If r, s are nonzero then
$\alpha_0 + \alpha_1 r^2 + \alpha_2 s^2 = 0 \quad \alpha_0 + \alpha_1 s^2 + \alpha_2 r^2 = 0$
As long as $\alpha_1 \neq \alpha_2 \Rightarrow r^2 = s^2$.

I will leave as an exercise the stability theory for these patterns.

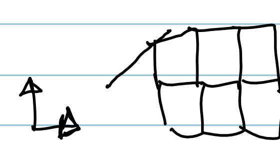
Bifurcations on a lattice

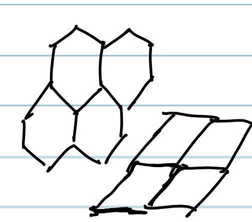
Consider a planar equation, such as RD or Neural network, eg

$$u_t = F(u,\lambda) + D\nabla^2 u$$

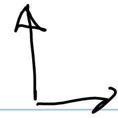
$u : \mathbb{R}^2 \times \mathbb{R} \xrightarrow{\times \Delta} \mathbb{R}^n$. As usual let $u=0$ be a steady state homogeneous soln & linearize to get
$M_k = (A - Dk^2)\xi$ where $u = e^{\nu t} e^{i(k_1 x + k_2 y)} \xi$

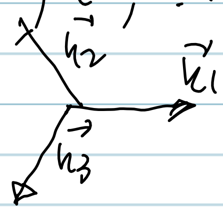
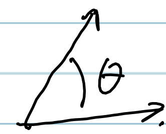
$k^2 = k_1^2 + k_2^2$. If M_{k^*} has, say a zero eigenvalue then we lose stability to k^*. But there are infinitely many choices for k_1, k_2 & Nullspace has a continuum of members. Thus to apply our nice theory, we need to make the nullspace finite dimensional. One standard way to do this, is to ~~put~~ restrict the patterns on a lattice, that is to doubly periodic patterns in a plane. There are 3 ways to tessalate the plane with regular polygons, the square, rhombus, & hexagon. These are all defined by the angles between two vectors $\vec{k}_1, k_2$ such that
$|k_1| = |k_2| = k$





(a) $k_1 \cdot k_2 = 0$ squares
(b) $k_1 \cdot k_2 = k^2 \cos 2\pi/3$ Hexagons
(c) $k_1 \cdot k_2 \neq 0$, $k^2 \cos \frac{1\pi}{3}$, $k^2 \cos \frac{\pi}{3}$ Rhombi

Squares $e^{ikx}, e^{iky}, \ldots$ cc 

Hexagons  Rhombi  $\theta \neq \frac{2\pi}{3}, \frac{\pi}{2}$

Each has different symmetries & results different amplitude equations which we will derive

Nullspace is 4 dimensional for rhombus/square and 6 dimensional for Hexagon.

We want to use symmetry to derive equations.

Symmetries of our equation are rotation + translation and reflection in space. ~~Let label null~~

SQUARE $z_1 e^{ikx}, z_2 e^{iky}, z_3 e^{-ikx}, z_4 e^{-iky}$

We want real solutions so have $z_3 = \bar{z}_1$ $z_4 = \bar{z}_2$

~~Let call them $(z, w, \bar{z}, \bar{w})$~~

On ~~reflection $\dot{z}_1 = F$~~ reflection $z_1 \leftrightarrow z_3$, $z_2 \leftrightarrow z_4$

We have
$$\dot{z}_1 = F_1(z_1, z_2, z_3, z_4)$$
$$\dot{z}_2 = F_2(z_1, z_2, z_3, z_4)$$
$$\dot{z}_3 = F_3(z_1, z_2, z_3, z_4)$$
$$\dot{z}_4 = F_4(z_1, z_2, z_3, z_4)$$

rotation $z_1 \to z_2$ $z_3 \to z_4$

(a) $\bar{F}_3 = F_1$ $F_4 = F_2$ reality condition

(b) $F_1(z_1, z_2, z_3, z_4) = F_3(z_3, z_2, z_1, z_4)$ reflection

$F_2(z_1, z_2, z_3, z_4) = F_4(z_1, z_4, z_3, z_2)$

$F_1(z_1, z_2, z_3, z_4) = F_2(z_2, z_1, z_4, z_3)$ rotation

Thus we can express all equations in terms of F_1!

$F_2(z_1, z_2, z_3, z_4) = F_1(z_2, z_1, z_4, z_3)$
$F_3(z_1, z_2, z_3, z_4) = \overline{F_1(z_3, z_2, z_1, z_4)}$
$F_4(z_1, z_2, z_3, z_4) = \overline{F_1(z_2, z_1, z_4, z_3)}$

we next use the translation invariance:

$T_g e^{ikx} = e^{igk} e^{ikx}$, etc. $g = (g_1, g_2)$
$= e^{ikg_2} e^{iky}$

Let us set $z_1 = z$, $z_2 = w$, $z_3 = \bar{z}$, $z_4 = \bar{w}$

$F_1 = \sum \alpha_{pqrs} z^p w^q \bar{z}^r \bar{w}^s$

$T_g F_1 = e^{ikg_1} F_1 = F_1(e^{ikg_1} z, e^{ikg_2} w, e^{-ikg_1} \bar{z}, e^{-ikg_2} \bar{w})$

$\Rightarrow e^{ikg_1} = e^{ikg_1 p} e^{ikg_2 q} e^{-ikg_1 r} e^{-ikg_2 s} \quad \forall g_1, g_2$

