More pursuit curves:

$$\frac{(A-P)}{|A-P|}\frac{\dot P}{|\dot P|}=1, \quad \frac{(P-A)}{|P-A|}\cdot\frac{\dot A}{|\dot A|}=\cos\varphi$$ Law of cosines so, $\varphi$ = angle of avoidance

$$\frac{(a-x)\dot x+(b-y)\dot y}{d\cdot v}=1 \qquad \frac{(x-a)\dot a+(y-b)\dot b}{d\cdot u}=\cos\varphi$$

$d=|A-P|$, $u=|\dot A|$, $v=|\dot P|$

as before, we get for $\dot x,\dot y$:

$\dot x=\frac{a-x}{d}$, $\dot y=\frac{b-y}{d}$, but what about $(a,b)$

$$(x-a)^2\dot a^2+(y-b)^2\dot b^2+2\dot a\dot b(x-a)(y-b)=\cos^2\varphi\,(\dot a^2+\dot b^2)\,((x-a)^2+(y-b)^2)$$

$$\dot a=\frac{\cos\varphi\,(x-a)+\sin\varphi\,(y-b)}{d}\sigma$$

$$\dot b=\frac{-\sin\varphi\,(x-a)+\cos\varphi\,(y-b)}{d}\sigma$$ , where $\sigma$ is A's max speed

This gives really cool dynamics. Let $u=x-a$, $v=y-b$, $d=u^2+v^2$

$$\dot u=\dot x-\dot a=\frac{-u-(\cos\varphi\,u+\sin\varphi\,v)\sigma}{d}$$

$$\dot v=\dot y-\dot b=-\frac{v+(\sin\varphi\,u-\cos\varphi\,v)\sigma)}{d}$$

If $\sigma=1$ + $\varphi=\pi$
Then $u=v=0$
Never catch up

For $\sigma$ large enough, can you prove $\exists\,\varphi$ st $u,v$ have a periodic solution?

$u=R\cos\omega t$, $v=R\sin\omega t$, plug it in, the R's cancel, so need $\omega$'s to work out! Can get spirals etc. If $\varphi=0$, then collide + if $\varphi=\pi$, Never catch up as you see

what if you make $\sigma=\sigma(R)$, for example

suppose, $\sigma$ is big if close + small if far? Or suppose $\varphi=\varphi(R)$ Then what happens. (E.g. $\varphi=\hat\varphi e^{-kd}$) $\longleftarrow$ Get stable Limit cycle!!

Now, we will start talking about swarms

Many models start at the individual level, so that is where we begin

Suppose, first we operate in 1-dimensional space so that the agents can move freely on the line, possibly subject to some boundary conditions

$$m_i \ddot{x}_i = \alpha \dot{x}_i - \beta |\dot{x}_i|^2 \dot{x}_i - \frac{\partial}{\partial x_i} U(x_1,..,x_N) + f_i$$

where $m_i$ is the mass of the ith agent, $\alpha, \beta$ are friction constants. Note that if $\alpha, \beta > 0$ then $\dot{x} \sim \pm\sqrt{\frac{\alpha}{\beta}}$, so the animals want to move. We could set $\beta = 0$ + $\alpha < 0$ for just plain friction!

$f_i$ is an external influence, say wind or gravity. The key force is the interaction potential, $U(\vec{x})$. In real swarms the agents don't want to get too far apart, but also, don't want to get too close, so in this case, we want a distance dependent force that is attractive at long distances + repulsive at short distances. The so called Morse potential is often used:

$$U = \sum_i \sum_{j \neq i} C_r e^{-|x_i - x_j|/\ell_r} - C_a e^{-|x_j - x_i|/\ell_a}$$

$$-\nabla_i U = \sum_{j \neq i} \left[\ell_r C_r e^{-|x_i - x_j|/\ell_r} - \ell_a C_a e^{-\frac{|x_j - x_i|}{\ell_a}}\right] \text{sign}(x_i - x_j)$$

So, e.g. if $x_i - x_j > 0$ and within repulsion zone then get pushed away else get pulled together. Typically, $\ell_a > \ell_r$, so that if far away then will be attracted but if too close will be repelled.

