So we are very good for this since $\ln|\nu_i| < 0$ for $i = 2, .., N$ & thus we have here that $\lambda + \ln|\nu_i| < 0$ even if $\lambda > 0$.

Dani suggested this notation which is better!

IN GENERAL:

$$x_i(t+1) = \sum_{j=1}^{N} m_{ij}\, F(x_j(t)) \qquad F : \mathbb{R}^m \to \mathbb{R}^m$$

when $\sum_{j=1}^{N} m_{ij} = 1 \quad \forall i$.

and

$u(t+1) = F(u(t))$ is a sequence (periodic or perhaps, not periodic)

Then to study stability of synchrony state Let ν_k be eigenvalue of $M = (m_{ij})$

Then you must look at

$$\lambda_k \equiv \ln \nu_k + \lim_{t \to \infty} \frac{1}{t} \underbrace{\sum_{s=1}^{t} \ln(\|A(t)\|)}_{\gamma}$$

Where $A(t) = D_x F(u(t))$

and $\|A(t)\|$ is some matrix norm, for example

$$|A| \quad \sup_{\cancel{\text{rows of } A}\; i} \sqrt{\sum_{j=1}^{m} |A_{ij}|^2}$$

This can be done best numerically since you cannot usually find γ.

If $u(t)$ is periodic, then γ is ≤ 0 and so any coupling of this form will synchronize.

CAVEAT: "OF THIS FORM"

This is a very special type of coupling. Let me clarify this.

Suppose you have the two-dimensional map:

$$u(t+1) = f(u(t), v(t))$$
$$v(t+1) = g(u(t), v(t))$$

$$\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix} \quad \vec{F} = \begin{pmatrix} f \\ g \end{pmatrix}$$

$$\begin{pmatrix} u_i(t+1) \\ v_i(t+1) \end{pmatrix} = \sum_{j=1}^{N} m_{ij} \begin{bmatrix} f(u_j(t), v_j(t)) \\ g(u_j(t), v_j(t)) \end{bmatrix}$$

says that species u + species v migrate with exactly the same probabilities.

Instead, the more general equation is

$$u_j(t+1) = \sum_{j=1}^{N} m_{ij}\, f(u_j^{(t)}, v_j^{(t)})$$

$$v_i(t+1) = \sum_{j=1}^{N} n_{ij}\, g(u_j(t), v_j(t))$$

So, if $n_{ij} \neq m_{ij}$ then we cannot apply the previous results.

Indeed, we will see later with continuous differential equations, that different migration rates can have profound effects on synchrony.

Let's return to the map scalar case and look at two different geometries of connectivity.

"All to All" $\quad m_{ij} = \begin{cases} 1 - c & i = j \\ \frac{c}{N-1} & i \neq j \end{cases}$

Here c is the coupling rate

$$M = \begin{bmatrix} 1-c & \frac{c}{N-1} & \cdots & \frac{c}{N-1} \\ \frac{c}{N-1} & 1-c & & \vdots \\ \vdots & & \ddots & c/N-1 \\ \frac{c}{N-1} & \cdots & \frac{c}{N-1} & 1-c \end{bmatrix}$$

What are eigenvalues?

Aside:

$$M = \begin{bmatrix} a & b & b & b \\ b & \ddots & & \vdots \\ \vdots & & \ddots & b \\ b & \cdots & b & a \end{bmatrix} = (a-b)I + b\begin{bmatrix} 1 & \cdots & 1 \\ & & \\ 1 & \cdots & 1 \end{bmatrix}$$

Eigenvalues of $\begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \\ 1 & \cdots & 1 \end{bmatrix}$ are N, 0 ($N-1$ times)

eigenvalues of M are

$a-b+bN$ and $a-b$ ($N-1$) times

so for us: $a = 1-c$ $\quad b = \frac{c}{N-1}$

$$a-b+bN = 1-c-\frac{c}{N-1}+\frac{cN}{N-1} = 1$$

$$a-b = 1-c-\frac{c}{N-1}$$

so $\nu_1 = 1$ as usual

$$\nu_{2,\dots,N} = 1-c-\frac{c}{N-1}$$

and theorem says synchrony will be stable if

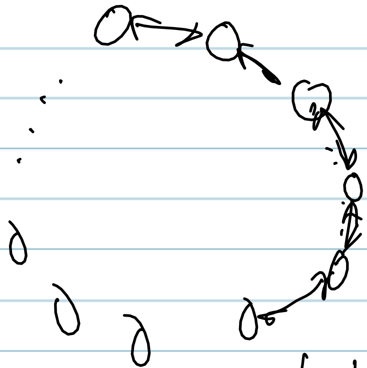
$$\lambda + \ln\left|1-c-\frac{c}{N-1}\right| < 0$$

For example, Logistic map

$u(t+1) = r\,u(t)(1-u(t))$ $\quad r = 3.9$

$\lambda = 0.492$ so need strong coupling

Example 2 for coupling.