$\Rightarrow p - r = 1$ & $q - s = 0 \Rightarrow p = r+1$, $q = s$

so

$$\sum_{p,s} \alpha_{ps} z (z\bar{z})^p (w\bar{w})^s$$

To lowest order

$$\alpha_{00} z + \alpha_{10} z^2 \bar{z} + \alpha_{01} z w \bar{w}$$

$F_1(z, \bar{z}, w, \bar{w}) = \cancel{\alpha_{00} z + \alpha_{10} z}\ z[\alpha_{00} + \alpha_{10} z\bar{z} + \alpha_{01} w\bar{w}]$

reflection $\bar{F}_3 = F_1$ but also reflection so
$\alpha_{00}, \alpha_{10}, \alpha_{01}$ are all real since $F_1(z, \bar{z}, w, \bar{w}) = F_1(\bar{z}, z, \bar{w}, $

Thus $z_\tau = z(\alpha_{00} + \alpha_{10} z\bar{z} + \alpha_{01} w\bar{w})$
$w_\tau = w(\alpha_{00} + \alpha_{10} w\bar{w} + \alpha_{01} z\bar{z})$

What are the possible solutions.
Set $z = re^{i\theta}$, $w = se^{i\varphi}$

$r_\tau = r(\alpha_{00} + \alpha_{10} r^2 + \alpha_{01} s^2)$
$s_\tau = s(\alpha_{00} + \alpha_{10} s^2 + \alpha_{01} r^2)$

Horiz rolls

$\to$ trivial

$r=s=0$, $\left(r=0, s=\sqrt{-\frac{\alpha_{00}}{\alpha_{10}}}\right)$

vert rolls $\left(r=\sqrt{-\frac{\alpha_{00}}{\alpha_{10}}}, s=0\right)$ $\left(r=s=\sqrt{\frac{-\alpha_{00}}{\alpha_{10}+\alpha_{01}}}\right)$ checks

WLOG lets assume $\alpha_{00}>0$, ~~CLEaok~~ Note $\alpha_{00}(\lambda)$ ~~if α_{00}~~ so as ~~..~~ when $\lambda=0$ $\alpha_{00}(0)=0$ so $\alpha_{00}=\alpha_{00}\lambda$, Assume $\alpha_{00}>0$ & so as λ increase $(0,0)$ becomes unstable.

~~lets linearize~~:

Lets make notation simpler $\alpha_{00}=\lambda$ $\alpha_{10}=-b$, $\alpha_{01}=-c$
so equation are:

$\dot{r}=r(\lambda-br^2-cs^2)$ $\quad\dot{s}=s(\lambda-bs^2-cr^2)$

$$D_f=\begin{pmatrix}\lambda-3br^2-cs^2 & -2crs\\ -2crs & \lambda-3bs^2-cr^2\end{pmatrix}$$

~~$br^2+cs^2=\lambda$~~ $(b+c)r^2=\lambda$ when r,s are nonzero
$\Rightarrow r^2=\frac{\lambda}{b+c}$ These are called "squares"

$r=0, s\neq0$ or $s=0, r\neq0$ are called "rolls"

H Rolls & V rolls have same stability & existence.

Squares

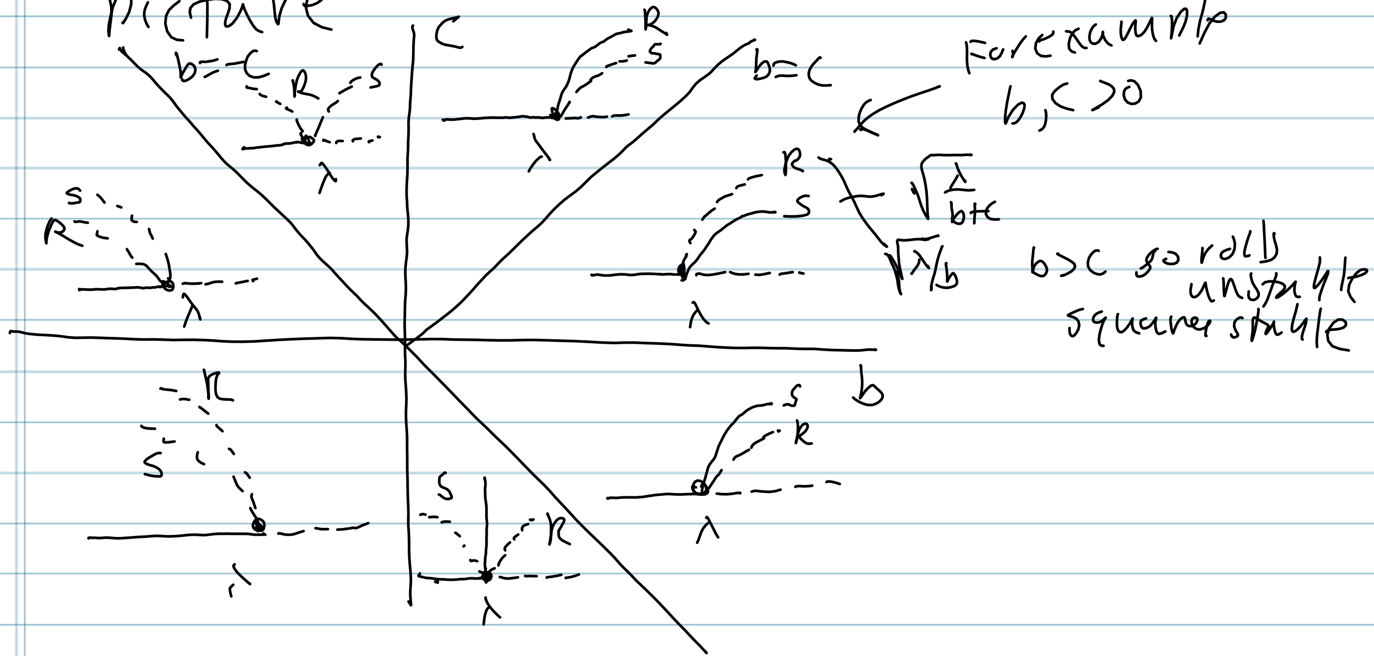
$$D_f=\begin{pmatrix}\frac{-2b\lambda}{b+c} & -\frac{2c\lambda}{b+c}\\ \frac{-2c\lambda}{b+c} & -\frac{2b\lambda}{b+c}\end{pmatrix}\qquad r^2=s^2=\frac{\lambda}{b+c}$$

