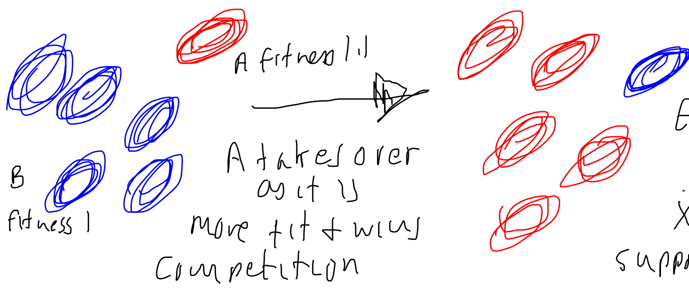
# Evolutionary Games

Game Theory was invented by Oskar Morgenstern & John von Neuman & mathematically characterized by John Nash (Nobel Prize). William Hamilton & Robert Trivers applied it to biology & John Maynard Smith to evolution.




Here fitness is a somewhat abstract concept, which we will formalize later.

Example 1.
$\dot{x} = x(1-x)$ $x \to 1$ now introduce $y$
$\dot{x} = x(1-x-ay)$ $\dot{y} = y(1-y-bx)$
suppose $a>1$, $b<1$, small amount of $y$ takes over.

more general idea. Let $\vec{x} = (x_A, x_B)$, & let $f_A(\vec{x})$ be fitness of A, $f_B(\vec{x})$, fitness of B. Suppose $x_A, x_B$ are fraction of pop that is A, B resp.

Let $\phi = x_A f_A + x_B f_B$ be the average fitness. Then we write:

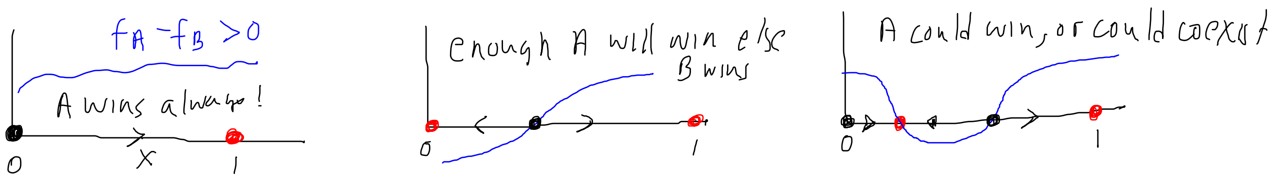
$\dot{x}_A = x_A(f_A(\vec{x}) - \phi)$, $\dot{x}_B = x_B(f_B(\vec{x}) - \phi)$. Since $x_A + x_B = 1$, check:

$\dot{x}_A + \dot{x}_B = x_A f_A + x_B f_B - \phi(x_A + x_B) = \phi(1-(x_A+x_B)) \Rightarrow x_A + x_B = 1$

Let $x_A = x$, $x_B = 1-x$, so $f_A(\vec{x}) = f_A(x)$, $f_B(\vec{x}) = f_B(x) \Rightarrow$

$\dot{x} = x(f_A(x) - (x f_A + (1-x) f_B)) = x(f_A(1-x) - (1-x) f_B) = x(1-x)(f_A(x) - f_B(x))$

So: $x=0$, $x=1$, $f_A(x) = f_B(x)$ are all the fixed points




Clearly: $x=0$ stable iff $f_A(0) - f_B(0) < 0$, $x=1$ stable iff $f_A(1) > f_B(1)$, interior point $x^*$ is stable, iff $f_A'(x^*) - f_B'(x^*) < 0$

# Games + Payoff Matrices

A game with two strategies, A, B is described by a payoff matrix:

$$\begin{array}{c} \\ A \\ B \end{array}\begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

A + B are <u>NOT</u> the players, but the strategies

If each player plays strategy A, then both get "a"
If both play "B", they get "d"
If one plays A + the other plays B, the one playing A, gets "b"
+ the one playing B gets "c"

In evolutionary game theory the fitness is determined by the expected payoff. So, a player that takes strategy A, can expect

$f_A = a x_A + b x_B$ (where $x_A, x_B$ are fraction playing A, B)

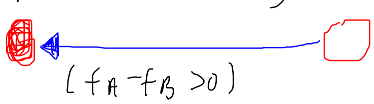
$f_B = c x_A + d x_B$. These are a linear fitness model.