Nearest neighbor coupling

$0 < c \leq \frac{1}{2}$

$$M = \begin{bmatrix} 1-2c & c & 0 & \cdots & 0 & c \\ c & 1-2c & c & 0 & \cdots & 0 \\ \cdots & \cdots & c & 1-2c & c & \cdots \\ c & 0 & \cdots & \cdots & c & 1-2c \end{bmatrix}$$

Periodic ring of N elements.

<u>Aside:</u> (Always with the asides!)

Circulant matrices

Let $\vec{c} := [c_0 \ c_1 \ c_2 \ \cdots \ c_{N-1}]$

Let $M = \begin{bmatrix} c_0 & c_1 & \cdots & & c_{N-1} \\ c_{N-1} & c_0 & c_1 & \cdots & c_{N-2} \\ \vdots & & & & \\ c_1 & c_2 & \cdots & c_{N-1} & c_0 \end{bmatrix}$

M is called a <u>circulant matrix</u>

$$(M\vec{x})_i = \sum_{j=0}^{N-1} c_j x_{i+j}$$ ← Like a discrete convolution

or alternately

$$(M\vec{x})_i = \sum_{j=1}^{N} C_{j-i}\, x_j$$

where we take $j-i$ modulo N

eg if $j-i = -3$ then add N to make it $N-3$.

we have identified all elements to lie in a circle which is why it is called a circulant matrix.

coupling from j to i depends only on $j-i$!!

Eigenvalues of circulant matrices are easy to find.

Let $z_k^N = 1$, so z is an N^{th} root of 1

For example $N=2$ $z = \pm 1$

$N=3$ $z = e^{\frac{2\pi i}{3}}, 1, e^{\frac{-2\pi i}{3}}$

$N=4$ $z = +1, -1, i, -i$

in general $z = e^{\frac{2\pi i k}{N}}$ $k = 0, \ldots, N-1$

since $z^N = \left(e^{\frac{2\pi i k}{N}}\right)^N = e^{2\pi i k} = 1$

Claim that

$$\vec{v} = \begin{bmatrix} 1 \\ z_k^1 \\ z_k^2 \\ \vdots \\ z_k^{N-1} \end{bmatrix} \quad \text{is eigenvector}$$

proof

$$\nu e^{\frac{2\pi i l k}{N}} \stackrel{?}{=} \sum_{j=0}^{N-1} c_j e^{\frac{2\pi i k(l+j)}{N}}$$

$$= \sum_{j=0}^{N-1} c_j e^{\frac{2\pi i kj}{N}} e^{\frac{2\pi i k l}{N}}$$

$$\Rightarrow \nu = \sum_{j=0}^{N-1} c_j e^{\frac{2\pi i kj}{N}} \quad \text{is ind of } l!$$

so ν is eigenvalue

<u>Proposition</u> If M is a circulant matrix with first row $c_0 \ldots c_{N-1}$

then the N eigenvalues are

$$\boxed{\nu_k = \sum_{j=0}^{N-1} c_j e^{\frac{2\pi i kj}{N}}}$$

For nearest neighbor coupling

$c_0 = 1-2c \quad c_1 = c \quad c_{N-1} = c \quad c_j = 0$

so

$$\nu_k = c_0 \cdot 1 + c_1 e^{\frac{2\pi j k}{N}} + c_{N-1} e^{\frac{-2\pi j k}{N}}$$

$$= 1 - 2c + 2c \cos\frac{2\pi k}{N}$$

$$= 1 - 2c\left(1 - \cos\frac{2\pi k}{N}\right)$$

$k = 0, \ldots, N-1$

so 1 is eigenvalue & all the others are less than 1

but for N large $1-\cos\frac{2\pi k}{N}$ is very close to zero

The largest of these eigenvalues is

$\nu_{max} \quad 1 - 2c\left(1 - \cos\frac{2\pi}{N}\right) \qquad (\nu_0 = 1$ is largest$)$

