Beyond pattern forming instabilities

- The standard model for field equations

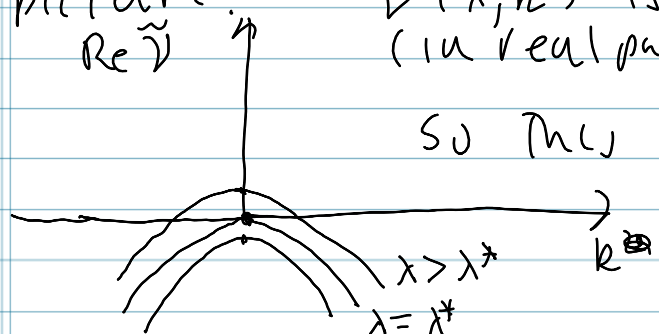
I will start with another reaction-diffusion example and try to derive a classic model from this, the Ginzburg Landau equations.

$$\frac{\partial u}{\partial t} = F(u, \lambda) + D \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty.$$

and I will work on the infinite domain

We suppose that $u_0(\lambda)$ is an equilibrium point and that we have the following picture: $\tilde{\nu}(\lambda, k^2)$ is maximum eigenvalue (in real part sense)

F1



So this is a Hopf or zero with ~~non~~ zero wave number $k^* = 0$

Typically we ignore space completely since eigenspace is either

$$z e^{i\omega_0 t} \vec{\xi} + cc \quad \text{or} \quad r \vec{\xi}$$

with $z = z(\tau)$, $r = r(\tau)$, $\tau = \varepsilon^2 t$ or εt

But clearly, space exists in this model so how do we incorporate it.

For simplicity, suppose it is a Hopf bifurcation, so that in absence of space, we have:

$$Z_\tau = Z(\alpha(\lambda-\lambda^*) - \beta Z\bar{Z}) \qquad (\star\star)$$

We have already figured out how to compute α, β.

Let us alter our ansatz. Note that for $\lambda > \lambda^*$, $|k|$ small but nonzero ~~p~~ ~~ly~~ leads to instability & growth. Thus we will assume that there are "long wave" effects and thus look for patterns that depend on slow modulations in space. Lets introduce a spatial effect

$$\zeta = \varepsilon X$$

so that $\frac{\partial^2}{\partial X^2} = \varepsilon^2 \frac{\partial^2}{\partial \zeta^2}$

~~Without belaboring the calculations, we get a new term in our perturbation.~~

We suppose that ~~instead of~~

$$u(x,t) = \varepsilon Z(\tau,\zeta)\, e^{i\omega_0 t}\, \vec{\xi} + cc + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots$$

Without belaboring calculations, we get a new term in the $O(\varepsilon^3)$ equations:

$$Z_\tau e^{i\omega_0 t} \vec{\xi} = A_0 u_3 - \frac{\partial u_3}{\partial t} + \overset{\text{Nonlinear stuff}\rightarrow}{\cdots} + D \vec{\xi} Z_{\zeta\zeta} e^{i\omega_0 t}$$

so that when we apply the Fredholm Alt we get the spatial analogue of $(\star\star)$

$$\frac{\partial z}{\partial t} = z(\alpha(\lambda-\lambda^*) - \beta z\bar{z}) + d\, z_{33} \quad (2)$$

Where $d = \bar{\eta}^T D \xi$

We can now prove that if the picture is as in $\boxed{F}$, that $\mathrm{Re}\, d > 0$.

To see why this is, note that

$\mathrm{Re}\, \tilde{\nu}(k^2) \approx -B k^2$ at $\lambda = \lambda^*$ by hypothesis (see Figure) where $B > 0$

Let $A_0 - k^2 D \equiv M(k^2)$ and we will examine $\tilde{\nu}$ for k^2 small

write $M(k^2)\phi(k^2) = \tilde{\nu}(k^2)\phi(k^2)$

differentiate wrt k^2: $\qquad \phi_0 = \vec{\xi}$

$$-D\phi_0 + A_0\phi' = i\omega_0\phi' - B\phi_0 + iC\phi_0$$

multiply by $\bar{\eta}^T$ to get

$$d \equiv \bar{\eta}^T D \xi = B - iC$$

So $\mathrm{Re}\, d > 0$.

Equation (2) is called complex Ginzberg-Landau equation or CGL

Let me absorb $\lambda - \lambda^*$ into α and write

$\alpha = a_1 + i a_2$, $\beta = b_1 - i b_2$, $d = d_1 + i d_2$

$z = r e^{i\theta}$

$$r_t = (a_1 - b_1 r^2) r + d_1(r_{33} - r\theta_3^2) - d_2(2r_3\theta_3 + r\theta_{33})$$

$$\theta_t = a_2 + b_2 r^2 + d_1\left(\theta_{33} + \frac{2 r_3 \theta_3}{r}\right) + d_2\left(\frac{r_{33}}{r} - \theta_3^2\right)$$

Even when $d_2 = 0$, this equation has many interesting solutions (most of which are dynamically unstable)

Let's explore some of them.