One easy limit is to set $\beta = 0$, $\alpha < 0$ (linear damping) + $m_i = 0$ (no inertia)

Then we get a system of first order ODEs:

$$\alpha \dot{x}_i = \sum_{j \neq i} \left[\ell_r C_r e^{-\frac{|x_j - x_i|}{\ell_r}} - \ell_a C_a e^{-\frac{|x_j - x_i|}{\ell_a}}\right] \text{sign}(x_i - x_j)$$

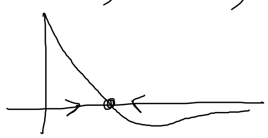
Lets start with $N=2$: (set $\alpha=1$)

$\dot{x}_1 = f(|x_1-x_2|)\,\text{sign}(x_1-x_2)$ Let $y = x_1 - x_2$

$\dot{x}_2 = f(|x_1-x_2|)\,\text{sign}(x_2-x_1)$

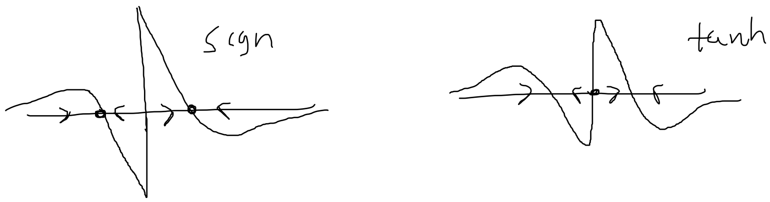
$$\dot{y} = \overbrace{2f(|y|)\,\text{sign}(y)}^{g(y)} \qquad f(r) = \frac{C_r}{\ell_r} e^{-|r|/\ell_r} - \frac{C_a}{\ell_a} e^{-|r|/\ell_a}$$

Eg $\ell_r = 1$, $\ell_a = 2$, $C_r = .8$, $C_a = .2$, $f(r) = .8e^{-r} - .4e^{-r/2}$



They will "lock" in at a distance where $f(r^*) = 0$

Can replace sign(x) by $\tanh(\gamma x)$ where $\gamma \gg 1$ to make a smoother model

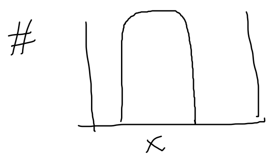



(N.B. If you use $\tanh(\gamma x)$, then your kernel will be $\exp(-\ln(\cosh bx)/b)$)

So, now, what happens for $N > 2$ in this case:

$$\dot{x}_i = \sum_{j \neq i} \left( \frac{C_r}{\ell_r} e^{-|x_i - x_j|/\ell_r} - \frac{C_a}{\ell_a} e^{-|x_i - x_j|/\ell_a} \right) \text{sign}(x_i - x_j)$$

Do $N = 200$, $\ell_a = 10$, $\ell_r = 1$, $C_a \ell_a = .5$, $C_r \ell_r = 1$




Several other cases of interest are as follows:

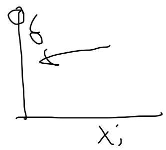
Suppose we have only repulsive interactions:

$$\dot{x}_i = \sum_{j \neq i} e^{-|x_i - x_j|}\,\text{sign}(x_i - x_j) + f_i$$

where $f_i$ is the external force that is $-\nabla_i F(x)$, where $F$ is an external potential.

No force, finite domain $\rightarrow$  Expect them to pile up at the edges − $\delta$-functions

Suppose gravity, $f_i = -g$, & you work on $x_i \geq 0$ so that $x=0$ is the ground. Then if gravity is strong all lie on the origin & you get

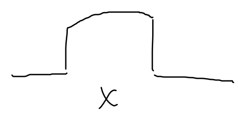



If gravity is weaker you find


On infinite domain, imagine you have a bunch of anti social guys and there is an attraction in the middle, say at $x=0$. Then

$$\dot{X}_i = \sum_{j \neq i} e^{-|x_i - x_j|} \operatorname{sign}(x_i - x_j) - \gamma X_i$$

This tends to pull them into the center. Then you might expect




a compact swarm centered where the attraction is.

