Oscillations in LV models

We've seen that in 2D LV models, there are no limit cycles, but can be nested periodic orbits. These are <u>nongeneric</u>, that is small changes in the equations will destroy them. In Exercise 10, you will be asked to analyze a modification of the LV model, where there is prey saturation + for which there are stable limit cycles. To better get a handle on conditions for existence we state the following important theorem <u>ASIDE</u>

Hopf Bifurcation Theorem

Consider $\dot{x} = f(x,p)$ $x \in \mathbb{R}^n$, $p \in \mathbb{R}$, $n \geq 2$. Let $\bar{x}(p)$ be an equilibrium, $f(\bar{x},p)=0$ and let $A(p) \equiv D_x f(\bar{x}(p),p)$ be the jacobian matrix. Suppose

1. There is a pair of eigenvalues of $A(p)$, $\lambda(p) = \alpha(p) \pm i\omega(p)$ such that $\lambda(p_0) = \pm i\omega_0$, $\omega_0 \neq 0$, $\left.\frac{d\alpha}{dp}\right|_{p_0} \neq 0$, for some $p_0$
2. No other eigenvalues of $A(p)$ have zero real part when $p = p_0$
3. A certain quantity $\gamma(p_0)$ is nonzero (we usually ignore this but it is important)

<u>Then</u> for $p - p_0$ small and either positive or negative (but not both) there exists a limit cycle with period near $\frac{2\pi}{\omega_0}$ + such that

$$|x(t,p) - \bar{x}(p_0)| = O\left(\sqrt{|p-p_0|}\right)$$

in fact: NORMAL FORM OF HB!

$$x(t) = \bar{x}(p_0) + \mathrm{Re}\left(z(t)\,\Phi\, e^{i\omega_0 t}\right)$$

where $A(p_0)\Phi = i\omega_0 \Phi$, $\frac{dz}{dt} = z\left((a+bi)(p-\bar{p}) + (\gamma+i\nu) z\bar{z}\right)$ $a = \left.\frac{d\alpha}{dp}\right|_{p_0}$

How can we easily look at eigenvalues of a matrix A?

We use the Routh-Hurwitz criterion!

Routh-Hurwitz

Write $\det(\lambda I - A) = \lambda^n + b_1\lambda^{n-1} + b_2\lambda^{n-2} + \dots + b_{n-1}\lambda + b_n$

Form the determinants:

$$\Delta_k = \begin{vmatrix} b_1 & 1 & 0 & \cdots & \cdots & 0 \\ b_3 & b_2 & b_1 & 1 & 0 & 0 \\ \vdots & & & & & \vdots \\ b_{2k-1} & b_{2k-2} & \cdots & \cdots & & b_k \end{vmatrix} \quad \text{Set } b_j = 0 \text{ for } j > n$$

If $\Delta_k > 0$ $k = 1, \dots, n$, then all the eigenvalues of $A$ have negative real parts.

If $\Delta_n = 0$, $\Delta_k \neq 0$, then there is a simple zero eigenvalue. If $\Delta_{n-1} = 0$ then there are imaginary eigenvalues, if $b_n > 0$

Examples

$n=2$ $\quad \lambda^2 + b_1\lambda + b_2 = 0$ $\quad \Delta_1 = b_1$, $\quad \Delta_2 = \begin{bmatrix} b_1 & 1 \\ 0 & b_2 \end{bmatrix}$ $\quad b_1 = 0 \Rightarrow \lambda^2 + b_2 = 0$, $\lambda = \pm i\, b_2$ $\quad \Delta_2 = 0 \Rightarrow b_2 = 0 \Rightarrow \lambda = 0$

$n=3$ $\quad \lambda^3 + b_1\lambda + b_2\lambda^2 + b_3 = 0$ $\quad \Delta_1 = b_1$ $\quad \Delta_2 = \begin{bmatrix} b_1 & 1 \\ b_3 & b_2 \end{vmatrix} = b_1b_2 - b_3$, $\quad \Delta_3 = \begin{vmatrix} b_1 & 1 & 0 \\ b_3 & b_2 & b_1 \\ 0 & 0 & b_3 \end{vmatrix} = b_3\,\Delta_2$

If $b_1 > 0$, $b_3 > 0$ $\quad b_1b_2 - b_3 > 0 \Rightarrow$ A.S.