Let $r = \rho$ $\theta(\xi, \tau) = \Omega\tau + \alpha\xi$

plug this in and we get:

$$0 = \rho(a_1 - b_1\rho^2) - d_1\rho\alpha^2$$

$$\Rightarrow \rho = \sqrt{\frac{a_1 - d_1\alpha^2}{b_1}} \quad (3)$$

$$\Omega = a_2 + b_2\rho^2 - d_2\alpha^2 \quad (4)$$

This is a family of traveling waves. When $\alpha = 0$, we have the synchronous spatial oscillation, but for $\alpha \neq 0$, we get nice traveling waves.

Notice that since $d_1 > 0$ $b_1 > 0$, $a_1 > 0$ as α increases amplitude decreases and thus, they exist ~~for~~ a limited band of α.

combining (3) & (4) we get:

$$\Omega = a_2 + \frac{a_1 b_2}{b_1} - \left(d_2 + \frac{b_2 d_1}{b_1}\right)\alpha^2$$

The so-called dispersion relation.

Stability is a b*tch, but we will try to get some insights.

To make life easier, I will make some assumptions and rescaling. Assume $b_1 > 0$ $a_1 > 0$ $d_1 > 0$ (not assumption)

Then we can rescale r, t, ζ (amplitude, time, space) & rotate θ in a time-dependent manner to get:

$$\begin{cases} r_t = r(1-r^2) + r_{\zeta\zeta} - r\theta_\zeta^2 - \delta[2r_\zeta\theta_\zeta + r\theta_{\zeta\zeta}) \\ \theta_t = qr^2 + \theta_{\zeta\zeta} + \frac{2r_\zeta\theta_\zeta}{r} + \delta\left[\frac{r_{\zeta\zeta}}{r} - \theta_\zeta^2\right] \end{cases}$$

Traveling wave is $r = \sqrt{1-\alpha^2}$ $\theta = \Omega t + \alpha\zeta$

$\Omega = q(1-\alpha^2) - \delta\alpha^2$ $(\theta_\zeta = \alpha + \varphi_\zeta)$ $\theta_{\zeta\zeta} = \varphi_{\zeta\zeta}$

write $r = \sqrt{1-\alpha^2} + \rho$, $\theta = \Omega t + \alpha\zeta + \varphi$ and linearize to obtain

$$\rho_t = (1-3r^2)\rho + \rho_{\zeta\zeta} - \rho\alpha^2 - 2r\alpha\varphi_\zeta - \delta[2\rho_\zeta\alpha + r\varphi_{\zeta\zeta}]$$

$$\varphi_t = 2qr\rho + \varphi_{\zeta\zeta} + \frac{2\alpha\rho_\zeta}{r} + \delta\left[\frac{\rho_{\zeta\zeta}}{r} - 2\alpha\varphi_\zeta\right]$$

Solutions are autonomous in ζ, t so of the form $e^{\nu t} e^{ikx}$ $\Rightarrow$

$$\nu\begin{bmatrix}\tilde{\rho} \\ \tilde{\varphi}\end{bmatrix} = \begin{bmatrix} -k^2-\alpha^2+1-3r^2-\delta 2ik\alpha & -2rik\alpha + r\delta k^2 \\ 2qr + \frac{2\alpha ik}{r} - \frac{\delta}{r}k^2 & -k^2 - 2i\alpha k\delta \end{bmatrix}\begin{bmatrix}\tilde{\rho} \\ \tilde{\varphi}\end{bmatrix}$$

Want eigenvalues of this complex matrix to have negative real part for all k

~~Lets first look at k = 0~~

$$\begin{bmatrix} \cancel{-2+2\alpha^2} & \\ \cancel{2qr} & \end{bmatrix}$$

Let's first look at small amplitude solution where $\alpha \approx 1$ so that r_0 is small

$$\begin{bmatrix} -k^2 - 2ik\delta\alpha & -2rik\alpha + \delta r k^2 \\ \frac{1}{r}\left[2\alpha i k - \delta k^2\right] & -k^2 - 2\alpha i k \delta \end{bmatrix}$$

For $\delta = 0$ (say, diagonal diffusion)

$Tr = -k^2 \quad \det \approx k^4 - 4\alpha^2 k^2$

For k small, this is negative so there will be a positive eigenvalue

I won't go through details but when $\delta = 0$ we have following result:

