

Homework 1

1. Just because synchrony is stable does not mean it is the only attractor. For example, if you consider the logistic map with, say, $r = 3.5$, then as we saw in class, there is a period 4 solution. Let's call the cycle, u_1, u_2, u_3, u_4 which then repeats. Suppose we start $x(0) = u_1$ and $y(0) = u_2$. If they are not coupled then they will produce a pattern that is shifted by 1. This is a stable pattern even when coupled. Thus, if the coupling is sufficiently small, then the pattern will not be able to change much (this is a feature of any smooth dynamical system that is "normally hyperbolic"). Thus, we can expect to see a solution for very small coupling, that is close to the shifted period-4 solution you see with no coupling. So, use simulations to figure out the minimum value of coupling, c such that

$$\begin{aligned}x(t+1) &= (1-c)f(x) + cf(y) \\y(t+1) &= (1-c)f(y) + cf(x)\end{aligned}$$

synchronizes no matter what the initial conditions. You could start $x(0) = u_1$ and let y vary over a range of initial conditions between 0 and 1 and then look at $|x - y|$ after, say 100 iterations. If you cannot figure out how to code this up, I can give you my XPP code.

2. The theory does not require that the coupling is symmetric; it just makes it easier to decompose the eigenspace. Consider the following type of coupling for N patches. Migration into patch i occurs from exactly k other patches that are randomly chosen. Thus $m_{ij} = (1-c)$ for $j = i$ and c/k for j among the randomly chosen patches. Let \hat{M} be the $N \times N$ connectivity matrix. That is \hat{m}_{ij} is 1 if patch j can visit patch i . Express the eigenvalues of M in terms of \hat{M}, k, c .

Suppose that we now consider

$$x_i(t+1) = \sum_j m_{ij} f(x_j(t))$$

and suppose that the synchronous solution, $u(t)$ satisfies

$$u(t+1) = f(u(t))$$

and has Lyapunov exponent

Create a 20×20 connection matrix such that each row has 5 1's and 15 0's and there are no diagonal elements. Divide each row by 5. Use MatLab or Octave to compute the eigenvalues of your matrix. Compute the magnitude of the eigenvalues. Consider the following equations:

$$x_i(t+1) = (1-c) * f(x_i) + c \sum_{j=1}^{20} m_{ij} f(x_j).$$

Suppose that λ is the maximal Lyapunov exponent for the system

$$u(t+1) = f(u(t)),$$

Compute the Lyapunov exponents for the synchronous solution. Now, use this matrix to couple 20 logistic maps and start with initial condition, $x_i = 0.2 + 0.01z_i$ where z_i is a random number between 0 and 1. This is near synchrony. Use the following equations:

and $f(x) = rx(1-x)$ with $r = 3.1, 3.5, 3.9$. I will post XPP and MatLab code for computing the matrices and the eigenvalues.

3. All of our analysis was for a system with one-dimensional dynamics. Suppose that each patch obeys two-dimensional dynamics

$$\begin{aligned} x(t+1) &= f(x(t), y(t)) \\ y(t+1) &= g(x(t), y(t)) \end{aligned}$$

Let $(u(t), v(t))$ be a steady state sequence $u(t+1) = f(u, v), v(t+1) = g(u, v)$. Form the Jacobian matrix

$$A(t) := \begin{bmatrix} f_x(u(t), v(t)) & f_y(u(t), v(t)) \\ g_x(u(t), v(t)) & g_y(u(t), v(t)) \end{bmatrix}$$

where f_x is the derivative of f with respect to x etc. For any matrix norm, $|A|$, you want to use, we can compute a quantity

$$\lambda = \lim_{T \rightarrow \infty} \frac{1}{T} \sum_{j=0}^T \log |A(j)|$$

just like we did for the scalar case. (You may want to review matrix norms, the simplest is just the maximum absolute row sum of the matrix.)

- (a) Suppose that there are N patches and the equations are

$$\begin{aligned} x_i(t+1) &= \sum_j m_{ij} f(x_j(t), y_j(t)) \\ y_i(t+1) &= \sum_j m_{ij} g(x_j(t), y_j(t)). \end{aligned}$$

where m_{ij} is as in class. Find conditions that make synchrony stable given the eigenvalues of the matrix $M = (m_{ij})$.