For N Large $\cos x = 1 - \frac{x^2}{2}$ (x small)

so $1-2c\left(1-\cos\frac{2\pi}{N}\right) \approx 1 - c\frac{4\pi^2}{N^2}$

and $\ln \nu_{max} \approx -c\frac{4\pi^2}{N^2}$ (N large)

so we can only overcome a slightly positive λ

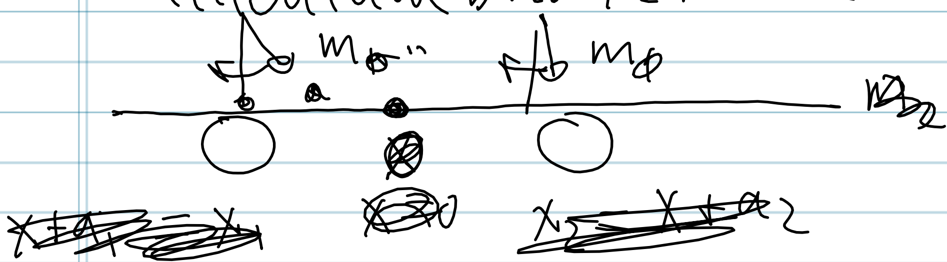
For example in our Logistic map example when $r = 3.9$ & $\lambda = .492$, we will not be able to synchronize large rings, since $\ln \nu_{max} \approx -c\, 4\pi^2/N^2$ is small $(0 \le c \le \frac{1}{2})$

MORAL: Global (all:all) coupling is much better for synchrony than local.

However, in many systems, you can only access local information, such as Fireflies. Thus there will not always be synchrony even for identical elements.

For HW I will have you look at a bunch of other forms of matrices including the mammilary & sparse cases

Millenium bridge + the metronomes



Two metronomes or for now, pendulums
That rest on a board of mass m_2
Let X be center of mass of board
Let pendulums base be at $X + a_i$ $\quad i = 1, 2$
Let θ_1, θ_2 be angle of bobs & let
l = length & let m be mass of pendulums
Potential energy is just due to gravity
(we will drive them later)

$$-mgl[\cos\theta_1 + \cos\theta_2] = P.E$$

$$y_1 = -l\cos\theta_1, \quad y_2 = -l\cos\theta_2$$

$$x_1 = X + a_1 + l\sin\theta_1 \quad x_2 = X + a_2 + l\sin\theta_2$$

$$\dot{x}_1 = \dot{X} + \dot{\theta}_1 l\cos\theta_1 \quad \dot{x}_2 = \dot{X} + \dot{\theta}_2 l\cos\theta_2$$

$$\dot{y}_1 = l\dot{\theta}_1\sin\theta_1 \quad \dot{y}_2 = l\dot{\theta}_2\sin\theta_2$$

$$K.E. = \frac{m}{2}\left[\dot{x}_1^2 + \dot{x}_2^2 + \dot{y}_1^2 + \dot{y}_2^2\right] + \frac{M}{2}\dot{x}^2$$

$$= \frac{m}{2}\left[2\dot{x}^2 + \dot{\theta}_1^2 + \dot{\theta}_2^2 + 2\dot{x}\dot{\theta}_1 l\cos\theta_1 + 2\dot{x}\dot{\theta}_2 l\cos\theta_2\right] + \frac{M}{2}\dot{x}^2$$

Lagrangian = ~~P.E. - K.E.~~ K.E - P.E. $= \mathcal{L}$

Dynamics:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\partial \mathcal{L}}{\partial x} \qquad \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} = \frac{\partial \mathcal{L}}{\partial \theta_1} \qquad \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}_2}\right) = \frac{\partial \mathcal{L}}{\partial \theta_2}$$

$$\mathcal{L} = mgl\left[\cos\theta_1 + \cos\theta_2\right] + \frac{2m+M}{2}\dot{x}^2 + \frac{ml^2}{2}\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right) + m\cancel{2}l\dot{x}\left[\dot{\theta}_1\cos\theta_1 + \dot{\theta}_2\cos\theta_2\right]$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = (2m+M)\dot{x} + ml\left[\dot{\theta}\cos\theta_1 + \dot{\theta}_2\cos\theta_2\right]$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0 \Rightarrow \frac{\partial \mathcal{L}}{\partial \dot{x}} = C \Rightarrow$$