Waves are stable iff:

$$\boxed{\alpha^2 < \frac{1}{2+q^2}}$$

Note q makes it <u>worse</u>

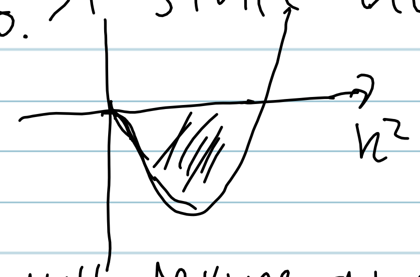
Synchronous solution. $\alpha = 0$, $r_0 = 1$

$$\begin{bmatrix} -k^2 - 2 & \delta k^2 \\ 2q - \delta k^2 & -k^2 \end{bmatrix}$$

$$\text{Det} = k^2(2+k^2) + \delta^2 k^4 - 2q\delta k^2$$
$$= 2k^2[1 - q\delta] + k^4[1+\delta^2]$$

This is really cool if $q\delta > 1$ Then for k small, even synchronous solution is unstable. This is a well known instability & can lead to chaotic behavior as I will show you via numerics or that you can do yourself.

Notice this is classic pattern formin if $q\delta > 1$ since have



We will derive another set of equations called Kuramoto Shivashinsky, shortly.

How general is this idea?

We can replace the diffusion operator with any spatial isotropic operator. For example, consider

$$z(x) := \int_{-\infty}^{\infty} k(x-y)\, u(y)\, dy$$

we rewrite this as $\displaystyle\int_{-\infty}^{\infty} k(y)\, u(x-y)\, dy$

Now suppose u depends on slow ~~time~~ space $\varepsilon x \equiv \xi$.

Then

$$z = \int_{-\infty}^{\infty} k(y)\, u(\xi - \varepsilon y)\, dy \approx \int_{-\infty}^{\infty} k(y) \left[u(\xi) - \varepsilon y u'(\xi) + \frac{\varepsilon^2 y^2}{2} u''(\xi) + \cdots \right] dy$$

$$= k_0 u(\xi) + \varepsilon k_1 u'(\xi) + \frac{\varepsilon^2}{2} k_2 u''(\xi) + \cdots$$

where $k_0 = \int_{-\infty}^{\infty} k(y)\, dy$, $k_1 = \int_{-\infty}^{\infty} y\, k(y)\, dy$, $k_2 = \int_{-\infty}^{\infty} y^2 k(y)\, dy$

If $k(x)$ is isotropic, then $k_1 = 0$ + we recover the same eqns as with the reaction diffusion model.

Now we turn to the following:

$$\frac{\partial u}{\partial t} = F(u) + D \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty$$

but we assume:

$\frac{du}{dt} = F(u)$ has an asymptotically stable periodic solution, $u_0(t+T) = u_0(t)$.

Aside F.A. alternative for periodic systems.

Consider $\frac{dy}{dt} = A(t) y + b(t)$

and suppose $A(t+T) = A(t)$, $b(t+T) = b(t)$

Homogeneous problem:

$\frac{dy}{dt} = A(t)y$ Floquet theory says solutions to this have the form

$y(t) = e^{\mu t} p(t)$ where $p(t+T) = p(t)$

Note that $y(t+T) = e^{\mu(t+T)} p(t+T) = e^{\mu T} y(t) = \gamma y(t)$

γ is a complex number called Floquet multiplier

If $\gamma = 1$ then there is a T-periodic solution to Homogeneous problem (nontrivial, since 0 is always a solution.)

This means

$$Ly \equiv \frac{dy}{dt} - A(t)y$$

has a ~~null~~ nontrivial null space in the space of T-periodic functions. So we can solve

$\frac{dy}{dt} - A(t)y = b(t)$ iff $\int_0^T y^*(t) \cdot b(t)\,dt = 0$
T-periodic

for every solution $y^*(t)$ to the adjoint equation:

$\frac{dy^*}{dt} + A^T(t)y^* = 0$

Back to equation $\frac{du}{dt} = F(u)$ Let $u(t) = u_0(t) + y$

Then $\frac{dy}{dt} = A(t)y$ where $A(t) = D_u F(u_0(t))$

Note $A(t+T) = A(t)$. Note also that $u_0'(t)$ satisfies

$\frac{dy}{dt} = A(t)y$ since $\frac{du_0}{dt} = F(u_0(t))$

so

$$\frac{d^2u_0}{dt^2} = \frac{d}{dt}[u_0'(t)] = D_u F(u_0(t)) \frac{du_0}{dt} = A(t) u_0'(t)$$

So this means that ~~the solution~~ there will be a solution to

$$\frac{dy}{dt} - D_u F(u_0(t))\, y = b(t)$$

iff $\int_0^T y^*(t) \cdot b(t)\, dt = 0$, where

$$\frac{dy^*}{dt} + D_u F(u_0(t))^T y^*(t) = 0 \quad .$$

XPP can compute $y^*(t)$ for you given $u_0(t)$ & $F(u)$.