- (b) Now, in the above case, the migration of each of the species, x, y is exactly the same. Suppose that the corridors or paths that are available to the two species are the same, but they move along them at different rates. Let's write

$$\begin{aligned} x_i(t+1) &= \sum_j m_{ij} f(x_j(t), y_j(t)) \\ y_i(t+1) &= \sum_j q_{ij} g(x_j(t), y_j(t)). \end{aligned}$$

Thus, there are two matrices, Q, M . We interpret the assumption that they both use common corridors to mean that the *eigenvectors* of the matrices Q, M are the same but that they have different eigenvalues. Note that the assumptions on M, Q imply that $[1, 1, \dots]^T$ is an eigenvector of each with eigenvalue 1. Let \vec{v} be another eigenvector and let μ, ν be the corresponding eigenvalues of M, Q . Can you draw any conclusions about the stability of synchrony? I didn't think so. So, let's make it simpler. Suppose that the synchronous solution is just a fixed point. That is $u(t+1) = u(t), v(t+1) = v(t)$ for all t . Then there is only one matrix, $A(t)$, and since it is two by two, we write it as

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The linearized for the synchronous solution are

$$\begin{aligned} x_i(t+1) &= \sum_j m_{ij} [ax_j(t) + by_j(t)] \\ y_i(t+1) &= \sum_j q_{ij} [cx_j(t) + dy_j(t)]. \end{aligned}$$

Since m_{ij}, q_{ij} have the same eigenvector, \vec{v} but different eigenvalues, we write

$$\begin{aligned} \vec{x}(t) &= w(t)\vec{v} \\ \vec{y}(t) &= z(t)\vec{v} \end{aligned}$$

Show

$$\begin{aligned} w(t+1) &= \mu [aw(t) + bz(t)] \\ z(t+1) &= \nu [cw(t) + dz(t)]. \end{aligned}$$

We have replaced the study of a $2N$ dimensional system to the study of N 2 dimensional systems for each of the N eigenvalue pairs μ, ν . That is, you replace the matrix A above with

$$A(\mu, \nu) := \begin{bmatrix} \mu a & \mu b \\ \nu c & \nu d \end{bmatrix}$$

Given that μ, ν are between $-1, 1$, and real (we are making life easy), can you conclude that synchrony is always stable? let me pose this in an easier fashion. Suppose that $A(1, 1)$ has all its eigenvalues inside the unit circle. Then does this guarantee that $A(\mu, \nu)$ has all its eigenvalues inside the unit circle? The following may help you. A 2×2 real matrix has all its eigenvalues inside the unit circle iff

$$2 > 1 + \det > |\text{Tr}|$$

where \det is the determinant and Tr the trace of the matrix.

- (c) Suppose instead of a fixed point, the system has a synchronous period two orbit, $u(t+2) = u(t), v(t+2) = v(t)$. Then the matrix $A(t)$ has two values, A_1, A_2 and the period-2 orbit is stable if the eigenvalues of $B := A_1 A_2$ are in the unit circle. Try to find a matrix $B(\mu, \nu)$ that you need to study to determine stability of synchrony for the period-2 case. (Hint: let $D = \text{diag}(\mu, \nu)$. Then, show $B(\mu, \nu) = D A_1 D A_2$.) An open problem (as far as I know) would be to find conditions of $A_{1,2}$ that assure the eigenvalues of $B(\mu, \nu)$ are inside the unit circle whenever the eigenvalues of $B(1, 1)$ are for $|\mu|, |\nu|$ less than 1.
4. Consider a migration matrix formed as follows. Patch 1 is the “mother” patch. $c/(N-1)$ is the migration rate in and out of this patch from the other $N-1$ patches, so $m_{11} = 1 - c$, $m_{1j} = c/(N-1)$, $m_{jj} = 1 - c/(N-1)$, $j > 1$, and $m_{j1} = c/(N-1)$. All other entries are zero. Compute the eigenvalues for M . Does this foster synchrony better, worse, or the same as the nearest-neighbor coupling and the all-to-all coupling? (Note that $0 \leq c \leq 1$.)