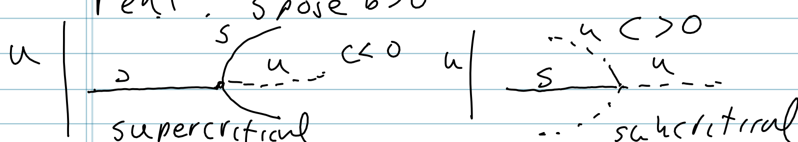
$Z_\tau = z(\nu b + c z\bar{z})$. c, ν are real so

$\bar{Z}_\tau = \bar{z}(\nu b + c z\bar{z})$ write $z = re^{i\theta}$ + get

$r_\tau = r(\nu b + c r^2)$ $\quad r=0, \quad r = \sqrt{\frac{-\nu b}{c}}$

Stability: $s_\tau = \nu b s + 3 c r^2 s$
$= (\nu b - 3\nu b)s = -2\nu b s \quad (r = \sqrt{\frac{-\nu b}{c}}$

Thus stable iff $\nu b > 0 \Rightarrow c < 0$ since r must be real. Suppose $b > 0$



Summary for reaction diffusion equations:

- Find ν_0, ϕ, ψ, k_0
- Expand in cubic + Quadratic
- Solve for ξ_0, ξ_2
- compute b, c

Example Brusselator:

Treat, say, ~~B as a parameter~~ ~~D as parameter~~ a as a parameter

$f(u,v) = \begin{bmatrix} a-(b+1)u + u^2 v \\ bu - u^2 v \end{bmatrix} \quad (u,v) = (a+\tilde{u}, \frac{b}{a} + \tilde{v})$

$f(\tilde{u},\tilde{v}) = \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix}\begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix} + \begin{bmatrix} 2a\tilde{u}\tilde{v} + \frac{b}{a}\tilde{u}^2 \\ -2a\tilde{u}\tilde{v} - \frac{b}{a}\tilde{u}^2 \end{bmatrix} + \begin{bmatrix} \tilde{u}^2\tilde{v} \\ -\tilde{u}^2\tilde{v} \end{bmatrix}$

Our bifurcation equations are:

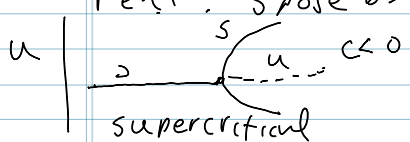
$z_\tau = z(\nu b + c z\bar{z})$. c, n are are real so

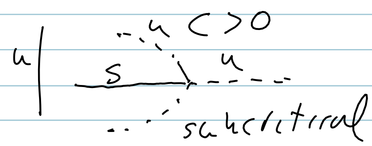
$\bar{z}_\tau = \bar{z}(\nu b + c z\bar{z})$ write $z = re^{i\theta}$ + get

$r_\tau = r(\nu b + cr^2)$ $r = 0$, $r = \sqrt{\frac{-\nu b}{c}}$

Stability: $s_\tau = \nu b s + 3 c r^2 s$
$= (\nu b - 3\nu b)s = -2\nu b s$ $\left(r = \sqrt{\frac{-\nu b}{c}}\right)$

Thus stable iff $\nu b > 0 \Rightarrow c < 0$ since r must be real. Spose $b > 0$





Summary for reaction diffusion equations:

- Find u_0, ϕ, ψ, λ_0
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Example Brusselator:

Treat, say, ~~B as a parameter~~ ~~D as parameter~~
a as a parameter

$$f(u,v) = \begin{bmatrix} a-(b+1)u + u^2 v \\ bu - u^2 v \end{bmatrix} \quad (u,v) = \left(a + \tilde{u}, \frac{b}{a} + \tilde{v}\right)$$

$$f(\tilde{u},\tilde{v}) = \begin{bmatrix} b-1 & a^2 \\ -b & -a^2 \end{bmatrix}\begin{pmatrix}\tilde{u} \\ \tilde{v}\end{pmatrix} + \begin{bmatrix} 2a\tilde{u}\tilde{v} + \frac{b}{a}\tilde{u}^2 \\ -2a\tilde{u}\tilde{v} - \frac{b}{a}\tilde{u}^2 \end{bmatrix} + \begin{bmatrix} \tilde{u}^2\tilde{v} \\ -\tilde{u}^2\tilde{v} \end{bmatrix}$$

How do we write these as quadratic + cubic forms?