Note $y^* \cdot u_0'(t) \equiv 1$

With this in mind, we will now analyze long wave perturbations in the RD equation when there is a bulk oscillation. As above, lets suppose

$$u(x,t) = u_0(t+\theta) + \varepsilon u_1 + \varepsilon^2 u_2 + \cdots$$

$\tau = \varepsilon^2 t \quad \zeta = \varepsilon x$

Where $\theta = \theta(\varepsilon^2 t, \varepsilon x)$ is a slowly varying phase.

$$\frac{\partial}{\partial t} u(x,t) = u_0'(t+\theta) + \varepsilon^2 u_0'(t+\theta)\frac{\partial \theta}{\partial \tau} + \cdots$$

$$\frac{\partial}{\partial x} u(x,t) = u_0'(t+\theta)\, \varepsilon \frac{\partial \theta}{\partial \zeta} \qquad \frac{\partial^2 u}{\partial x^2} = u_0''(t+\theta)\, \varepsilon^2 \left(\frac{\partial \theta}{\partial \zeta}\right)^2$$

$$+ \varepsilon^2 u_0'(t+\theta)\frac{\partial^2 \theta}{\partial \zeta^2}$$

So lets plug this assumption into equations.

$$u_0' + \varepsilon u_1' + \varepsilon^2 u_2' + \varepsilon^2 u_0' \frac{\partial \theta}{\partial \tau} = F(u_0) + \varepsilon D_u F(u_0)) u_1$$

$$+ \varepsilon^2 D_u F(u_0)) u_2 + \frac{\varepsilon^2}{2} D_{uu} F(u_0)[u_1, u_1] + \cdots$$

$$+ \varepsilon^2 \theta_\zeta^2 D u_0'' + \varepsilon^2 D u_0' \theta_{\zeta\zeta} + \cdots$$

$u_0' = F(u_0)$ has a solution $u_0(t+\theta)$ where θ is arbitrary phase

$$u_1' - A(t) u_1 = 0 \implies u_1 = 0$$

$$u_2' - A(t)u_2 = -u_0'\frac{\partial\theta}{\partial\tau} + D\left[u_0''\left(\frac{\partial\theta}{\partial z}\right)^2 + u_0'\frac{\partial^2\theta}{\partial z^2}\right]$$

This has a soln iff orthog to adjoint:

$$\frac{\partial\theta}{\partial\tau} = b\left(\frac{\partial\theta}{\partial z}\right)^2 + d\,\frac{\partial^2\theta}{\partial z^2}$$

where $b = \frac{1}{T}\int_0^T y^*(t)\, D\, u_0''(t)\, dt$, $d = \frac{1}{T}\int_0^T y^*(t)\, D\, u_0'(t)\, dt$

We apply this to our λ-ω system.

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 1-u^2-v^2 & -q(u^2+v^2) \\ q(u^2+v^2) & 1-u^2-v^2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \qquad \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos qt \\ \sin qt \end{pmatrix}$$

Adjoint is $\begin{pmatrix} u^* \\ v^* \end{pmatrix} = \begin{bmatrix} \cos qt - \frac{1}{q}\sin qt \\ \sin qt + \frac{1}{q}\cos qt \end{bmatrix}$

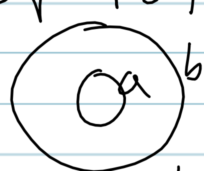
which we leave as an exercise.

The more general form of this equation is in n-dimensions:

$$\frac{\partial\theta}{\partial\tau} = \alpha(\nabla\theta)^2 + \beta\nabla^2\theta \qquad ☆$$

Lets consider the existence of rotating waves in an annulus

(r,ϕ)



In polar coordinates (☆) can be written as

$$\frac{\partial\theta}{\partial\tau} = \alpha\left[\left(\frac{\partial\theta}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\theta}{\partial\phi}\right)^2\right] + \beta\left[\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{1}{r^2}\frac{\partial^2\theta}{\partial\phi^2}\right]$$

We look for rotating waves. These have the form:

$$\Theta(r,\phi,t) = \omega t - \cancel{N\phi} + z(r)$$

where $\frac{dz}{dr}\Big|_{r=a,b} = 0$.

(For example, suppose $z(r)$ is just αr. Then $\omega t - N\phi + \alpha r$ defines an archimedean spiral at each fixed time.)