$$\boxed{\dot{x} = \frac{C - ml\left[\dot{\theta}_1\cos\theta_1 + \dot{\theta}_2\cos\theta_2\right]}{2m+M}}$$

$$\frac{\partial \mathcal{L}}{\partial \dot\theta_1} = m\cancel{g} l^2 \dot\theta_1 + m l \dot x \cos\theta_1$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta_1} = m l^2 \ddot\theta_1 + m l \ddot x \cos\theta - \cancel{m l \dot\theta_1 \dot x \sin\theta_1}$$

$$= \frac{\partial \mathcal{L}}{\partial \theta_1} = -mgl\sin\theta_1 - \cancel{m l \dot\theta_1 \dot x \sin\theta_1}$$

$$\boxed{m l^2 \ddot\theta_1 = -mgl\sin\theta_1 - m l \cos\theta_1 \ddot x}$$

$$m l^2 \ddot\theta_2 = -mgl\sin\theta_2 - m l \cos\theta_2 \ddot x$$

$$\ddot x = -\frac{ml}{2m+M}\left[\ddot\theta_1 \cos\theta_1 + \ddot\theta_2 \cos\theta_2 - \dot\theta_1^2 \sin\theta_1 - \dot\theta_2^2 \sin\theta_2\right.$$

$$m l \cos\theta_1 \ddot x = -\frac{m^2 l^2}{2m+M}\left[\ddot\theta_1 \cos^2\theta_1 + \ddot\theta_2 \cos\theta_1 \cos\theta_2 - \dot\theta_1^2 \sin\theta_1 \cos\theta_1 - \dot\theta_2^2 \sin\theta_2 \cos\theta_1\right.$$

$$m l \cos\theta_2 \ddot x = -\frac{m^2 l^2}{2m+M}\left[\ddot\theta_2 \cos^2\theta_2 + \ddot\theta_1 \cos\theta_1 \cos\theta_2 - \dot\theta_2^2 \sin\theta_2 \cos\theta_2 - \dot\theta_1^2 \sin\theta_1 \cos\theta_2\right.$$

$$ml^2 \begin{bmatrix} 1 - \frac{m}{2m+M}\cos^2\theta_1 & -\frac{m\cos\theta_1\cos\theta_2}{2M+M} \\ -\frac{m\cos\theta_1\cos\theta_2}{2m+M} & 1 - \frac{m}{2m+M}\cos^2\theta_2 \end{bmatrix} \begin{matrix} ml^2\ddot\theta_1 \\ ml^2\ddot\theta_2 \end{matrix}$$

Thus invertible

so we can solve for $\ddot\theta_1$, $\ddot\theta_2$

$$\mathcal{L} = \frac{1}{2}(M+2m)\dot{x}^2 + m\dot{x}l(\dot{\theta}_1\cos\theta_1 + \dot{\theta}_2\cos\theta_2) + \frac{1}{2}ml^2(\dot{\theta}_1^2+\dot{\theta}_2^2) + mgl(\cos\theta_1+\cos\theta_2) + \frac{k}{2}x^2$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{x}} = \frac{\partial\mathcal{L}}{\partial x} \Rightarrow$$

$$\frac{d}{dt}\left[(M+2m)\dot{x} + ml(\dot{\theta}_1\cos\theta_1 + \dot{\theta}_2\cos\theta_2)\right] = -kx - B\dot{x}$$

B is damping

We can rewrite this as

$$(M+2m)\ddot{x} + B\dot{x} + kx = -ml[\sin\theta_1 + \sin\ddot{\theta}_2]$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\dot{\theta}_k} = \frac{\partial\mathcal{L}}{\partial\theta_k} \Rightarrow$$

$$ml^2\ddot{\theta}_k + mgl\sin\theta_k = -ml\ddot{x}\cos\theta_k - bml^2\dot{\theta}_k + ml^2\hat{f}_k$$

Here b is friction or damping &

$\hat{f}_k$ is restoring force for the clock

Let $Y = X/l$ $\quad \tau = t\sqrt{g/l}$ Then we get

$$\theta_k'' + 2\gamma\theta_k' + \sin\theta_k = -Y''\cos\theta_k + f_k$$
$$Y'' + 2\Gamma Y' + \Omega^2 Y = -\mu(\sin\theta_1 + \sin\theta_2)''$$

Where $\gamma = b\sqrt{l/4g}$ $\quad \Gamma = \dfrac{B\sqrt{l/4g}}{M+2m}$ $\quad \Omega^2 = \dfrac{K}{M+2m}$

$\mu = m/(M+2m)$

This is a messy non linear equation.