$\cancel{\begin{pmatrix} 2a\tilde{u}\tilde{v} \\ 2a\tilde{u}\tilde{v} \end{pmatrix} = \begin{bmatrix} a \\ -a \end{bmatrix}(\tilde{u},\tilde{v})}$ Think of $\tilde{u}\tilde{v}$ as comp$_1$ * comp$_2$ and since this is bilinear we can drop the 2 and

$\cancel{Q(\vec{\alpha}_1,\vec{\beta}_2) = \begin{bmatrix} a \\ -a \end{bmatrix}}$ $$Q(\vec{\alpha},\vec{\beta}) = \begin{bmatrix} a \\ -a \end{bmatrix}[\alpha_1\beta_2 + \alpha_2\beta_1]$$

since it must be symmetric $$+ \frac{b}{a}\begin{bmatrix} 1 \\ -1 \end{bmatrix}[\alpha_1\beta_1]$$

$$C(\vec{\alpha},\vec{\beta},\vec{\gamma}) = \frac{1}{3}\begin{bmatrix} 1 \\ -1 \end{bmatrix}[\alpha_1\beta_1\gamma_2 + \alpha_1\beta_2\gamma_1 + \alpha_2\beta_1\gamma_1]$$

This is really only an issue if we want to apply general formula to a specific problem. However starting from a specific problem all this stuff about quadratic + cubic forms can be obviated.

~~Exam~~

What happens if "c" is small. When $c=0$ must go to 5th order:

$$z_\tau = z(b\nu + cz\bar{z} + d\,z^2\bar{z}^2)$$ where c has been rescaled accordingly (since it is small)

Write: $z \to re^{i\theta}$.

$$r_\tau = r(b\nu + cr^2 + dr^4)$$

suppose $d<0$ and rescale r by $(-d)^{1/4}$ so, + $\nu \Rightarrow b\nu$ we get:

$$r_\tau = r(\nu + cr^2 - r^4)$$ with approp redef of c

HW compute bifurcation diagram for $c<0$ + $c>0$

HW Simulate

$$u_t = \lambda u + \beta u^2 + \gamma u^3 - u'''' - 2u''$$

in regimes where pitchfork is subcritical but $\underline{\underline{\gamma<0}}$

Hopf Bifurcation

$\dot{\tilde{X}} = \tilde{F}(\tilde{X},\lambda)$ Let $\tilde{X} = u_{ss}(\lambda) + X$ where

$F(u_{ss}(\lambda),\lambda) = 0$ $F(X,\lambda) = \tilde{F}(u_{ss}+X,\lambda)$

$\dot{X} = F(X,\lambda)$ $F(0,\lambda) = 0$

Let $D_X F(0,\lambda) = A(\lambda)$, $A_0\varphi = i\omega\varphi$
$A_1^T\psi = -i\omega\psi$ $\bar{\psi}\cdot\varphi = 1$

$F(X,0) = A_0 X + Q(X,X) + C(X,X,X)$

A_0 has no other eigenvalues with zero real part.

$X = \varepsilon z\varphi e^{i\omega t} + \varepsilon^2 X_2 + \varepsilon^3 X_3 + \cdots + c.c.$

$\tau = \varepsilon^2 t$ $\lambda = \varepsilon^2\hat{\lambda}$ $A(\lambda) = A_0 + \lambda A_1$

Assume $\mathrm{Re}\,\bar{\psi}\cdot A_1\varphi > 0$ (means as λ increases $\mathrm{Re}\,\mu(\lambda)$ crosses axis)

ε^2 order

$$\frac{\partial X_2}{\partial t} - A_0 X_2 = z^2 e^{2i\omega t} Q(\varphi,\varphi) + \bar{z}^2 e^{-2i\omega t} Q(\bar{\varphi},\bar{\varphi}) + 2Q(\varphi,\bar{\varphi})$$

Note $\partial y/\partial t - A_0 y = 0$ has soln $e^{\pm i\omega t}\varphi$ so NOT invertible

However X_2 equation is solvable since $\pm i\omega$ is only eigenvalue with zero real part for A_0.