Plugging this into our equation, we see that

$$\omega = \alpha\left[\left(\frac{dz}{dr}\right)^2 + \frac{N^2}{r^2}\right] + \beta\left[\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr}\right]$$

$$\frac{dz}{dr}(a) = \frac{dz}{dr}(b) = 0 \qquad 0 < a < b < \infty$$

If we assume $\alpha \neq 0$ we can assume $\alpha > 0$ WLOG since we can let $\Theta \to -\Theta$ in $\star$. We can ~~divide by~~ rescale space and divide by α in $\star$ to make $\alpha, \beta = 1$ WLOG.

Let $u = dz/dr$. Then

$$\omega = u^2 + \frac{N^2}{r^2} + u' + \frac{u}{r}$$

we want $u(a) = u(b) = 0$

Lets see why we expect a solution

$$\cancel{u' + u^2 = \omega - \frac{N^2}{r^2} - \frac{u}{r}}$$

Let $u = \frac{v}{r}$. Then $u' = \frac{v'}{r} - \frac{v}{r^2}$ so we get $\omega = \frac{v^2}{r^2} + \frac{N^2}{r^2} + \frac{v'}{r}$

Which we rewrite as:

$$v' + \left(\frac{v}{r}\right) v = r\left(w - \frac{N^2}{r^2}\right)$$ and using an

integrating factor $e^{\int \frac{v}{r}}$ we get

$$\left(v e^{\int \frac{v}{r}}\right)' = e^{\int \frac{v}{r}} r\left(w - \frac{N^2}{r^2}\right)$$

So we can now get some shooting figures

Suppose $w > \frac{N^2}{a^2}$. $v(a) = 0$ so

$\left(v e^{\int \frac{v}{r}}\right)' > 0$ for $r \geq a$ since $\frac{N^2}{r^2} \leq \frac{N^2}{a^2}$

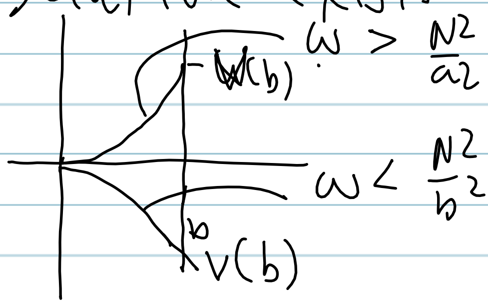
Thus $v > 0$ for all r and we have

Now suppose $w < \frac{N^2}{b^2}$.

Then $v < 0$ for all r

Pick b small and greater than a.

solution exists for b small enough



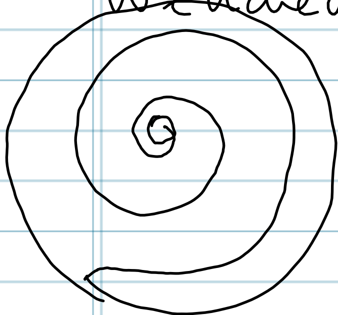
By cont wrt parameter $\exists w$ st $v(b) = 0$.

Can show (with more effort), that infact

there is an w st $u(\infty) = 0$

So, infact for all b there is a solution

~~We have~~ also found that $\omega \approx \frac{k}{a^2}$ as $a \to 0$



← Typical solution found using shooting (on computer)

Target patterns

$$\frac{\partial \theta}{\partial t} = A(\vec{r}) + \nabla^2 \theta + |\nabla \theta|^2$$

$A(\vec{r})$ is defined as follows:

$$\frac{\partial u}{\partial t} = F(u) + \varepsilon G(u, \vec{r}) + \varepsilon D \nabla^2 u$$

Then $A(\vec{r}) = \frac{1}{T} \int_0^T u^*(t)\, G(u_0(t), \vec{r})\, dt$

Let $\theta = \ln Q(r,t)$. ~~Then~~ (called Cole-Hopf)

$$\frac{\partial \theta}{\partial t} = \frac{\partial Q}{\partial t} \frac{1}{Q} \qquad \nabla \theta = \frac{\nabla Q}{Q} \qquad \nabla^2 \theta = -\frac{\nabla Q \cdot \nabla Q}{Q^2} + \frac{\nabla^2 Q}{Q}$$

so we get

$$\frac{\partial Q}{\partial t} = \nabla^2 Q + A(\vec{r})\, Q$$

$Q(\vec{r},t) = e^{\lambda t} p(\vec{r})$ we get

$\lambda p = \nabla^2 p + A(\vec{r}) p$ which is linear eigenvalue problem. Note $\ln e^{\lambda t} \sim \lambda t$ so it is ok for θ since it is a phase & can grow linearly in time. (and in fact, should)

Lets consider a single radially symmetric source $A(\vec{r}) = A(r)$. Then we get

$$\lambda P = \frac{d^2 P}{dr^2} + \frac{1}{r}\frac{dP}{dr} + A(r)P \qquad \text{want } P(r) > 0$$

(since $\theta = \ln(P) + \lambda t$. want $P(r)$ finite as $r \to 0$ and $r \to \infty$.