$x = x_A$, $1-x = x_B$, $\phi = x f_A + (1-x) f_B$

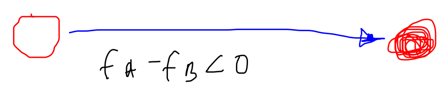
$\dot{x} = x(1-x)[f_A - f_B] = x(1-x)([a-b-c+d]x + b-d) = x(1-x)((a-c)x + (b-d)(1-x))$

$x^* = \dfrac{b-d}{c-a+b-d} = \dfrac{d-b}{a-b-c+d}$

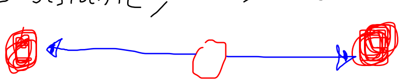
A dominates B $a>c$, $b>d$




B dominates A, $c>a$, $d>b$

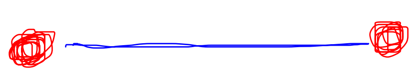



A, B bistable, $a>c$, $d>b$



coexistence, $a<c$, $d<b$

$a=c$, $b=d$ <u>Neutral</u> (inf # equ)



A is stable when $a>c$, B is stable when $d>b$
(If you play A against A, it is always better than playing B against A)

# Nash Equilibrium

A Nash Equilibrium is a strategy such that if any player deviates, he will not increase his payoff.
(decrease → strict)

$$\begin{array}{c c} & A \quad B \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \end{array}$$

You can think of a strategy as a vector $\vec{x}$ of prob, say $(1,0) \equiv$ pure A

(1) A is a strict Nash equil if $a > c$
(2) A is a N.E. if $a \geq c$
(3) B is a SNE if $d > b$
(4) B is a NE if $d \geq b$

These are "local", just like stability of equilib

(Recall that A will be a stable equil of the fitness eqn if $a > c$, sim B $(d > b)$)

$$\begin{array}{c c} & A \quad B \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 3 & 0 \\ 5 & 1 \end{pmatrix} \end{array}$$

B is a strict Nash Equilibrium, but note it is not the best global strategy – both playing A is better

However, if you play A + I switch to B, I do much better (I get 5, you get 0!) This is an example of the prisoners dilemma where it pays to "cheat"

$$\begin{array}{c c} & A \quad B \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 3 & 1 \\ 5 & 0 \end{pmatrix} \end{array}$$

If both play A, one can improve by switching to B
If both play B, then again one can improve by switching to A

So neither is Nash Equilibrium

$$\begin{array}{c c} & A \quad B \\ \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} 5 & 0 \\ 3 & 1 \end{pmatrix} \end{array}$$

Here cannot improve if both play A, nor can you improve if both play B. A, B are Nash equilibria

## Evolutionary Stable Strategy (ESS) (John Maynard Smith)

Suppose a lot of A players + a single mutant B player arises. Will A be able to hold off invasion of B? Let $\varepsilon$ be freq of B + $1-\varepsilon$ = freq of A

$$\underbrace{a(1-\varepsilon) + b\varepsilon}_{f_A} > \underbrace{c(1-\varepsilon) + d\varepsilon}_{f_B} \Rightarrow a > c \text{ or } a = c + b > d. \text{ A is ESS.}$$

## More Than 2 strategies

Let $E(S_i, S_j)$ be payoff for strategy $S_i$ against strategy $S_j$

(i) $S_k$ is a strict Nash equilibrium if

$$E(S_k,S_k) > E(S_i,S_k) \quad \forall i \neq k$$

(ii) Nash Eq if $E(S_k,S_k) \geq E(S_i,S_k) \quad \forall i$

(iii) ESS if $E(S_k,S_k) > E(S_i,S_k) \ \forall i$

or $E(S_k,S_k) = E(S_i,S_k)$ & $E(S_k,S_i) > E(S_i,S_i)$ ∯

ESS ⇒ selection will oppose any potential invader

Same true for strict N.E. but not for a N.E. If $E(S_k,S_k) = E(S_j,S_k)$ and $E(S_k,S_j) < E(S_j,S_j)$ Then $S_k$ is still a N.E. but selection will favor $S_j$ invading $S_k$.

(iv) "Weak ESS" $E(S_k,S_k) > E(S_i,S_k) \ \forall i \neq k$ or

$E(S_k,S_k) = E(S_i,S_k)$ & $E(S_k,S_i) \geq E(S_i,S_i)$  Strict NE ⇒ ESS ⇒ weak ESS ⇒ NE

"Unbeatable strategy" $E(S_k,S_k) > E(S_i,S_k)$ & $E(S_k,S_i) > E(S_i,S_i) \ \forall i \neq k$

dominates all other strategies

### General Replicator dynamics

For $n$ strategies, There is an $n \times n$ matrix, $A = [a_{ij}]$ which is the payoff

Let $x_i$ be the frequency of strategy $i$. The fitness is $f_i = \sum_{j=1}^{n} a_{ij} x_j \equiv [Ax]_i$

The mean fitness is $\sum_{i=1}^{n} x_i f_i = X \cdot A X \overset{\text{def}}{=} \phi$, so we write:

$\dot{x}_i = x_i(f_i - \phi)$ or $\dot{x}_i = x_i\left((A\vec{x})_i - \vec{x}\cdot A\vec{x}\right)$, we will show this is equivalent to an $(n-1)$ dim LV system. Note it is a cubic ODE not quadratic as is LV. For $n=2$, we see This reduces to a one-d ODE. There is an interior fixed point iff

$$\frac{d-b}{a-b-c+d} > 0 \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \to \frac{a_{22}-a_{12}}{a_{11}-a_{21}-a_{12}+a_{22}} > 0 \iff (a_{11}-a_{21})(a_{12}-a_{22}) < 0$$

(prove This!!)