From $\varnothing$, we get

$$X_2 = z^2 \xi_2 e^{2i\omega t} + 2z\bar{z}\,\xi_0 + \bar{z}^2\bar{\xi}_2 e^{-2i\omega t}$$

where

$\xi_2 = (2i\omega - A_0)^{-1} Q(\varphi,\varphi)$ and $\xi_0 = -A_0^{-1} Q(\varphi,\bar{\varphi})$

So far $X = \varepsilon(z\varphi e^{i\omega t} + \bar{z}\bar{\varphi} e^{-i\omega t}) + \varepsilon^2[2\xi_0 z\bar{z} + \xi_2 z^2 e^{2i\omega t} + \bar{\xi}_2 \bar{z}^2 e^{-2i\omega t}$

ε^3 terms of
$Q(x,x) = z^2\bar{z}e^{i\omega t}\{4Q(\varphi,\xi_0) + 2Q(\bar{\varphi},\xi_2)\} +$ other stuff

ε^3 terms of $C(x,x,x) \to 3z^2\bar{z}e^{i\omega t}\,C(\varphi,\varphi,\bar{\varphi}) + \cdots$

So

$$\frac{\partial x_3}{\partial t} - A_0 x_3 = -\frac{\partial z}{\partial \tau}e^{i\omega t}\varphi + \hat{\lambda}A_1\varphi z e^{i\omega t} + z^2\bar{z}e^{i\omega t}\Big[\underbrace{3C(\varphi,\varphi,\bar{\varphi}) + 4Q(\varphi,\xi_0) + 2Q(\bar{\varphi},\xi_2)}_{\eta}\Big] + cc$$

HAS usual cft RHS orthogonal to Adjoint Nullspace

$\Rightarrow$

$$0 = -\frac{\partial z}{\partial \tau} + \alpha_1 z\hat{\lambda} + \gamma_3 z^2\bar{z} \quad \text{where}$$

$$\alpha_1 = \bar{\psi}\cdot A_1\varphi, \quad \gamma_3 = \bar{\psi}\cdot\eta$$

plus cc the equations. Write α_1 as $a+ib$
write $\gamma_3 = p+iq$. Let $z = re^{i\theta}$. Then

$$r_\tau = r(\hat{\lambda}a + pr^2) \qquad \theta_\tau = \hat{\lambda}b + qr^2$$

From this we see that
(1) if $p>0$ $\exists$ unstable periodic orbit $r = \sqrt{-\frac{\hat{\lambda}a}{p}}$
for $\hat{\lambda}a<0$ and Freq is $\omega + \varepsilon^2[\hat{\lambda}b - q\hat{\lambda}a/p]$
(2) if $p<0$ $\exists$ stable periodic orbit $r = \sqrt{-\hat{\lambda}a/p}$ for
$\hat{\lambda}a>0$ + Freq $\omega + \varepsilon^2[\hat{\lambda}b - q\hat{\lambda}a/p]$

($\boxed{\text{HW}}$ is to prove these two statements.

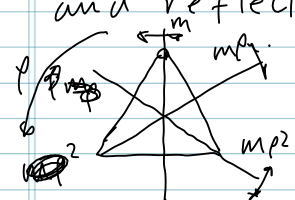
A group Γ consists of a set of elements $\{g_1, g_2, \ldots\}$ together with an operator ("multiplication") st ① $g_1 g_2 \in \Gamma$ $\forall g_1, g_2$
② $(g_1 g_2) g_3 = g_1 (g_2 g_3)$ ③ $\exists\, e \in \Gamma$ st $g_1 e = e g_1 = g_1$

(4) $\exists\, g_1^{-1}$ st $g_1 g_1^{-1} = g_1^{-1} g_1 = e$ $\quad \forall g_1 \in \Gamma$.