Eigenvalues are $-\frac{2\lambda(b\pm c)}{b+c}$

$r^2=\frac{\lambda}{b}$ $s=0$

Rolls $D_f=\begin{pmatrix}-2\lambda & 0\\ 0 & \lambda\left(1-\frac{c}{b}\right)\end{pmatrix}$

From this we deduce the following picture



Note rhombic lattice is similar
How do we compute coefficients for these problems?

~~I will do this for the RD equation~~ but ~~in general~~

I will do this fairly abstractly:
Lets consider the following problem:

$$u_t = L_0 u + \lambda L_1 u + Q(u,u) + C(u,u,u) + \cdots$$

Where L_0, L_1 is a linear operator on some Banach space with planar symmetries, Q are quadratics, C are cubics, etc

I will do the rhombic & the square lattices simultaneously
We rescale space so eigen ~~vectors~~ vectors are

$e^{\frac{ix}{3}}$, $e^{\frac{i(\cos\theta x + \sin\theta y)}{3}}$, cc $\qquad \theta = \frac{\pi}{2}$ is square, $\theta \neq \frac{2\pi}{3}$ or $\frac{4\pi}{3}$

$L_0(v e^{i\vec{k}\cdot\vec{x}}) = (\tilde{L}_0(|k|)\, v) e^{i\tilde{k}\cdot\tilde{x}}$ because of Euclidean invariance. Similarly for L_1 and $Q_1 + C$ (since these are bi (tri) linear)

As we usually do write

$$u(x,t) = \varepsilon\left(z e^{ix} + w e^{i k(\theta)\cdot\vec{x}}\right) + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots$$

with z, w function of $\tau = \varepsilon^2 t$, $\lambda = \varepsilon^2 \tilde{\lambda}$.

(write $c = \cos\theta$, $s = \sin\theta$)

ε order

$$\cancel{L_0 \xi = 0} \qquad L_0(1)\xi = 0$$

Let $L_0^{*}(1)\,\eta = 0 \quad \eta\cdot\xi = 1$

ε^2 order

$$0 = L_0 u_2 + Q(\xi,\xi)\Big[z^2 e^{2ix} + w^2 e^{2i k(\theta)\cdot\vec{x}} + 2\bar{z} w e^{i((c-1)x+sy)} + 2z\bar{z} + 2w\bar{w} + 2zwe^{i((1+c)x+sy)}$$

$+$ c.c. $z\bar{w}$, $\bar{z}\bar{w}$, $\bar{z}^2$, $\bar{w}^2$

$$L_0\, e^{i(k_1x+k_2y)} v = \left(\hat{L}_0(k_1^2+k_2^2)\, v\right) e^{i(k_1x+k_2y)}$$

$\hat{L}_0(4)$ (covers z^2, w^2) $\hat{L}_0(0)$ covers $z\bar{z}, w\bar{w}$

$(c-1)^2 + s^2 = c^2 + s^2 + 1 - 2c = 2(1-c) \quad \hat{L}_0(2(1-c))$

$(c+1)^2 + s^2 \to 2(1+c)$.

All of these are invertible by hypothesis so we can solve quadratic part

Note what if $c = \pm\frac{1}{2}$?? UH OH but that is Hexagonal lattice! so $c \neq \pm\frac{1}{2}$.

Thus we get

$$u_2(x) = V_{20} e^{2ix} z^2 + V_{20} e^{2i(\vec{k}\cdot\vec{x})} w^2 + 2V_{11}(\theta) z w e^{i(x+\vec{k}\cdot\vec{x})}$$

$$+ 2V_{1,-1}(\theta) e^{i(x-\vec{k}\cdot\vec{x})} z\bar{w} + 2V_{00}(w\bar{w} + z\bar{z}) + cc$$

where $\hat{L}(4) V_{20} + Q(\xi,\xi) = 0$ (independent of θ)

$\hat{L}(0) V_{00} + Q(\xi,\xi) = 0$ (ind of θ)

$\hat{L}(2(1+\cos\theta)) V_{11} + Q(\xi,\xi) = 0$ dep on θ

$\hat{L}(2(1-\cos\theta)) V_{1,-1} + Q(\xi,\xi) = 0$ " " "

So onto cubic terms

$$(z_\tau e^{ix} + w_\tau e^{i\vec{k}\cdot\vec{x}})\xi = L_0 u_3 + \lambda \hat{L}_1(1)\xi(z e^{ix} + w e^{i\vec{k}\cdot\vec{x}})$$