We want to study these cases analytically, but to do that we will proceed to the continuum limit

Let $\rho(\vec{x},t)$ be the density of guys at $\vec{X}$ and time $t$, in some arbitrary domain $D \subset \mathbb{R}^n$. We assume that the mass is conserved so that the only way to lose or gain mass is through the boundary of $D$. Thus we have

$$\int_D \frac{\partial \rho}{\partial t} dx = \frac{\partial}{\partial t} \int_D \rho \, dx = -\int_{\partial D} \rho \vec{v} \cdot \vec{n} \, dS \qquad \vec{v}\cdot\vec{n} > 0 \text{ means escaping out of the domain}$$

rate of change of mass

Flux out of D with velocity V

Apply divergence theorem: $\int_{\partial D} (\rho \vec{v}) \cdot \vec{n} \, dS = \int_D \nabla \cdot (\rho \vec{v}) \, d\vec{x}$

so $\int_D \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) \right] d\vec{x} = 0$; since $D$ is arbitrary we get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$

Continuity equation

$\rho \vec{V} \equiv \vec{J}$ is the flux

what is $\vec{V}$?

$\dot{X}_i = F_i(X_1, \ldots, X_N) \rightarrow$ This has the dimensions that we want (It is a velocity)

Let me write it explicitly:

$$F_i = \sum_j q(X_i - X_j) - f(X_i)$$

Lets Approximate $\rho$ as a sum of $\delta$ Functions:

$$\rho(x,t) = \sum_i \delta(x - X_i(t)) \rightarrow \text{continuous density as } N \to \infty$$

$$F_i = \sum_j q(X_i - X_j) + f(X_i) = \int q(X_i - y) \sum_j \delta(y - X_j(t))\, dy + f(X_i) = \int q(X_i - y)\rho(y,t)\, dy + f(X_i)$$

So in $N \to \infty$ limit, $F_i = f(X_i) + \int q(X_i - y)\rho(y,t)\,dy$ is the velocity. Since $X_i$ is arbitrary, $V(x) = f(x) + \int q(x-y)\rho(y,t)\,dy$

and we get: (in One dimension)

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left[\rho V(x,t)\right] \quad \text{where } V(x,t) = f(x) + \int q(x-y)\rho(y,t)\,dy$$

This is the object of our analysis!

Note: This works <u>only</u> because each guy $X_i$ is the same in the sense of being one in a crowd & subject to all the same forces as the other guys.

Swarms in multiple dimensions obey the same equations:

$$\frac{\partial \rho}{\partial t} = -\nabla(\rho V)$$

We now concentrate on 1-dimensional swarms & attempt to better understand the numerical results. & in particular look at say repulsive interactions with an attractant in the middle, or just a att/rep swarm.

We look for stationary solutions, $\rho = \rho(x)$ ind of time. This leads to:

$$\frac{d}{dx}(\rho V) = 0$$

$\Rightarrow$ $\rho V = C$ where $C$ is a constant. Suppose that $C \neq 0$. Then $V = C/\rho$. But since $\rho \geq 0$ & $\int_{-\infty}^{\infty} \rho(x)dx = 1$, this means $\rho(x) \to 0$ for large $x \Rightarrow C/\rho(x) \to \infty \Rightarrow V \to \infty$ but $V$ is a nice bounded function (convolution with $\rho$ + some source terms) $\Rightarrow$ $V$ is bded $\Rightarrow$ $C = 0$ is only possibility. So we see that

$$\rho V = 0$$

Either $\rho(x) = 0$ or $V(x) = 0$.