However, if θ_n, X are small, we can approximate it by a linear equation

Before doing so, we need to discuss f_n

The restoring force of the clocks (otherwise they will just damp to zero!)

We introduce a simple mechanism. Whenever the pendulum reaches a threshold angle $\pm\varphi$, the angular velocity reverses direction & the magnitude changes according to

$$|\theta_k'| \rightarrow (1-c)|\theta_k'| + \varepsilon$$

c, ε are small. Since $M >> m$, the impulse has neglible effect on the platform

The linear approximation is

$$\theta_1'' + 2\gamma\theta_1' + \theta_1 = -Y'' + f_1$$
$$\theta_2'' + 2\gamma\theta_2' + \theta_2 = -Y'' + f_2$$
$$Y'' + 2\Gamma Y' + \Omega^2 Y = -\mu(\theta_1'' + \theta_2'')$$

We rewrite this as

$$\theta_1'' + y'' = \zeta_1 = -2\gamma\theta_1' + \theta_1 + f_1 \quad \text{where } \mu \ll 1$$

$$\theta_2'' + y'' = \zeta_2 = -2\gamma\theta_2' - \theta_2 + f_2$$

$$y'' + \mu\theta_1'' + \mu\theta_2'' = \zeta_3 = -2\Gamma y' - \Omega^2 y$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ \mu & \mu & 1 \end{bmatrix} \begin{pmatrix} \theta_1'' \\ \theta_2'' \\ y'' \end{pmatrix} = \begin{pmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}$$

$$\theta_1'' = \frac{\zeta_1 + \mu(\zeta_2 - \zeta_1) - \zeta_3}{1-2\mu}$$

$$\theta_2'' = \frac{\zeta_2 + \mu(\zeta_1 - \zeta_2) - \zeta_3}{1-2\mu}$$

$$y'' = \frac{\zeta_3 - \mu(\zeta_1 + \zeta_2)}{1-2\mu}$$

This is linear between impulses.
I tried solving it but only found
out of phase locking!

Before moving on, I want to look at a variation with nonlinear friction in the oscillators:

$$\theta'' + 2\gamma\theta' + \sin\theta = 0$$

This cannot self sustain. Suppose we consider:

$$\theta'' + \gamma(\theta^2 - \theta_0^2)\theta' + \theta = 0$$

θ_0 is small (rescale $\theta = \theta_0\varphi$)

$$\theta_0\varphi'' + \gamma\theta_0^3(\varphi^2-1)\varphi' + \theta_0\varphi = 0$$

$$\Rightarrow \boxed{\varphi'' + \tilde{\gamma}(\varphi^2-1)\varphi' + \varphi = 0}$$

This is self sustained - VDP oscillator

Exact same as before but

$$z_1 = -\gamma(\theta_1^2 - \theta_0^2)\theta_1' - \theta_1$$

$$z_2 = -\gamma(\theta_2^2 - \theta_0^2)\theta_2' - \theta_2$$

Later we will explore this in detail using some weakly nonlinear perturbation analysis.

FLOWS

So, what can we say about Flows

Review of Limit cycles + stability

- Linear systems

$$\dot{x} = Ax \qquad A \text{ is constant} \qquad (1)$$

$$x(t) = e^{tA} x(0)$$

e^{tA} = exponential of matrix. If $Av = \lambda v$

then $x = ve^{\lambda t}$ is a solution.