Examples – multiplication with non zero real numbers, addition of integers, addition modulo N

- Group is called Abelian or commutative if $g_1 g_2 = g_2 g_1$

- Symmetry group of equilater triangle. D_3. It is generated by 2 elements rotation ρ by $2\pi/3$ and reflection



Elements are e, ρ, ρ^2 $(\rho^3 = e)$
$m, \rho m \rho, m \rho^2$ <u>order</u> 6

	e	ρ	ρ^2	m	$m\rho$	$m\rho^2$
e	e	ρ	ρ^2	m	$m\rho$	$m\rho^2$
ρ	ρ	ρ^2	e	$m\rho^2$		
ρ^2	ρ^2					
m	m					
$m\rho$	$m\rho$					
$m\rho^2$	$m\rho^2$					

Fill in rest of this for Homework

Dihedral group D_n has order $2n$
rotation $2\pi/n$ + reflection through center and corner or midpoint of a side
$e, \rho, \rho^2, \ldots, \rho^{n-1}, m, m\rho, \ldots, m\rho^{n-1}$

Euclidean group $E(2)$ translations, reflections + rotations of a plane

Let ρ be reflection in an axis containing origin or rotation about the origin + $\vec{t}$ be translation

Let $\vec{x} \in \mathbb{R}^2$ $\quad \vec{x}' = (\rho, \vec{t})x \equiv \rho\vec{x} + \vec{t}$

$(\rho_1, t_1)(\rho_2, t_2)\vec{x} = \rho_1\rho_2\vec{x} + \rho_1\vec{t_2} + \vec{t_1}$ so

$(\rho_1, t_1)(\rho_2, t_2) = (\rho_1\rho_2, \rho_1 t_2 + t_1)$

(HW) prove associativity, identity, inverse exist

Dihedral + $E(2)$ are main groups in pattern formation!

Subgroup is subset of a group closed under multiplication. Eg translations are subgroup of $E(2)$ rotation/reflections also.

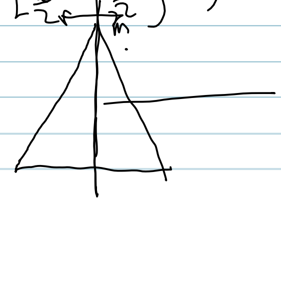
A <u>representation</u> of a group is a mapping

$\Theta : \Gamma \to n \times n$ matrices of $\mathbb{R}$ or $\mathbb{C}$ (in our context)

(specifically, a homomorphism which must satisfy $\Theta(\gamma_1\gamma_2) = \Theta(\gamma_1)\Theta(\gamma_2)$ $\quad \Theta(e) \to$ identity ~~ez~~ matrix
o ~~Texg~~ r.

Example D_3 $\qquad M_e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M_m = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$M_\rho = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$, you can fill out rest



$-\bar{z}$ $\quad$ $\cdot z$

$-z$ $\quad$ $\cdot \bar{z}$

Could also consider $z \in \mathbb{C}$ + ~~D_3~~

$\rho: z \to z e^{2\pi i/3}$ $\quad$ $m: z \to -\bar{z}$

<u>HW</u> ① Make a table for the symmetry group of the square

② show symmetry group of a rectangle is abelian

~~③ ...~~

③ Show that integers $0, 1, \ldots, n-1$ form group with addition modulo n (called $\mathbb{Z}_n$)

Let's see how this applies to ODEs etc

Consider our old pal:

$$\dot{x} = f(x,y) \qquad \dot{y} = f(y,x)$$

and the group $\underset{T_e, T_\alpha}{\mathbb{Z}_2}$ $\quad (x,y) \xrightarrow{e} (x,y) \quad (x,y) \xrightarrow{\alpha} (y,x)$

Write this as $\dot{\vec{X}} = F(\vec{X}) \quad X \in \mathbb{R}^{2n}$

Note that $T_\alpha F = F T_\alpha$. That is F commutes with the representation of the group.

~~Theorem~~ Theorem (Sattinger)

The operator $L_0 = D_u G(0,0)$ commutes with T_g & the null space, η is invariant under T_g.