Recall $V(x) = f(x) + \int q(x-y)\rho(y)dy$

so $V(x) = 0$ is a linear integral equation for $\rho(x)$. In general we have seen that swarms seem to be compact so we will generally make the ansatz that $\rho(x) = 0$ outside some compact region say $(a,b)$ and then we must solve:

$$f(x) + \int_a^b q(x-y)\rho(y)dy = 0$$

It may be easier (since $q(x)$ is discontinuous, e.g.) to integrate this eqn obtaining $F(x) + \int_a^b Q(x-y)\rho(y)\,dy = \gamma$ $(F' = f, Q' = q)$

for example, suppose $q(x)$ is a repulsive kernel + $f(x)$ is an attraction

$$\dot{x}_i = \sum e^{-|x_i-x_j|} \, sgn(x_i - x_j) - 2\gamma x_i$$

$$\Rightarrow q = e^{-|x|} sgn(x) \Rightarrow Q = -e^{-|x|} \ + \ F(x) = -\gamma x^2$$

which gives us:

$$\frac{ax^2}{2} - Cr\int_a^b e^{-|x-y|}\rho(y)dy = \lambda$$

a, b are unknown but it is clear that the swarm should be symmetric about the origin:


so we want to solve:

$$\gamma x^2 + \int_{-\beta}^{\beta} e^{-|x-y|}\rho(y)dy = \lambda$$

How do we determine the unknown $\beta$?

We can write the integral as

$$\int_\alpha^x e^{y-x}\rho(y)dy + \int_x^\beta e^{x-y}\rho(y)dy \equiv I(x)$$

$$\frac{dI}{dx} = -\int_\alpha^x e^{y-x}\rho(y)dy + \int_x^\beta e^{x-y}\rho(y)dy, \quad \frac{d^2 I}{dx^2} = -2\rho(x) + I(x)$$

$$I(x) = \lambda - F(x)$$

$$I''(x) = -2\rho(x) + I(x) = -F''(x)$$

$$\Rightarrow \rho^*(x) = \frac{F''(x)}{2} + \frac{I(x)}{2} = \frac{F''(x)}{2} + \frac{\lambda}{2} - \frac{F(x)}{2} \quad !!$$

This is a necessary condition but not sufficient + to verify it is a solution, we must substitute back into the integral:

$$I(\rho^*) = \int_\alpha^\beta e^{-|x-y|}\left[\frac{\lambda}{2} - \frac{1}{2}[F(y) - F''(y)]\right]dy$$

$$= \lambda - F(x) + \frac{e^{\alpha-x}}{2}[F(\alpha) - F'(\alpha) - \lambda] + \frac{e^{x-\beta}}{2}[F(\beta) + F'(\beta) - \lambda]$$

Notice we have terms multiplying $e^{-x}, e^{x}$ which are in the nullspace of the integral operator. These can vanish only when

$$F(\alpha) - F'(\alpha) = \lambda + F(\beta) + F'(\beta) = \lambda$$

This will restrict $\alpha, \beta$ + give us $\lambda$. We also need to preserve mass $\int \rho(x)dx = M$, the total initial mass.

Back to $F(x) = \gamma x^2$. Then we get

$$\rho^*(x) = \frac{\lambda + 2\gamma - \gamma x^2}{2}$$

$$A = \frac{\lambda - \gamma\alpha^2 + 2\gamma\alpha}{2} = 0, \quad B = \frac{\lambda - \gamma\beta^2 - 2\gamma\beta}{2} = 0 \Rightarrow \alpha = -\beta.$$ We write $\beta = H, \alpha = -H$

we have $\lambda = \gamma H^2 + 2\gamma H = \gamma(H+1)^2 - \gamma$

$$\int_{-H}^{H} \rho^*(x)dx = M = \int_0^H (\lambda + 2\gamma - \gamma x^2)dx = (\lambda + 2\gamma)H - \frac{\gamma}{3}H^3 = M$$

$$\Rightarrow \lambda = \frac{M + \gamma(H^2 + H^3/3)}{1+H} \ ?$$

$$\Rightarrow H = \left(\frac{3M}{2\gamma} + 1\right)^{1/3} - 1$$ Note that as $\gamma \to \infty$ $H \to 0$ as expected and $H \uparrow$ as $M \uparrow$ + as $\gamma \to 0$ $H \to \infty$ as expected.