If ~~some~~ $\forall \lambda \in \Sigma(A)$ ← spectrum of A have negative real parts then all solutions to (1) decay to 0 as $t \to \infty$. $x = 0$ is Asym stable

- Linear periodic systems.

$$\dot{x} = A(t)x \qquad A(t+T) = A(t) \qquad 2)$$

Floquet Theorem There

Let $\Phi(t)$ be the fundamental matrix for (2)

That is $\dot{\Phi}(t) = A(t)\Phi(t)$ and suppose

$\Phi(0) = I$. Then $\exists$ matrix $M, P(t)$ such that

$$\Phi(t) = P(t)e^{tM} \qquad \text{where } P(t+T) = P(t)$$

$$\Phi(0) = I \quad ~~\Phi(T)~~ \Rightarrow P(0) = I \Rightarrow \Phi(T) = e^{TB}$$

so $\Phi(nT) = P(nT)\, e^{nTB} = \left(e^{TB}\right)^n$

call $M = e^{TB}$

$M^n \to 0$ as $n \to \infty$ iff all eigenvalues of M are in unit circle.

We call $\rho \in \Sigma(M)$ a <u>Floquet Multiplier</u>

If we write $\rho = e^{\lambda T}$, then λ is called a Floquet <u>exponent</u>

These are defined up to multiples of $2\pi i$

If $\operatorname{Re} \lambda < 0 \iff |\rho| < 1 \iff \rho^n \to 0$

<u>Theorem</u> If all Floquet exponents have negative real parts then all solutions to (2) decay to 0 as $t \to \infty$.

<u>Theorem</u> If there is a nontrivial T-periodic solution to (2) then there must be at least one multiplier, ρ st $|\rho| = 1$

Autonomous Systems & L.C.

Consider

$$\dot{u} = F(u)$$

& suppose that $u_0(t+T) = u_0(t)$ is T-periodic solution.

Write $u(t) = u_0(t) + Y$ where Y is small.

Then

$$\dot{u} = \dot{u}_0 + \dot{Y} = F(u_0(t) + Y(t))$$
$$\approx D_u F(u_0(t))\, Y(t) + u_0(t) + O(|Y|^2)$$

$D_u F(u_0(t)) \equiv A(t)$ is T-periodic.

So ... What happens to $Y(t)$?

Remark $\dot{Y} = A(t) Y$ has a multiplier $\rho = 1$

Proof $\frac{du_0}{dt} = F(u_0(t))$

Differentiate:

$$\frac{d}{dt}\dot{u}_0(t) = \frac{d^2 u_0}{dt^2} = D_u F(u_0(t)) \frac{du_0}{dt} \equiv A(t)\, \dot{u}_0$$

Thus $\exists$ T-periodic solution to $\dot{Y} = A(t) Y$

$\Rightarrow \rho = 1$

We say a limit cycle is Asymptotically stable if remaining floquet multipliers are inside unit circle

Example

$$\dot{x} = x(1 - x^2 - y^2) - y$$
$$\dot{y} = y(1 - x^2 - y^2) + x$$

$x = \cos t$ $y = \sin t$ is limit cycle

Exercise: Find the nonzero Floquet ~~ex~~ exponent

Helpful hint:

Let $\dot{\Phi}(t) = A(t)\,\Phi(t)$ be a fundamental matrix then

$$\rho_1 \rho_2 \cdots \rho_N = e^{\int_0^T \operatorname{Tr} A(s)\,ds} \det \Phi(0)$$

(This is a well-known identity from ODE's - I think it's called Abel's formula?)

With this background, we are now ready to study coupled systems.

$$\dot{X}_1 = F(X_1) + K(X_2 - X_1)$$
$$\dot{X}_2 = F(X_2) + K(X_1 - X_2)$$
eg

$F: \mathbb{R}^n \to \mathbb{R}^n \qquad K \in \mathbb{R}^{n\times n}$

$\dot{u}_0 = F(u_0(t))$ has a Asy stable T-periodic LC solution

$X_1 = X_2 = u_0(t)$ is a synchronous solution

Write $X_j = u_0(t) + Y_j(t)$

$\Rightarrow$
$$\dot{Y}_1 = A(t)\,Y_1 + K(Y_2 - Y_1)$$
$$\dot{Y}_2 = A(t)\,Y_2 + K(Y_1 - Y_2)$$
where $A(t) = D_u F(u_0(t))$

$$\begin{bmatrix} \dot{Y}_1 \\ \dot{Y}_2 \end{bmatrix} = \begin{bmatrix} A(t) & 0 \\ 0 & A(t) \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} + \begin{bmatrix} -k & +k \\ k & -k \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