$b_1b_2 - b_3 = 0$ $\quad b_3 > 0 \Rightarrow \pm i\omega$! Way to check for HB!!

END ASIDE

With this theorem + the HBT, we can find conditions on LV models for oscillations

One way to write the equations is:

$$\dot{x}_i = x_i\left(\sum_{j=1}^{n} a_{ij}(1 - x_j)\right)$$

which guarantees an interior equilibrium $(x_i = 1)$ +

$J = -A = -(a_{ij})$ is the stability matrix. I will give you a cool exercise on this. Once you have 3 or more dimensions, chaos + other exotica are possible.

I now state some results on specific models

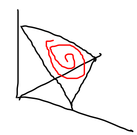
Competitive models:

We can write a general competitive model as:

$$\dot{x}_i = r_i\Big(1 - x_i - \sum_{j\neq i} a_{ij} x_j\Big) \text{ where } a_{ij} \geq 0,\ r_i > 0$$

Moe Hirsch proved that solutions to this all ended up on a simplex. That is there is a closed invariant set, C, which is a global attractor & is topologically homeomorphic to the n-1 dim simplex

$$\Sigma_n = \{x_i \mid x_i \geq 0, \sum_{i=1}^{n} x_i = 1\}$$



Example
$$\dot{x}_1 = x_1(1 - x_1 - a x_2 - b x_3)$$
$$\dot{x}_2 = x_2(1 - x_2 - a x_3 - b x_1)$$
$$\dot{x}_3 = x_3(1 - x_3 - a x_1 - b x_2)$$

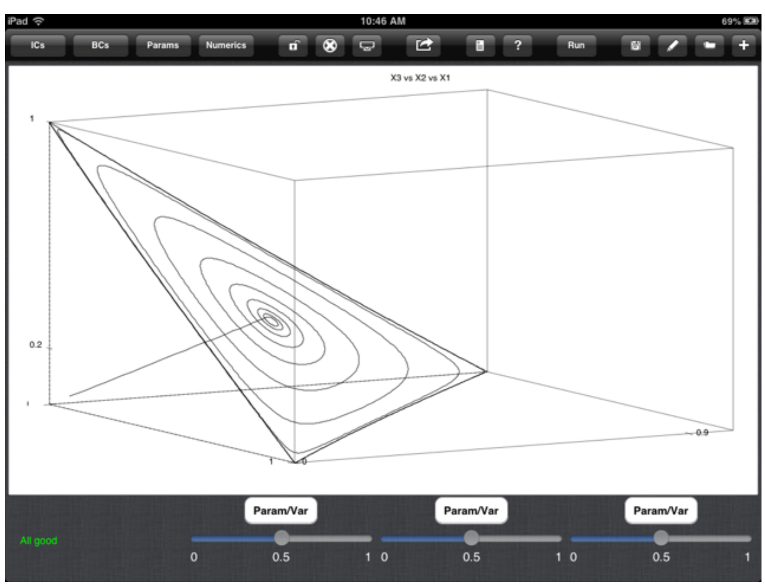
$a < 1 < b$

called Voter's paradox

eg $(x_1, x_2)$ $x_2$ wins

$(x_1, x_3)$ $x_1$ wins, $(x_3, x_2)$ $x_3$ wins

$$\left(\bar{x}_1 = \bar{x}_2 = \bar{x}_3 = \frac{1}{1+a+b}\right)$$




$a = .8$

$b = 1.27$

($1 + a + b > 2$ then $\bar{x}$ is unstable, if $a + b < 2$, then it is stable)

&

$$J_{(\bar{x},\bar{x},\bar{x})} = \begin{pmatrix} -\bar{x} & -a\bar{x} & -b\bar{x} \\ -b\bar{x} & -\bar{x} & -a\bar{x} \\ -a\bar{x} & -b\bar{x} & -\bar{x} \end{pmatrix}$$

clearly $\lambda_1 = -1$, $\vec{v} = \begin{pmatrix}1\\1\\1\end{pmatrix}$

ASIDE Cyclic systems:

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots \\ \vdots & & & \vdots \\ a_1 & a_2 & \cdots a_{n-1} & a_0 \end{pmatrix}$$

Let $\vec{V}_\ell = \begin{pmatrix} 1 \\ r \\ r^2 \\ \vdots \\ r^{n-1} \end{pmatrix}$ where $r = e^{\frac{2\pi i \ell}{n}}$ is $n^{th}$ root of unity

Claim This is an eigenvector for this matrix

$A_{ij} = A_{j-i \bmod n}$ 1st row clearly works, look at row 2 $A_{21} = A_{-1} = a_{n-1}$ etc

Check eigenvectors work.

Row 1

$\lambda = a_0 + a_1 r + \cdots + a_{n-1} r^{n-1}$

$\lambda r = a_{n-1} + a_0 r + a_1 r^2 + \cdots + a_{n-2} r^{n-1}$

divide by $r$, $r^{-1} = r^{n-1}$ since $r^n = 1$

$\lambda = a_{n-1} r^{n-1} + a_0 + a_1 r + \cdots + a_{n-2} r^{n-2}$

and so on!

Thus $\lambda_\ell = \sum_{j=0}^{n-1} r_\ell^j a_j$, $r_\ell$ is an $n^{th}$ root of 1

END OF ASIDE

$a_0 = -\bar{x}$, $a_1 = -a\bar{x}$, $a_2 = -b\bar{x}$ $r = 1$, $r = e^{\frac{2\pi i}{3}}$

$\lambda_2 = -\bar{x} - a\bar{x} e^{\frac{2\pi i}{3}} - b\bar{x} e^{-\frac{2\pi}{3} i}$

$= -\bar{x} [ 1 + a \cos\frac{2\pi}{3} + b \cos\frac{2\pi}{3} + (a-b) i \sin\frac{2\pi}{3} )$

$= -\bar{x} ( 1 - \frac{a+b}{2} ) + i \bar{x} (b-a) \frac{\sqrt{3}}{2}$

So, it looks like HB when $a+b=2$. Lets see That it is not. suppose $a+b=2$. Then, claim $x_1+x_2+x_3=1$ is invariant:

$$\dot{x}_1+\dot{x}_2+\dot{x}_3 = x_1(1-x_1-ax_2-bx_3) + x_2(1-x_2-ax_3-bx_1) + x_3(1-x_3-ax_1-bx_2)$$

$$=(x_1+x_2+x_3)(1-(x_1+x_2+x_3)) \text{ if } a+b=2 \quad \left(\begin{array}{c}\text{easy}\\ \text{to show}\end{array}\right)$$