Now we turn to the case where there is <u>NO</u> extrinsic potential but the kernel has attraction + repulsion:

$$q(x) = G e^{-|x|/\ell} \operatorname{sgn}(x) - e^{-|x|} \operatorname{sgn}(x)$$

$$Q(x) = -G\ell e^{-|x|/\ell} + e^{-|x|}$$

$0 < G < 1$, $\ell > 1$. We are interested in the case where $G\ell^2 > 1$ (we will see what that means presently.

First, note that FT of $e^{-|x|}$ is $2/(1+k^2)$ and $G\ell e^{-|x|/\ell}$ is $\frac{2G\ell^2}{1+k^2\ell^2}$

So we can think of the integral operators as inverse operators $\frac{1}{1+k^2} = (1-\partial_{xx})^{-1}$, $\frac{1}{1+\ell^2 k^2} = (1-\ell^2\partial_{xx}^2)^{-1}$

so our problem is generally:

$\int_\Omega Q(x-y)\rho(y)\,dy + F(x) = \lambda$ with $Q$ as above $\hat{Q} = \frac{2}{1+k^2} - \frac{2G\ell^2}{1+\ell^2 k^2}$

Note $\hat{Q}(0) = 2(1-G\ell^2)$ & we will assume this is negative since if it is positive $\Rightarrow$ repulsion dominates & the "swarm" will spread out. Won't agregate — in language of pattern formation - Not a Mexican hat. Let $\mathcal{L}_1 = \partial_{xx} - 1$, $\mathcal{L}_2 = \ell^2\partial_{xx} - 1$. Then we see that

$\int Q(x-y)\rho^*(y)\,dy = \lambda - F(x) \Rightarrow$ formally:

$-\frac{2G\ell^2}{1+\ell^2 k^2}\hat{\rho} + \frac{2}{1+k^2}\hat{\rho} = \widehat{\lambda - F} \Rightarrow -2G\ell^2(1+k^2)\hat{\rho} + 2(1+\ell^2 k^2)\hat{\rho} = (1+k^2)(1+\ell^2 k^2)\widehat{(\lambda - F)}$

or $\nu\rho^*_{xx} + \varepsilon\rho^* = -\mathcal{L}_1\mathcal{L}_2(\lambda - F(x))$ where $\nu = 2L^2(1-G)$, $\varepsilon = 2(GL^2 - 1)$

By hypothesis $G<1$, & we now see why we want $GL^2 - 1 > 0$ since this means solutions to $\nu\rho^*_{xx} + \varepsilon\rho^*$ are cosines & sines.

Now we want $F(x) = 0$ in this example so:

$\nu\rho^*_{xx} + \varepsilon\rho = -\lambda$, solutions are $\rho(x) = C\cos\mu x + D\sin\mu x - \frac{\lambda}{\varepsilon}$

where $\mu = \sqrt{\frac{\varepsilon}{\nu}} = \sqrt{\frac{GL^2-1}{L^2(1-G)}}$

without loss of generality we can center the swarm at the origin & then we find that, $D = 0$ since the solution is symmetric

Hence $\rho = C\cos\mu x - \frac{\lambda}{\varepsilon}$, we need to determine $H$, $C$, $\lambda$. $[-H, H] = \mathrm{supp}(\rho)$ We also must have $\int_{-H}^{H}\rho(x)\,dx = M$. We plug this into the integral equation to get

$\frac{2}{1+\mu^2}[\cos\mu H - \mu\sin\mu H]\,C - \frac{1}{G\ell^2 - 1}\lambda = 0$

$\frac{2}{1+\mu^2\ell^2}[\cos\mu H - \mu\ell\sin\mu H]\,C - \frac{1}{G\ell^2 - 1}\lambda = 0$

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{pmatrix} C \\ \lambda \end{pmatrix} = 0 \Rightarrow a_{11}a_{22} - a_{12}a_{21} = 0 \Rightarrow \frac{\cos \mu H}{\sin \mu H} = \frac{G\ell - 1}{\sqrt{(1-G)(G\ell^2-1)}}$

Mass constraint $\Rightarrow \left(\frac{2}{\mu}\sin \mu H\right) C - \frac{H}{G\ell^2-1}\lambda = M$

we also have $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{bmatrix}\begin{pmatrix} C \\ \lambda \end{pmatrix} = 0$ so 2 equations for $C, \lambda$

$C = \frac{M}{2(H+\ell+1)} \frac{\sqrt{G}(\ell^2-1)}{\ell(1-G)}, \quad \lambda = \frac{M(1-G\ell^2)}{H+\ell+1}$

Cool, huh!