NOTE "6 vs 3"

$$+ C(\xi,\xi,\xi)[3z^2\bar{z} e^{ix} + 6 w\bar{w} z e^{ix} \ldots] \quad e^{ix}$$

$$+[4Q(\xi,V_{00}) z^2\bar{z} + 4Q(\xi,V_{11}) z w\bar{w} + 4Q(\xi,V_{1,-1}) z w\bar{w}$$

$$+ 2Q(\xi,V_{20}) z^2\bar{z}] \text{ and other stuff}$$

Apply Fredholm alternative & get

$$z_\tau = z[\lambda \alpha_{00} + z\bar{z}\alpha_{10} + w\bar{w}\alpha_{01}]$$

$$\alpha_{00} = \eta \cdot \hat{L}_1(1)\xi \neq 0$$

$$\alpha_{10} = \eta \cdot [3C(\xi,\xi,\xi) + 4Q(\xi,V_{00}) + 2Q(\xi,V_{20})]$$

$$\alpha_{01} = \eta \cdot [6C(\xi,\xi,\xi) + 4\underbrace{Q(\xi,V_{11}) + 4Q(\xi,V_{1,-1})}_{\theta \text{ dependent}} + 4Q(\xi,V_{00})]$$

This is pretty cool – only the $w\bar{w}$ terms in the z equation contain Lattice dependence. Also if NO quadratic terms, then no lattice dependence and also there will never be stable squares since $6>3$.

~~Hexagons~~

Note that as $\theta \to 0$ $\quad V_{11} \to V_{20}$ and
$\quad V_{1-1} \to V_{00}$

so

$$\alpha_{10} \equiv \mathcal{L}(0) + \alpha_{01} = 2\mathcal{L}(\theta)$$

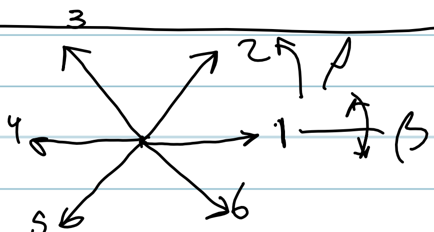
where $\mathcal{L}(\theta) = 3C(\xi,\xi,\xi) + 2Q(\xi, V_{11}(\theta))$
$\qquad + 2Q(\xi, V_{1,-1}(\theta)) + 2Q(\xi, V_{00})$

so check $\mathcal{L}(0) = 3C(\xi,\xi,\xi) + 2Q(\xi, V_{20}) + 4Q(\xi, V_{0,0})$
This is called the lattice function.

This was proven by Sattinger by using the symmetry & L.S. reduction. We have shown it with direct calculation!

θ-dependence arises only from the quadratic terms. These also allow one to get squares.
(My proof is much simpler)

Hexagonal Lattice



Here $u(x,t) = \sum_{j=1}^{6} z_j(t) e^{i \vec{k}_j \cdot \vec{x}}$

Rotation: $(z_1\ z_2\ z_3\ z_4\ z_5\ z_6) \to (z_2, z_3, z_4, z_5, z_6, z_1)$
reflection $(z_1, z_2\ z_3\ z_4\ z_5\ z_6) \to (z_1\ z_6\ z_5\ z_4\ z_3\ z_2)$
can do more rotations to get rest of equations, thus we have all equations once we have F_1. Note that $\underline{z_4 = \bar{z}_1 \quad z_5 = \bar{z}_2 \quad z_6 = \bar{z}_3}$

$F_4(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_1, z_2, z_3, z_4, z_5, z_6)$

~~Also $(x,y) \to (-x,-y) \Rightarrow$~~
reflection about y-axis gives $z_1 \to z_4 \quad z_2 \to z_3$
$z_5 \to z_6$

$F_4(z_1, z_2, z_3, z_4, z_5, z_6) = F_1(z_4, z_3, z_2, z_1, z_6, z_5)$
and so on.

Lets recall the following $k_2 + k_6 = k_1$
and so on.

Translation $(x_1+g_1, x_2+g_2) \to e^{i\vec{k_j}\cdot\vec{g}} z_j$

$$e^{i\vec{k_1}\cdot\vec{g}} \sum z_1^n z_2^m z_3^l \bar{z}_1^p \bar{z}_2^q \bar{z}_3^r =$$

$$\sum e^{ik_1 \cdot g n} e^{ik_2\cdot g m} e^{ik_3 \cdot g l} e^{-ik_1\cdot g p} e^{-k_2\cdot g q} e^{-k_3 \cdot g r} \alpha z_1^n z_2^m z_3^l \bar{z}_1^p \cdots$$

so this means

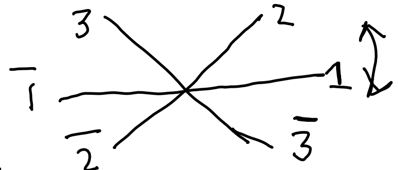
$$\vec{k_1} = (n-p)\vec{k_1} + \vec{k_2}(m-q) + \vec{k_3}(l-r)$$

Clearly if we choose $n-p=1$, $m-q=0$, $l-r=0$ we will satisfy this. However we also have $k_2 + k_6 = k_2 - k_3 = k_1$ so ~~must have~~ could have $m=1$, $r=1$ which give quadratic term. Thus up to cubic terms we have

$$F_1 = \alpha_{0} z_1 + \alpha_1 z_1 \bar{z}_1 + \alpha_2 z_2 \bar{z}_2 + \alpha_3 z_3 \bar{z}_3 + \beta z_2 \bar{z}_3$$

Flip around x-axis means interchanging $(z_2, \bar{z}_3), (z_3, \bar{z}_2)$ doesnt change F_1