The bifurcation equations $F(\lambda, v)$ are covariant with respect to the finite dimensional representation T_g restricted to η. That is $T_g F(\lambda, v) = F(\lambda, T_g v)$

~~How could we use group theory to arrive at the correct equations?~~

Proof $T_g G(\lambda, u) = G(\lambda, T_g u)$ so

differentiate wrt u

$$T_g G_u(\lambda, u) = G_u(\lambda, T_g u) T_g$$

set $\lambda = 0$, $u = 0$ + get $T_g L_0 = L_0 T_g$.
Since T_g commutes with L_0, if $\phi \in \mathcal{N}$ then $T_g \phi \in \mathcal{N}$ as well ($L_0(T_g \phi) = T_g L_0 \phi = T_g 0 = 0$)

so $T_g|_{\mathcal{N}}$ is a finite dimensional representation since $\mathcal{N}$ is finite dimensional.
P + $Q = I - P$ commute with T_g (recall P, Q show up in Liap Schm reduction)

(✱) $Q G(\lambda, v + \psi) = 0$ so

$$T_g Q G(\lambda, v + \psi(v, \lambda)) = Q G(\lambda, T_g v + T_g \psi(\lambda, v)) = 0$$

on the other hand replacing v with $T_g v$ we get

$$Q G(\lambda, T_g v + \psi(\lambda, T_g v)) = 0$$

since soln to ✱ is unique $T_g \psi(\lambda, v) = \psi(\lambda, T_g v)$

~~Apply T_g to~~

recall $P G(\lambda, v + \psi) = 0$ $\Rightarrow$

$$F(\lambda, v) \equiv P G(\lambda, v + \psi(\lambda, v)) = 0$$

so

$$\begin{aligned} T_g F(\lambda, v) &= T_g P G(\lambda, v + \psi(\lambda, v)) \\ &= P G(\lambda, T_g v + T_g \psi(\lambda, v)) \\ &= P G(\lambda, T_g v + \psi(\lambda, T_g v)) \\ &= F(\lambda, T_g v) \quad !! \end{aligned}$$

Back to our old Friend The symmetric problem.

$$\dot{x}_j = F(x_j, x_n) \qquad g \in \mathbb{Z}_2$$

~~Tg is the interchange~~.

recall $T_g \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$ + $T_e \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$

our nullspace is $\begin{pmatrix} v \\ -v \end{pmatrix}$ $T_g \varphi = -\varphi$ so

$$T_g F(\lambda, r) = -F(\lambda, r) = F(\lambda, T_g r) = F(\lambda, -r)$$

Thus $-F(\lambda, r) = F(\lambda, -r)$ $\forall \lambda$ so F is odd Function of r!

$$F(\lambda, r) = a_1(\lambda) r + a_3(\lambda) r^3 + a_5(\lambda) r^5 + \cdots$$

Lets Apply This to The Hopf Bifurcation

$$\frac{du}{dt} - G(u, \lambda) = 0$$

Let g be translations $T_g u(t) = u(t+g)$ where $g \in \mathbb{R}$

$T_g N(u, \lambda) = N(T_g u, \lambda)$ since $\frac{du(t+g)}{dt} = G(u(t+g), \lambda)$

is translation invariant., This is also sym under ~~Nullspace~~ complex conjugation.

Null space is $z e^{i\omega t}, \bar{z} e^{-i\omega t}$

$T_g z e^{i\omega t} = e^{i\omega g} z e^{i\omega t}$ $\qquad T_g \bar{z} e^{-i\omega t} = e^{-i\omega g} \bar{z} e^{i\omega t}$

The Bifurcation equations are

$$F_1(z, \bar{z}) = 0 \qquad F_2(z, \bar{z}) = 0$$

since eqns ~~commut~~ commute with complex conj.

Since $\bar{z}$ must follow z dynamics $\quad F_2(z, \bar{z}) = \overline{F_1(z, \bar{z})}$

=> only have to compute $F_1(z,\bar{z}) = F(z,\bar{z})$

Write $F_1(z,\bar{z}) = \sum_{n,m} \alpha_{nm} z^n \bar{z}^m$

$T_g F = F T_g$

$T_g F = e^{i\omega g} F(z,\bar{z}) = e^{i\omega g} \sum \alpha_{nm} z^n \bar{z}^m$

$F(T_g z, T_g \bar{z}) = \sum \alpha_{nm} e^{i\omega g n} e^{-i\omega g m} z^n \bar{z}^m$

This must hold for all g so we must have

$i\omega g = i\omega g[n-m] \quad \forall g \Rightarrow n-m=1$ so

$F(z,\bar{z}) = \sum_{n=0}^{\infty} \alpha_n z (z\bar{z})^n$!!!