We can rewrite this as

$$\left(r\frac{d^2P}{dr^2} + \frac{dP}{dr}\right) + \left(rA(r) - \lambda r\right)P = 0$$

$$= \frac{d}{dr}\left(r\frac{dP}{dr}\right) + \left(rA(r) - \lambda r\right)P = 0$$

with the $\frac{dP}{dr}(0) = 0$ a necessary condition.

Existence of solutions come from Sturm Liouville theory but could also be found by shooting

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1-dim solutions:

HW is to study inhomogeneties in finite domain

$$\theta_t = \alpha\theta_x^2 + \beta\theta_{xx}$$

Traveling wave $\theta = \omega t - kx \Rightarrow \omega = \alpha k^2$

Are there other solutions? consider $\theta = \omega t + q(x)$ Then

$$\omega = \alpha q_x^2 + \beta q_{xx} \qquad \text{Let } q_x = P$$

HW set $\alpha = \beta = 1$

$$\omega - P^2 = P_x$$

Prove $\exists$ solution $P(-\infty) = -\sqrt{\omega}$, $P(+\infty) = \sqrt{\omega}$ which have 1-dim targets!!

Consider an arbitrary volume V

enclosed by surface S

Let $C(x,t)$ be the concentration of some substance (could be probability or density etc)

The rate of change of "stuff" in the volume is equal to the flow into and out of the volume plus the production within the volume:

$$\frac{\partial}{\partial t}\int_V c(\vec{x},t)\,dv = -\int_S \vec{J}\cdot d\vec{S} + \int_V f\,dv$$

Apply divergence theorem to get continuity equation:

$$\int_V \left(\frac{\partial c}{\partial t} + \nabla\cdot J - f\right) dv = 0$$

Since V is arbitrary:

$$\boxed{\frac{\partial c}{\partial t} + \nabla\cdot J - f = 0}$$

Diffusion

$J = -D\nabla c$

+ we get

$$\frac{\partial c}{\partial t} = f + \nabla(D\nabla c)$$

. If D is constant then pull it out to get std RD equation.

more crowded
more dispersal

We have already studied this in detail. Some models for animal dispersal have D dependent on n, eg $D(n) = D_0\left(\frac{n}{n_0}\right)^m$ with $m>0$. where n is density of animals

Pure dispersal of insects: in one dimension:

$$\frac{\partial n}{\partial t} = D_0 \frac{\partial}{\partial x}\left(\left(\frac{n}{n_0}\right)^m \frac{\partial n}{\partial x}\right)$$

$$n(x,t) = \begin{cases} \frac{n_0}{\lambda(t)}\left[1 - \left\{\frac{x}{r_0\lambda(t)}\right\}^2\right]^{1/m} & |x| \le r_0\lambda(t) \\ 0 & |x| > r_0\lambda(t) \end{cases}$$

$$\lambda(t) = \left(\frac{t}{t_0}\right)^{\frac{1}{2+m}} \qquad r_0 = \frac{Q\,\Gamma\left(\frac{1}{m}+\frac{3}{2}\right)}{\left\{\pi^{\frac{1}{2}} n_0 \Gamma\left(\frac{1}{m}+1\right)\right\}}$$

$$t_0 = \frac{r_0^2 m}{2D_0(m+2)}$$

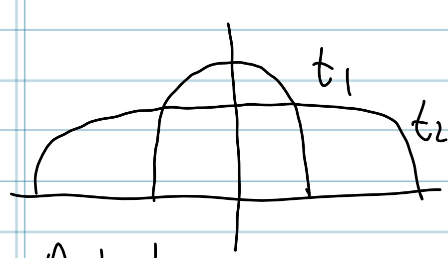
You can check that this is a solution. Interesting question: ask what happens as $m \to 0$. $Q = n(0,0)$

r_0 is from requiring conservation of insects

Recall pure diffusion:

$$n(x,t) = \frac{Q}{2(\pi D t)^{1/2}} e^{-\frac{x^2}{4Dt}}$$

In pure diffusion at any finite $t > 0$, $n(x,t) > 0$ for all x - instant communication. Here it is finite speed. ~~xxx~~ (This is also porous medium equation)



can solve this in radial coordinates as well

At low pops, insects tend to aggregate so might have negative diffusion at low pops

$$J = un - D(n)\frac{\partial n}{\partial x}$$

$u = -u_0\,\mathrm{sgn}(x)$ where origin is point of aggregation

At steady state $n(x,t) \to n(x)$

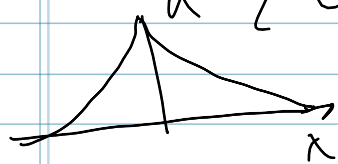
$$0 = u_0 \frac{\partial}{\partial x}[n \operatorname{sgn}(x)] + D_0 \frac{\partial}{\partial x}\left[\left(\frac{n}{n_0}\right)^m \frac{\partial n}{\partial x}\right)$$

integrate (Need $n(x) \to 0$ as $x \to \pm\infty$ so

$$0 = u_0 n \operatorname{sgn}(x) + D_0 \left(\frac{n}{n_0}\right)^m \frac{dn}{dx}$$