Notationally, it is sometimes nice to write this as the Kronecker product of matrices:

$$\cancel{[A(t) \otimes I + k \otimes Q]\, \vec{Y}}$$

$$\cancel{I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad A(t) \otimes I = \begin{bmatrix} A(t)\cdot 1 & A(t)\cdot 0 \\ A(t)\cdot 0 & A(t)\cdot 1 \end{bmatrix}}$$

$$Q = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \equiv \cancel{\text{"adjacency matrix"}}$$

Oops

$$I \otimes A(t) + Q \otimes K$$

Where $Q = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \equiv$ adjacency matrix

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I \otimes A(t) = \begin{bmatrix} 1\cdot A & 0\cdot A \\ 0\cdot A & 1\cdot A \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}$$

$$1 \rightleftarrows 2$$

Adjacency matrix

who is coupled to who

Let $Z = Y_1 - Y_2 \quad W = Z_1 + Z_2$

$$\dot{Z} = \dot{Y}_1 - \dot{Y}_2 = A(t)Y_1 - A(t)Y_2 + K(Y_2 - Y_1) - K(Y_1 - Y_2)$$

$$= A(t)Z - 2KZ$$

$$W = A(t)W$$

As we did earlier, we have reduced the system to two equations of smaller dimension.

① $\dot{Z} = [A(t) - 2K]Z(t)$

② $\dot{W} = A(t)W$

② is just the single isolated limit cycle. Since this is stable, we know that (2) has all multipliers inside the unit circle, except for 1 which is due to ~~...~~ u_0.

So we can say that synchrony will be stable if all solutions to ① decay to zero as $t \to \infty$ since

$Z(t) \to 0 \Rightarrow Y_1 - Y_2 \to 0 \Rightarrow X_1 - X_2 \to 0$ as $t \to \infty \Rightarrow$ synchrony!

So, it is hard to say what happens with Equation ①

Rem: Notice that the eigenvalues of Q are 0 + -2! (Sound familiar??)

Suppose $K = \sigma I$, That is K is a scalar multiple of the identity. Then we can draw some conclusions.

Let $\vec{\zeta}(t)$ satisfy

$$\dot{\zeta}(t) = A(t)\,\zeta(t)$$

~~and suppose $\zeta(t)$~~

Then we can see that

$$\eta(t) = e^{-2\sigma t}\,\zeta(t) \text{ satisfies:}$$

$$\begin{aligned}\frac{d\eta}{dt} &= -2\sigma e^{-2\sigma t}\zeta(t) + e^{-2\sigma t}\dot{\zeta}(t) \\ &= -2\sigma\eta + e^{-2\sigma t}A(t)\,\zeta(t) \\ &= A(t)\,\eta(t) - 2\sigma\,\eta(t)\end{aligned}$$

That is $\eta(t)$ solves (1) (only when $K = \sigma I$!)

So if $\zeta(t)$ is periodic then $\eta(t)$ will decay for $\sigma > 0$ + grow exponentially if $\sigma < 0$

If $\zeta(t)$ decays then so will $\eta(t)$ as long as $\sigma > 0$.

From this we can conclude that scalar "diffusive" coupling of oscillators will always synchronize them!

The main interest comes from the non scalar coupling case.

(N.B. for chaotic systems, the matrix $A(t)$ is not periodic and so you must look at the long term growth of

$$\dot{y} = A(t)y \quad (3)$$

When there is chaos, there is always a solution to (3) such that $\|y(t)\| = Ce^{\gamma t}$ where ~~γ real~~ $\gamma > 0$. This number is called the maximal Lyapunov exponent.

The variational equation for the coupled system with scalar diffusive coupling is

$$\dot{z} = A(t)z - 2\sigma z, \quad (4)$$

so if $2\sigma > \gamma$ then all solutions to (4) will decay + synchrony will be stable.

With scalar coupling of oscillators magnitude does not matter but with chaos it does(!))

Non scalar coupling is much more interesting

So how do we analyse it?

~~Let~~

Pecora-Carroll Master stability equation.

$$\dot{Z} = A(t) Z + (\alpha + i\beta) K Z$$

Find the regions in (α, β) where $Z(t) \to 0$ as $t \to \infty$

If we write $Z = R + iS$ then

$$\frac{dR}{dt} = A(t) R + \alpha K R - \beta K S$$

$$\frac{dS}{dt} = A(t) S + \alpha K S + \beta K R$$

Can just integrate this or ~~assume~~ use for periodic $A(t)$ you can compute the monodromy matrix as a function of (α, β)