So we get $\alpha_2 = \alpha_3$
Invariance under $(z_1, z_2, z_3) \to (\bar{z}_1, \bar{z}_2, \bar{z}_3)$ (reflection) $\Rightarrow \bar{F}_1() = F_1(\bar{\ })$
$\Rightarrow$ all coefficients are real
(Note this is different from reality)



Thus we get after some manipulation & scaling

$$\frac{dz_1}{d\tau} = z_1 \lambda + a z_2 \bar{z}_3 - b z_1 |z_1|^2 - z_1 c\left(|z_2|^2 + |z_3|^2\right)$$

$$\frac{dz_2}{d\tau} = z_2 \lambda + a z_3 z_1 - b z_2 |z_2|^2 - z_2 c\left(|z_1|^2 + |z_3|^2\right)$$

$$\frac{dz_3}{d\tau} = z_3 \lambda + a z_2 \bar{z}_1 - b|z_3|^2 - z_3 c\left(|z_2|^2 + |z_1|^2\right)$$

a, b, c are all real.

What are solutions?
we choose $a > 0$ WLOG (since if $a < 0$ then & change $z_j \to -z_j$ & get eqns back with $a > 0$)

Write $z_j = r_j e^{i\theta_j}$ & we get

$$\dot{r}_1 = \lambda r_1 + a r_2 r_3 \cos(\theta_2 - \theta_3 - \theta_1) - b r_1^3 - c(r_1 r_2^2 + r_1 r_3^2)$$

$$\dot{r}_2 = \lambda r_2 + a r_1 r_3 \cos(\theta_1 + \theta_3 - \theta_2) - b r_2^3 - c(r_2 r_1^2 + r_2 r_3^2)$$

$$\dot{r}_3 = \lambda r_3 + a r_1 r_2 \cos(\theta_2 - \theta_1 - \theta_3) - b r_3^3 - c(r_3 r_1^2 + r_3 r_2^2)$$

$$\left.\begin{aligned} r_1 \dot{\theta}_1 &= a r_2 r_3 \sin(\theta_2 - \theta_3 - \theta_1) \\ r_2 \dot{\theta}_2 &= a r_1 r_3 \sin(\theta_1 + \theta_2 - \theta_3) \\ r_3 \dot{\theta}_3 &= a r_1 r_2 \sin(\theta_2 - \theta_1 - \theta_3) \end{aligned}\right)$$

Take $\theta_j = 0$ & these are all fine!

So we are left with a bunch of algebraic equations.

Rolls, $r_2 = r_3 = 0$ & $r_1 = \pm\sqrt{\lambda / b}$

Hexagons $r_1 = r_2 = r_3 = R_0$ where
$\lambda + a R_0 - (b + 2c) R_0^2 = 0$ $\Rightarrow$ $\lambda > -\dfrac{a^2}{4(b+2c)}$ (solve quadratic!)

Finally, there are rectangles

$r_2 = r_3 \neq r_1 \Rightarrow$

(1) $\lambda r_1 + a r_2^2 - b r_1^3 - 2c r_1 r_2^2 = 0$

(2) $\lambda r_2 + a r_2 r_1 - r_2(b r_2^2 + c r_2^2 + c r_1^2) = 0$

Solve (2) for r_2 by dividing by r_2 first & getting r_2 as function of r_1. Substitute this into (1) to get cubic. You will find several soln to cubic, but one we will focus on is

$$r_1 = -\frac{a}{b-c}, \quad r_2 = r_3 = \pm\sqrt{\frac{1}{b+c}\left(\lambda - \frac{a^2}{(b-c)^2}\right)}$$

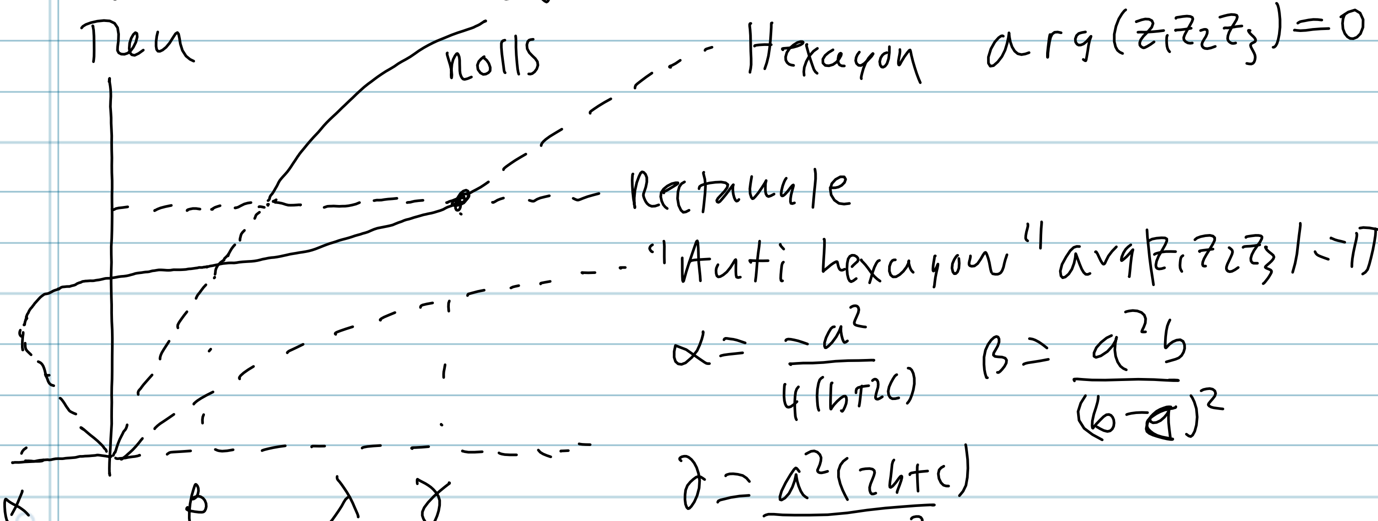
This branch does not bifurcate from zero since $r_1 \neq 0$ as $\lambda \to 0$.