~~Take x≥0~~ solve this is an easy exercise

$$n(x) = \begin{cases} n_0\left(1 - \frac{m u_0 |x|}{D_0}\right)^{1/m} & |x| \le \frac{D_0}{m u_0} \\ 0 & |x| > \frac{D_0}{m u_0} \end{cases}$$



Chemotaxis. Let $a(x,t)$ be an attractant in chemotaxis organisms move up or down according to the gradient of $a(x,t)$, ∇a, seeking a maximum (attract) or a minimum (repulsed) so that $J_c = n\chi(a)\nabla a$ χ is called the chemotactic constant. $J_{Total} = J_{Diff} + J_{chem}$

$$\underset{\text{Growth}}{\frac{\partial n}{\partial t} = f(n)} - \underset{\text{chemo}}{\nabla(n\chi(a)\nabla a)} + \underset{\text{Diff}}{\nabla(D\nabla n)}$$

Typically a has dynamics as well:

$$\frac{\partial a}{\partial t} = g(a,n) + \nabla(D_a \nabla a)$$

Keller-Segel equations, $g = hn - ka$ and $f = 0$

We will make a model of axon growth & guidance, for patterns in whisker barrels.

Let $n(x,t)$ = # thalamocortical axons in a barrel. We assume there's random motion, growth, pruning, & chemotactic attraction:

$$\begin{cases} \dfrac{\partial n}{\partial t} = \alpha_0 - \alpha_1 n + D_n \nabla^2 n - \chi \nabla n (\nabla c) \\ \dfrac{\partial c}{\partial t} = f(n) - \alpha_2 c + D_c \nabla^2 c \end{cases} \quad \text{in } \Omega \subset \mathbb{R}^2$$

$\nabla n = \nabla c = 0$ on $\partial\Omega$. $\qquad \underline{\underline{f'(n) > 0}}$ ws $(f(n) = hn)$

equilibria: $\bar{n} = \frac{\alpha_0}{\alpha_1}$ $\quad \bar{c} = \frac{1}{\alpha_2} f\left(\frac{\alpha_0}{\alpha_1}\right)$

Stability:

$$\frac{\partial u}{\partial t} = -\alpha_1 n + D_n \nabla^2 u - \chi \bar{n} \nabla^2 c$$

$$\frac{\partial c}{\partial t} = f'(\bar{n})n - \alpha_2 c + D_c \nabla^2 c$$

Let $-k^2$ be eigenvalue of ∇^2 on Ω $\quad \nabla^2 u = -k^2 u$

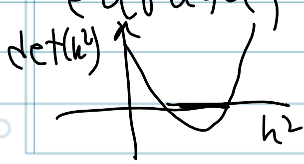
Then $(n,c) = (n_0, c_0) e^{\lambda t} u(\vec{x})$ and:

$$\lambda \begin{bmatrix} n_0 \\ c_0 \end{bmatrix} = \begin{bmatrix} -\alpha_1 - D_n k^2 & \chi \bar{n} k^2 \\ f'(\bar{n}) & -\alpha_2 - D_c k^2 \end{bmatrix} \begin{pmatrix} n_0 \\ c_0 \end{pmatrix}$$

Trace $= -\alpha_1 - \alpha_2 - (D_n + D_c)k^2 < 0$

$\det(k^2) = \alpha_1 \alpha_2 + (\alpha_1 D_n \ \alpha_2 D_c)k^2 + D_n D_c k^4 - \chi \bar{n} f'(n) k^2$

Clearly $\det > 0$ at $k = 0$ $\quad \det > 0$ for k large.
But for k in between, if $\chi f'(n)$ is large enough, then can get pattern formation!