Formal definitions:

We say a point $\hat{x} \in S_n$ $(x_i \geq 0 \mid \sum x_i = 1)$ is a Nash equilibrium if

$x \cdot A\hat{x} \leq \hat{x} \cdot A\hat{x}$ for all $x \in S_n$ + an Evolutionarily Stable State (ESS) if $\hat{x} A x > x A x$ for all $x \neq \hat{x}$ in a nbhd of $\hat{x}$

ASIDE Game Theory

Suppose there are N pure strategies, $R_1$ to $R_N$ + allow players to use mixed strategies as well, playing the pure strategies with probability $p_1 \cdots p_n$. A strategy, $\vec{p} \in S_N$. Corners of simplex are the pure strategies and the interior is a completely mixed strategy. Let's suppose only two players with $u_{ij}$ being the payoff for a player using pure $R_i$ against pure $R_j$. $U = (u_{ij})$ is the payoff matrix. An $R_i$ strategist obtains expected payoff $(U\vec{q})_i = \sum u_{ij} q_j$ against $\vec{q}$ strategist + a $\vec{p}$ vs $\vec{q}$ is then $p \cdot Uq = \sum_{i,j} p_i u_{ij} q_j$. Let $\beta(q)$ be the set of "best replies" to $q$. That is, the value of $p$ such that $pUq$ obtains maximum value. Thus we now see that a Nash Equilibrium is a strategy that is the best reply to itself.

END ASIDE

Before continuing with example games, I will show a relationship between replicator dynamics + LV. In one of your exercises, you prove that you can add a constant $c_j$ to each column $j$ without changing the dynamics, so, eg. the last row of $a_{ij}$ can be made zero with no loss in generality!

Theorem There exists a differentiable, invertible map from $\hat{S}_n = \{x \in S_n \mid x_n > 0\}$ onto $\mathbb{R}^{n-1}_+$ mapping the orbits of $\dot{x}_i = x_i((Ax)_i - x \cdot Ax)$ onto the orbits of the LV

$\dot{y}_i = y_i\left(r_i + \sum_{j=1}^{n-1} a'_{ij} y_j\right), \; i = 1, \dots, n-1$

where $r_i = a_{in} - a_{nn}$, $a'_{ij} = a_{ij} - a_{nj}$

Proof: Let $y_n \equiv 1$ & consider the transformation $y \to x$ given by $x_i = \frac{y_i}{\sum_{j=1}^n y_j}$ $i=1,\dots,n$

which maps $\{y \in \mathbb{R}^n_+ : y_n = 1\}$ onto $\hat{S}_n$. The inverse $x \mapsto y$ is

$$y_i = \frac{y_i}{y_n} = \frac{x_i}{x_n}, \; i=1,\dots,n$$

consider $\dot{x}_i = x_i((Ax)_i - x \cdot Ax)$ → $\phi$

With no loss in generality, we subtract the last row of A from every other row of A, so that the last row of A is now zero (see exercise)

$$\dot{y}_i = \dot{\left(\frac{x_i}{x_n}\right)} = \frac{\dot{x}_i x_n}{x_n^2} - \frac{x_i \dot{x}_n}{x_n^2} = \frac{x_i x_n [(Ax)_i - \phi] - x_n x_i [(Ax)_n - \phi]}{x_n^2} = \left(\frac{x_i}{x_n}\right)[(Ax)_i - (Ax)_n]$$

but $(Ax)_n = 0 \Rightarrow$

$$\dot{y}_i = y_i\left(\sum_{j=1}^n a_{ij} x_j\right) = y_i\left(\sum_{j=1}^n a_{ij} y_j\right) x_n.$$

Since $x_n > 0$, we can rescale time (see ODE 1) to get rid of $x_n$ without changing the phase portrait. Finally, recall that $y_n = 1$ so that

$$\dot{y}_i = y_i\left(a_{in} + \sum_{j=1}^{n-1} a_{ij} y_j\right).$$ Converse holds analogously. ▨

So - 3 strategy Replicator has no limit cycles!!