$$\Rightarrow x_1+x_2+x_3 \to 1 \Rightarrow x_3 = 1-x_1-x_2$$

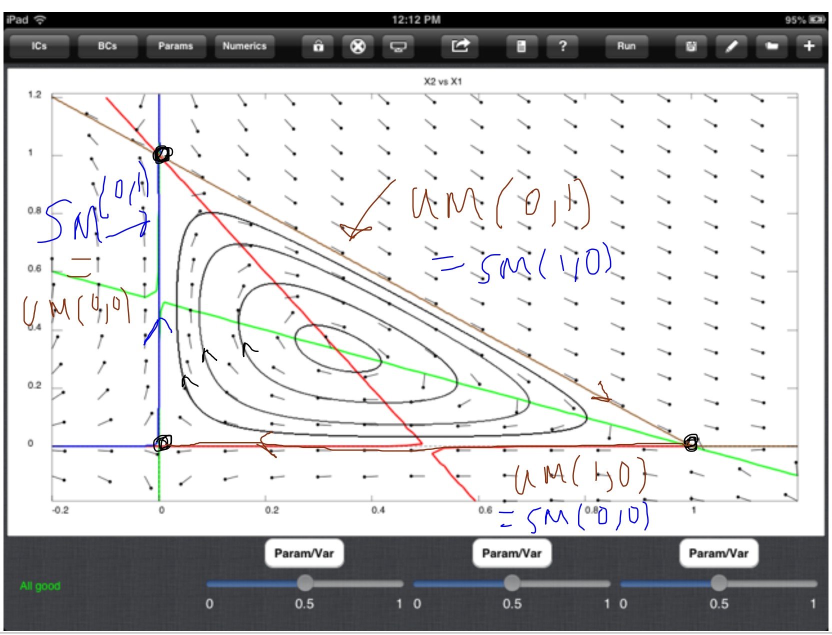
$\Rightarrow$

$$\dot{x}_1 = x_1(1-b-(1-b)x_1-2(1-b)x_2) \qquad b>1 \quad a<1$$

$$\dot{x}_2 = x_2(b-1-(b-1)x_2-2(b-1)x_1) \qquad \text{or } (b<1,\ a>1)$$

$x_1=x_2=\frac{1}{3}$ is interior equilibrium

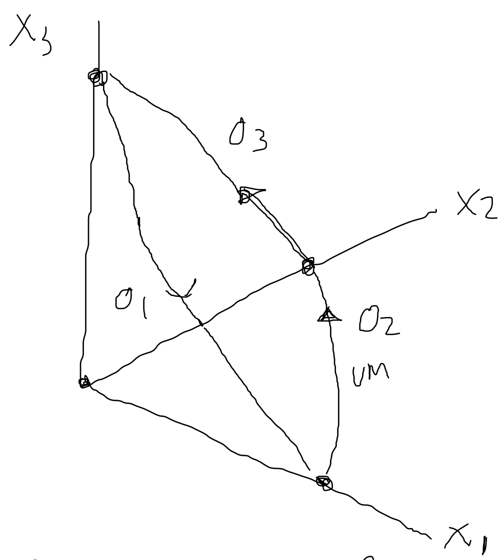
Trace $=0 \Rightarrow$ Prev Theorem it is conservative system!! Filled with periodic orbits but NOT LC



Let's now assume the case when the interior rest state is unstable, That is $a+b>2$

Restricted to $X_3=0$, the $X_1$-$X_2$ system is such That $0<b<1<a$, $a+b>2$, sorry This is reversed of what we had before, but I want to keep same conditions of the book.

When $X_3=0$ $X_2$ beats $X_1$. In addition to interior fixed pt, there are the points $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$. Stable manifold of $\vec{e}_2$ is Thus, $\{(x_1,x_2,x_3): x_1\geq 0,\ x_2>0,\ x_3=0\}$ while The unstable mfld of $\vec{e}_1$ is the orbit connecting $\vec{e}_1$ to $\vec{e}_2$, call it $O_2$; Let $F$ be the set consisting of $O_1$, $O_2$, $O_3$, & Three saddle $\vec{e}_1 - \vec{e}_3$




We'll show all orbits in $\mathbb{R}^3_+$, except diagonal have $F$ as $\omega$-limit set

Let $S=X_1+X_2+X_3$, $P=X_1X_2X_3$

$$\dot S = \dot x_1 + \dot x_2 + \dot x_3 = x_1 + x_2 + x_3 - [x_1^2 + x_2^2 + x_3^2 + (a+b)(x_1x_2 + x_2x_3 + x_3x_1)]$$

$$\dot P = \dot x_1 x_2 x_3 + x_1 \dot x_2 x_3 + x_1 x_2 \dot x_3 = P(3 - (1+a+b)S)$$

$$S^2 = x_1^2 + x_2^2 + x_3^2 + 2(x_1x_2 + x_1x_3 + x_2x_3)$$

$$\leq x_1^2 + x_2^2 + x_3^2 + (a+b)(x_1x_2 + x_1x_3 + x_3x_1) \Rightarrow \dot S \leq S - S^2 = S(1-S)$$

$\Rightarrow \dot S < S(1-S) \Rightarrow$ all populations must remain bded

$$\dot{\left(\frac{P}{S^3}\right)} = S^{-4} P\left(1 - \frac{a+b}{2}\right)\left[(x_1-x_2)^2 + (x_2-x_3)^2 + (x_3-x_1)^2\right] \leq 0$$

This is 0 iff $x_1 = x_2 = x_3 \Rightarrow$ any orbit that is not on the diagonal converges to the set where P vanishes that is the set F!