"Cross diffusion"

Example 2 predator prey

prey $\frac{\partial u}{\partial t} = f(u,v) + D_u \nabla^2 u + \chi_u \nabla(u \nabla v)$ ← runaway

pred $\frac{\partial v}{\partial t} = g(u,v) + D_v \nabla^2 v - \chi_v \nabla(v \nabla u)$ ← chase

simplest possible case $f(u,v) = a u(1-u-bv)$
$g(u,v) = c v(-1+du)$
For simplicity take $a=c=1$, $d=2$, $b=1$
So $u=v=\frac{1}{2}$ is coexistant state

Linearization:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \\ \frac{\partial v}{\partial t} &= \end{aligned}\begin{bmatrix} -\frac{1}{2} & -\frac{1}{2} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} D_u & \frac{\chi_u}{2} \\ -\frac{\chi_v}{2} & D_v \end{bmatrix} \nabla^2 \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{bmatrix} -\frac{1}{2} - D_u k^2 & -\frac{1}{2} - \frac{\chi_u}{2} k^2 \\ 1 + \frac{\chi_v}{2} k^2 & -D_v k^2 \end{bmatrix}$$

$Tr = -\frac{1}{2} - (D_u + D_v)k^2$

$\det = \left(\frac{1}{2} + D_u k^2\right) D_v k^2$

$\cancel{+\frac{1}{2} + \frac{\chi_u \chi_v k^4}{4}}$

$+\left(\frac{1}{2} + \frac{\chi_u}{2} k^2\right)\left(1 + \frac{\chi_v}{2} k^2\right)$ – No pattern forma –

Did I make a mistake.

Example 3 Patterns in Bacillus subtilis.

If B. subtilis is inoculated onto agar with very little nutrient, it can exhibit very cool patterns. Let $n(x,t)$ be nutrient and let $b(x,t)$ be bacteria concentration.

$$\frac{\partial n}{\partial t} = D_n \nabla^2 n - \frac{knb}{1+\gamma n} \leftarrow \text{consumption of nutrient}$$

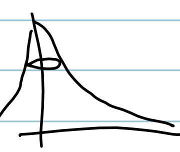
$$\frac{\partial b}{\partial t} = \nabla(D_b \nabla b) + \theta \frac{knb}{1+\gamma n} \leftarrow \text{growth rate of bacteria from nutrient}$$

Experimentally it is found that bacteria tend to move very little when nutrient is low at core of colony & also very little at edge where density of bacteria is low. So they suggest

$$D_b = \sigma n b$$

They also supposed $\sigma = 1 + \Delta$ where Δ is random fluctuation of movement.

initial data $n(x,0) = n_0$, $b(x,0) = b_0(x)$



can rescale the equation as

$$n^* = \left(\frac{\theta}{D_n}\right)^{\frac{1}{2}} n \quad b^* = \left(\frac{1}{\theta D_n}\right)^{1/2} b \quad \gamma^* = \left(\frac{D_n}{\theta}\right)^{1/2} \gamma$$

$$t^* = k(\theta D_n)^{1/2} t \quad x^* = \left(\frac{\theta k^2}{D_n}\right)^{1/4} x$$

to get simpler model & then set $\gamma = 0$ since n is pretty low.

$$\frac{\partial n}{\partial t} = \nabla^2 n - nb \qquad \frac{\partial b}{\partial t} = \nabla(\sigma n b \nabla b) + nb$$

σ + initial data are only parameters

Some examples

uniform
$\sigma_0 = 4$
ν_0 (initial food) $= 1.07$

complex branched fractal
colony
$\sigma = 1, \nu = .35$

colony growth approximation. Tips moving out at some rate

consider $\frac{dn}{dt} = -nb \quad \frac{db}{dt} = bn \sim \quad n + b \equiv$ constant

so $\frac{db}{dt} \approx b\left(1 - \frac{b}{N}\right)$ so maximum of N is

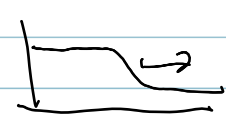
paper approximat

$$\frac{\partial b}{\partial t} = \nu_0 b\left(1 - \frac{b}{N}\right) + \frac{\partial}{\partial x}\left[\sigma \nu_0 b \frac{\partial b}{\partial x}\right]$$

$\nu_0 = 1$ WLOG (rescale time)
rescale b/N

Lets look for traveling waves. $b(x,t) = U(z)$
$z = x - ct$

$$-c\frac{db}{dz} = b(1-b) + \sigma\frac{d}{dz}\left(b\frac{db}{dz}\right)$$



$U(+\infty) = 0 \quad U(-\infty) = 1$

$$b\frac{db}{dz} = a$$

$$\sigma b\frac{da}{dz} = -c\frac{a}{b} - b^2(1-b)$$

$$\sigma\frac{da}{db} = \frac{-ca - b^2(1-b)}{a}$$

can solve this by shooting

Find $\exists$ waves if c is large enough

For the present problem

$$c \approx \left(\frac{\sigma_0 \nu_0^3}{2}\right)^{1/2} .$$