Note that when $\lambda = \frac{a^2 b}{b-c}$ $\quad r'_{ROLL} = r'_{RECTANGLE}$

and when

$$\lambda = \frac{(2b+c)a^2}{(b-c)^2},$$

they meet the Hexagon branch. Let's suppose $b-c<0$. Then



$$\alpha = \frac{-a^2}{4(b+2c)} \qquad \beta = \frac{a^2 b}{(b-c)^2}$$

$$\gamma = \frac{a^2(2b+c)}{(b-c)^2}$$

Before turning to the ~~a~~ stability analysis we consider a simpler version where $a=0$. This means no quadratic terms. It also says something about b & c (cf ~~not~~ square)

$$\dot{r}_1 = r_1(\lambda - b r_1^2 - c(r_2^2+r_3^2))$$
$$\dot{r}_2 = r_2(\lambda - b r_2^2 - c(r_1^2+r_3^2))$$
$$\dot{r}_3 = r_3(\lambda - b r_3^2 - c(r_1^2+r_2^2))$$

Rolls $r_1 = \sqrt{\lambda/b}$

$$\begin{bmatrix} \lambda-3br_1^2 & 0 & 0 \\ 0 & \lambda-cr_1^2 & 0 \\ 0 & 0 & \lambda-cr_1^2 \end{bmatrix} \rightarrow \begin{bmatrix} -2\lambda & 0 & 0 \\ 0 & \lambda(1-\frac{c}{b}) & 0 \\ 0 & 0 & \lambda(1-\frac{c}{b}) \end{bmatrix}$$

stable if $c > b > 0$, unstable if $b > c > 0$

Hex $r_1 = r_2 = r_3 = \sqrt{\frac{\lambda}{b+2c}}$ ~~$\lambda = b$~~ NB $\lambda = (b+2c)r^2$

$= ~~(b^2+\ldots)r^2~~$

$$M \quad \begin{bmatrix} \lambda-3br^2-2cr^2 & -2cr^2 & -2cr^2 \\ -2cr^2 & \lambda-3br^2-2cr^2 & -2cr^2 \\ -2cr^2 & -2cr^2 & \lambda-3br^2-2cr^2 \end{bmatrix}$$

Note $\alpha \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ has eigenvalues 0, 3α

So $M = (\lambda - 3br^2)I - 2cr^2 \mathbb{1}$ has eigenvalues

$\lambda - 3br^2$, $\lambda - 3br^2 - 6cr^2 = b+2c-3b-6c = -2(2c+b) < 0$

$r^2[\ldots]$ ~~$r^2[b^2+4cb+4c^2-3b]$~~ $r^2[b+2c-3b] = 2(c-b)r^2 < 0$ $c < b$ > 0 $c > b$

So in pure cubic case ~~dd~~ Hex stable when $\boxed{b > c > 0}$

Back to Full problem.

Rolls

$$\begin{bmatrix} \lambda - 3br^2 & 0 & 0 \\ 0 & \lambda - c\cancel{\lambda}r^2 & ar_0 \\ 0 & ar & \lambda - cr^2 \end{bmatrix} \quad r = \sqrt{\frac{\lambda}{b}} \Rightarrow r^2 = \frac{\lambda}{b} \Rightarrow \lambda = br^2$$

$$\Rightarrow \begin{bmatrix} -2br^2 & 0 & 0 \\ 0 & (b-c)r^2 & ar \\ & ar & (b-c)r^2 \end{bmatrix}$$

$\Rightarrow (b-c)\, r^2 \pm ar$ are eigenvalues. Take

$\Rightarrow$ ~~or~~ or better $(b-c)^2 r^4 - a^2 r^2 > 0 \quad r^2 = \frac{\lambda}{b}$

$\Rightarrow \frac{(b-c)^2}{b^2} \lambda^2 > \frac{a^2 \lambda}{b} \Rightarrow \lambda > \frac{a^2 b}{(b-c)^2} = \beta$

Trace $= 2(b-c)\, r^2 < 0$ if $b < c$ which is why we drew!! If $b - c > 0$ then rolls never stable

oops $\lambda \equiv \mu$

Hexagons

$$\begin{bmatrix} \mu - 3br^2 - 2cr^2 & -2cr^2 + ar & -2cr^2 + ar \\ -2cr^2 + ar & \mu - 3br^2 - 2cr^2 & -2cr^2 + ar \\ -2cr^2 + ar & -2cr^2 + ar & \mu - 3br^2 - 2cr^2 \end{bmatrix}$$

Writing this as $(\mu - 3br^2 - ar)\, I + (ar - 2cr^2)\, \mathbb{1}$ + using same trick as before, we get eigenvalues

$\mu - 3br^2 - ar, \quad \mu - 3br^2 - ar + 3(ar - 2cr^2)$ (twice)

using $\mu + ar - (b+2c)r^2 = 0$ we get

$$\left.\begin{array}{l} -2ar - 2(b-c)r^2 < 0 \\ ar - 2(b+2c)r^2 < 0 \end{array}\right) \text{ For stability}$$

Stability continued

$(*)$ $2ar > -2(b-c)r^2$ $\quad$ $b>c$ always true

$a > (b-c)r$ $\quad$ $b<c$ (drawn)