This says for all Hopf Bifurcations we always have

$z_\tau = z(\alpha_0 + \alpha_1 z\bar{z} + \alpha_2 (z\bar{z})^2 + \cdots)$

Now HW:

consider the discrete dynamical system:

$X_{n+1} = R(X_n, \lambda)$

Let $X_0 = R(X_0, \lambda)$ be a fixed point. Let $A(\lambda)$ be linearization:

$y_{n+1} = A(\lambda) y_n$

Let ν be eigenvalue of $A(\lambda)$. $A\varphi = \nu\varphi$

Let $y_0 = \varphi$. Then

$y_n = \nu^n \varphi$ so if $|\nu^n| \to 0$ as $n \to \infty$ $y_n \to 0$. $\Rightarrow |\nu| < 1$

Thus X_0 is linearly stable if all eigenval of A satisfy $|\nu| < 1$

How can we lose stability. $|\nu| = 1$

3 cases: $\nu = +1$, $\nu = -1$, $\nu = e^{i\theta}$

The equation $X_{n+1} = R(X_n, \lambda)$ commutes with the group T_g where $g \in \mathbb{Z}$ the integers:

$$X_{n+1+g} = R(X_{n+g}, \lambda).$$

Using this fact, deduce the bifurcation equations for (up to cubic)

1) $\nu = -1$ (Note it is real and

1) $\nu = e^{i\frac{\pi}{2}}$ ($\theta = \frac{\pi}{2}$)

2) $\nu = e^{2i\pi/3}$ ($\theta = \frac{2\pi}{3}$)

3) $\nu = e^{i\theta}$, θ irrational.

Hint when $\nu = e^{i\theta}$, null space is $z\varphi + \bar{z}\bar{\varphi}$

so F is just a function of $z, \bar{z}$.

Thus want

$$\cancel{e^{i\theta} \sum_{n,m} \alpha_{nm} z^n \bar{z}^m = \sum_{n,m}}$$

$$e^{ig\theta} \sum_{n,m} \alpha_{nm} z^n \bar{z}^m = \sum_{nm} \alpha_{nm} e^{ig(n-m)\theta}$$

for all $g \in \mathbb{Z}$!! You will get ~~dif~~ extra terms

for $\theta = \frac{\pi}{2} + \frac{2\pi}{3}$

~~Bifurcation to traveling waves~~

Comments:

For the Hopf we got

$$z_\tau = z(a + bz\bar{z}) \quad \text{wheren}$$

a, b are both complex.

However for bifurcation of stationary patterns on the line, we got

$$z_\tau = z(a + bz\bar{z}) \quad \text{where } a, b \text{ real}$$

why is this? what is different.

Lets revisit the bifurcation of stationary patterns on the line.

Eigenvectors are e^{ikx}, e^{-ikx}

We have translation symmetry, reality conditions

$$T_g e^{\pm ikx} = e^{\pm ikg} e^{\pm ikx}$$

But we also have reflexion symmetry in space. That is $x \to -x$ leaves equations unchanged:

$$F(u(x)) + D\frac{d^2u}{dx^2} \quad \text{is same under } x \to -x$$

Under reflection symmetry $e^{ikx} \leftrightarrow e^{-ikx}$.

So lets use symmetry to infer equations

$$\dot{z}_1 = F_1(z_1, z_2) \qquad \dot{z}_2 = F_2(z_1, z_2)$$

Reality $z_1 = z, \; z_2 = \bar{z} \quad \overline{F_1(z,\bar{z})} = F_2(z,\bar{z})$

Under reflection $z, \bar{z}$ interchange

so

$$F_1(z,\bar{z}) = F_2(\bar{z}, z)$$

Thus $\overline{F_1(z,\bar{z})} = F_2(z,\bar{z}) = F_1(\bar{z}, z)$

$F_1(z,\bar z) = \sum a_{nm} z^n \bar z^m$

$e^{ikg} F_1(z,\bar z) = F_1(e^{ikg} z, e^{-ikg} \bar z)$

$\Rightarrow$ $n-m=1$ as with Hopf so

$F_1(z,\bar z) = z \sum_{n=0}^{\infty} a_n (z\bar z)^n$

But $\overline{F_1(z,\bar z)} = F_1(\bar z, z) \Rightarrow \bar z \sum \bar a_n (z\bar z)^n = \bar z \sum a_n (z\bar z)^n$

$\Rightarrow a_n = \bar a_n$.

so reflection symmetry guarantees real coefficients!