$\bigstar$ $a < 2(b+2c)r$, $a > (c-b)r$

$\Rightarrow (c-b) < 2(b+c)$ which is always true if $b,c>0$

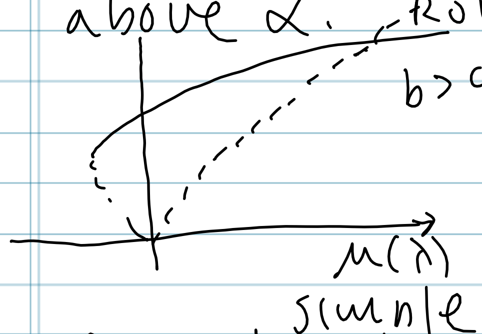
Note $r = \frac{1}{2(b+2c)}\left[a \pm \sqrt{a^2+4\mu(b+2c)}\right]$

Note $\bigstar$ says $r > \frac{a}{2(b+2c)}$ so $-$ root is unstable, $+$ root satisfies first part of $\bigstar$

we also need $\bigstar$ $r < \frac{a}{c-b}$. But when $r = \frac{a}{c-b} = -\frac{a}{b-c}$

This is when Hex hits the rectangles so we have proven most of our picture.
proving ~~stabil~~ unstability of rectangles is a pain.

when $b>c$. Then condition $(*)$ is always true and ~~rectangles~~ Hex are stable for all μ above α. Rolls are never stable when $b>c$



Beyond Lattices.

Recall in a finite square domain, $L \times L$ we can ~~have~~ have $k_c^2 = \frac{4\pi^2}{L^2}[n^2+m^2]$

or $n^2+m^2 = \frac{L^2}{4\pi^2}k_c^2$

Treat k_c^2 as fixed from the physical parameters & let L vary. For example let L be such that $\frac{L^2}{4\pi^2} k_c^2 = 1$

Then only possible (n, m) are $(\pm 1, 0)$, $(0, \pm 1)$

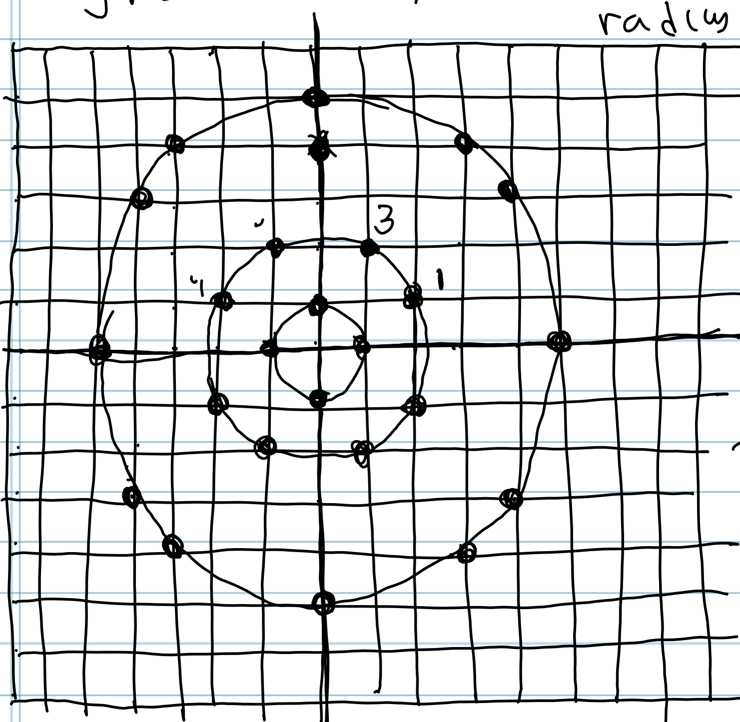
Suppose choose L a bit bigger so that

$\frac{L^2}{4\pi^2} k_c^2 = 5$, so $n^2+m^2=5$, $(\pm 2,1)$ $(\pm 1,2)$ & all negatives

Here is a picture of what is going on as L grows

radius 1   radius 2 ... radius 5

radius $\sqrt{2}$ radius $\sqrt{5}$



Note first interesting case is radius $\sqrt{5}$ with 8-dimensional null space

$$z_1 e^{i(2x+y)} + z_2 e^{i(-x+2y)} + z_3 e^{i(x+2y)} + z_4 e^{i(-2x+y)} + c.c$$

$\boxed{HW}$ is to sketch these & derive equations for ~~solutions to~~ bifurcation problem

~~radius~~ 25 has 12-dim

Curious case of mode interactions. In $\boxed{HW}$ for 2 weeks, you derive equations for bifurcation on a line when two modes go ~~back~~ unstable at once:

$$u(x,t) = z_1 e^{ix} + z_2 e^{2ix} + c.c.$$

You derive:
$$\frac{dz_1}{dt} = \mu_1 z_1 + a \bar{z}_1 z_2 - b_1 |z_1|^2 z_1 - b_2 |z_2|^2 z_1$$

$$\frac{dz_2}{dt} = \mu_2 z_2 + c z_1^2 - d_1 |z_2|^2 z_2 - d_2 |z_1|^2 z_2$$

WLOG let $a=1$, $c=\pm 1$ (rescaling amplitude, but cannot fix sign of c) all par are real.