Now we apply symmetry Theorems to bifurcation of spatio-temporal patterns on a line.

Suppose critical instability of a problem on the line is at an ~~e~~ imaginary eigenvalue so that the null space is 4 dimensional:

$$e^{ikx+i\omega t}, \; e^{-ikx+i\omega t}, \; e^{-ikx-i\omega t}, \; e^{ikx-i\omega t}$$

$\dot z_j = F_j(z_1, z_2, z_3, z_4)$

Reality $\Rightarrow$ $z_1 = z = \bar z_3$, $z_2 = w = \bar z_4$

so $\overline{F_1(z,w,\bar z,\bar w)} = F_3(z,w,\bar z,\bar w)$ ✓

$\overline{F_2(z,w,\bar z,\bar w)} = F_4(z,w,\bar z,\bar w)$ ✓

Only need to compute 2 equations! But it is even better, reflection invariance says we can interchange $1 \leftrightarrow 2$ + $3 \leftrightarrow 4$. This means

$F_1(w,z,\bar w,\bar z) = F_2(z,w,\bar z,\bar w)$ so only need to get F_1.

$T_{g,\tau}\, e^{i(kx+\omega t)} = e^{i(kg+\omega\tau)}\, e^{i(kx+\omega t)}$ for all g, τ

because time & space translation invariance

~~Thus~~ Write $F_1(z, w, \bar z, \bar w) = \sum a_{nmpq} z^n \bar z^m w^p \bar w^q$

This means

$$e^{i(kg+\omega\tau)} \sum a_{nmpq} z^n \bar z^m w^p \bar w^q = \sum a_{nmpq}$$

Aside set $k=\omega=1$ WLOG

$T_{g\tau} v_1 = e^{i(g+\tau)} v_1 \quad T_{g\tau} v_2 = e^{i(-g+\tau)} v_2 \quad T_{g\tau} v_3 = e^{-i(g+\tau)} v_3, \quad T_{g\tau} v_4 = e^{i(g-\tau)} v_4$

$\exp(i(g+\tau)) \sum a_{nmpq} z^n \bar z^m w^p \bar w^q =$

$\Rightarrow (g+\tau) = n(g+\tau) - m(g+\tau) + p(-g+\tau) + q(g-\tau)$

$\Rightarrow 1 = n-m-p+q \qquad 1 = n-m+p-q$

$\Rightarrow q=p$ & $m = n-1$ so general ~~solution~~ equation is

$F_1(z, w, \bar z, \bar w) = z \sum_{p,n=0} a_{n,p} (z\bar z)^n (w \bar w)^p$ ✓

Thus in the simplest form:

$$\begin{cases} z_\tau = z(a_{00} + a_{10} z\bar z + a_{01} w\bar w) \\ w_\tau = w(a_{00} + a_{10} w\bar w + a_{01} z\bar z) \end{cases}$$

There are only 3 coefficients / a_{ij} are complex

since we have reflection symmetry.

2 types of patterns (a) $z \neq 0, w = 0$; (b) $w \neq 0, z = 0$; (c) $w, z \neq 0$

(a) ~~exp~~ $\cos(kx+\omega t)$, (b) $\cos(kx-\omega t)$ (c) $\cos kx \cos \omega t$
wave wave standing oscill.

Are there other types of patterns.

Writing in terms of ~~real~~. $z = re^{i\theta}$, $w = se^{i\psi}$

$r_\tau = r(\alpha_0 + \alpha_1 r^2 + \alpha_2 s^2)$, $s_\tau = s(\alpha_0 + \alpha_1 s^2 + \alpha_2 r^2)$

$\theta_\tau = \beta_0 + \beta_1 r^2 + \beta_2 s^2$, $\psi_\tau = \beta_0 + \beta_1 s^2 + \beta_2 r^2$

If r, s are nonzero then
$\alpha_0 + \alpha_1 r^2 + \alpha_2 s^2 = 0 \qquad \alpha_0 + \alpha_1 s^2 + \alpha_2 r^2 = 0$
As long as $\alpha_1 \neq \alpha_2 \Rightarrow r^2 = s^2$.

I will leave as an exercise the stability theory for these patterns.