Let $z_1 = Re^{i\phi}$, $z_2 = Se^{i\psi}$, let $\cancel{\theta}$ $= \psi - 2\phi$

Get,

$$R' = \mu_1 R + RS\cos\theta - b_1 R^3 - b_2 S^2 R$$
$$S' = \mu_2 S \pm R^2 \cos\theta - d_1 S^3 - d_2 S R^2$$
$$\theta' = \left(\mp \frac{R^2}{S} - 2S\right)\sin\theta$$

Let $X = S\cos\theta$, $Y = S\sin\theta$, $Z = R^2$

$$X' = \mu_2 X \pm Z + 2Y^2 - d_1 X(X^2+Y^2) - d_2 X Z$$
$$Y' = \mu_2 Y - 2XY - d_1 Y(X^2+Y^2) - d_2 YZ$$
$$Z' = 2Z(\mu_1 + X - b_1 Z - b_2(X^2+Y^2))$$

(1) Pure modes $Y = Z = 0$ $X^2 = \frac{\mu_2}{d_1}$ $(R=0, \theta=0, S^2 = \frac{\mu_2}{d_1})$
~~X=Y=0~~ (Note $z_2 = 0$ is NOT invariant !!)

(2) Mixed modes $Y = 0$ $(\theta = 0, \pi)$
$\mu_1 + X - b_1 Z - b_2 X^2 = 0$ $\mu_2 X \pm Z - d_1 X^3 - d_2 X Z$

(3) Really cool! $c=-1$, travelling waves
~~$Z \neq 2(X^2+Y^2)$~~

Note $R\frac{d\phi}{dt} = RS\sin(\psi - 2\phi)$
$= RS\sin\theta$

If $\theta \neq 0, \pi$ then must have $\mp\frac{R^2}{S} - 2S = 0$
$\Rightarrow$ take $c=-1$ $\Rightarrow R^2 = 2S^2$
$\Rightarrow R = \sqrt{2}S$

$$0 = \mu_1 + S\cos\theta - 2b_1 S^2 - b_2 S^2$$
$$0 = \mu_2 - 2S\cos\theta - d_1 S^2 - 2d_2 S^2$$

multiply First by 2 + add to second

$2\mu_1 + \mu_2 - (4b_1 + 2b_2)s^2 - d_1 s^2 - 2d_2 s^2 = 0$

$$\boxed{(2\mu_1 + \mu_2)/(4b_1 + 2b_2 + d_1 + 2d_2) = s^2}$$

~~$\cos\theta = (2b_1 + b_2)(2\mu_1 + \mu_2)$~~

$$\cos\theta = \frac{-\mu_1 + (2b_1 + b_2)s^2}{s}$$ If this is <1

Then $\exists\,\theta$ and Thus there are travelling waves!! since $\boxed{\frac{d\phi}{dt} = 2 s^2 \sin\theta}$

since $\theta \neq 0, \pi$ & $s \neq 0$ $\frac{d\phi}{dt} = \omega \neq 0$ and

$\phi = \omega t$

$$\Rightarrow u(x,t) = \sqrt{2}\, s\, e^{i(x+\omega t + \phi_0)} + s\, e^{2i(x+\omega t + 2\phi_0)} + \underline{cc}$$

$\boxed{\psi_0 - 2\phi_0 = 0}$ pretty bizarre.

called Drift instability

There are many other mode interaction

For example if ~~two~~ modes are $n, n+1$

with $n \neq 1$ then it's back to our standard

Friends without resonant term

eg $(2,3)$ cannot get

$2 = n2 + m3$ except $n=1, m=0$

or $n=-2, m=2$ But this is not cubic?

So far we have spent all our time on square & lattice domains.

What about radially symmetric solutions.

For example Lets consider a reaction diffusion equation in polar coordinates!

$$\frac{\partial u}{\partial t} = F(u,\lambda) + D\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right]$$

Can we get bifurcation to radially symmetric solutions (These are like 1-d solutions)

want $\vec{u}(r) \to 0$ as $r \to \infty$

Linearized equation — radially symmetric

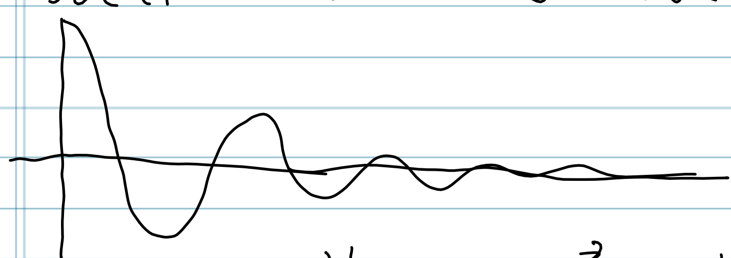
$$\frac{\partial V}{\partial t} = AV + D\left[\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r}\right]$$

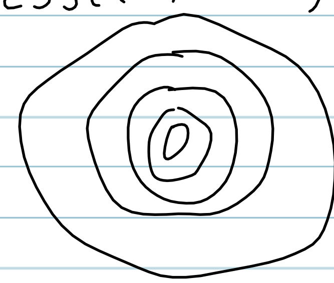
Look at eigenvalues of $\frac{d^2 y}{dr^2} + \frac{1}{r}\frac{dy}{dr} = -k^2 y$

$$r^2\frac{d^2 y}{dr^2} + r\frac{dy}{dr} + r^2 k^2 y = 0$$

Let $rk = x \Rightarrow x^2\frac{d^2 y}{dx^2} + x\frac{dy}{dx} + x^2 y = 0$

Solution to this is $J_0(x)$ Bessel Function





bull eye

So $V(r,t) = e^{\nu t} J_0(kr)\vec{\xi}$ where

$$\boxed{\nu\xi = A - k^2 D}$$ Looks Familiar!!