What about our model with the predator?

$\dot{x}_i = \sum_j q(x_i - x_j) + c_r e^{-|x_i - u|/\ell_r}\,\mathrm{sgn}(x_i - u)$

$\dot{u} = \frac{1}{N}\sum_j c_a e^{-|x_j - u|/\ell_a}\,\mathrm{sgn}(u - x_j)$ we get rid of $u$, $y_i = x_i - u$ ($c_a < 0$)

$\dot{y}_i = \dot{x}_i - \dot{u} = \sum_j q(y_i - y_j) + c_r e^{-|y_i|/\ell_r}\,\mathrm{sgn}\, y_i + \frac{c_a}{N}\sum_j e^{-|y_j|/\ell_a}\,\mathrm{sgn}(y_j)$

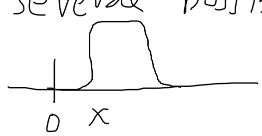
$v = \int q(x-y)\rho(y)\,dy + c_r e^{-|x|/\ell_c}\,\mathrm{sgn}(x) + c_a \int e^{-|y|/\ell_a}\,\mathrm{sgn}(y)\rho(y)\,dy$

We need to set $v = 0$

We integrate this to get: $\int_\alpha^\beta Q(x-y)\rho(y)\,dy - c_r e^{-|x|/\ell_r} + kx = \lambda$

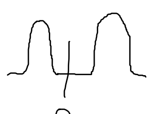
$k = c_a \int_\alpha^\beta e^{-|y|/\ell_a}\,\mathrm{sgn}(y)\rho(y)\,dy$

There seem to be several possible solutions to this:

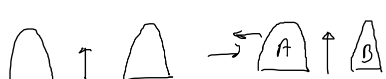
"push" 

all prey ahead of predator ($c_a$ weak)

push pull:



some prey ahead + some prey behind, but NOT balanced ($c_a$ a bit stronger)

I found as $c_a$ gets stronger you can get push pull - there is a symmetry breaking bifurcation: 


push pull - herd A gets pushed by predator, herd B likes herd A so much that it tries to join with it + gets pulled along

For higher $C_a$ seem to get an oscillation where herds no longer move but stay fixed on average - get $\bigcap \leftrightarrow \bigcap$ wolf oscillating and some sheep jumping!

Non classical solutions: Finite domain with repulsive interactions

Consider repulsive force, no extrinsic force + bounded domain $[-d, d]$

Recall $\rho^*(x) = \frac{\lambda}{2} - [F - F'']$

is the classical part of the solution so that we will take

$\rho^*(x) = \frac{\lambda}{2}$, but substitute this into the equation!

$$\frac{\lambda}{2}\int_{-d}^{d} e^{-|x-y|}\,dy = \frac{\lambda}{2}\int_{-d}^{x} e^{y-x}\,dy + \frac{\lambda}{2}\int_{x}^{d} e^{x-y}\,dy = \frac{\lambda}{2}\left[1 - e^{-d-x} + 1 - e^{x-d}\right]$$

$= \lambda$ + exponential terms which only vanish when $\lambda = 0$! So something is missing. We need stuff piled up at the edge

Candidate solution: $\rho^*(x) = \frac{\lambda}{2} + A\,\delta(x+d) + B\,\delta(x-d)$

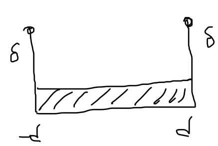
Plug this in:

$$\int_{-d}^{d} e^{-|x-y|}\rho^*(y)\,dy = \lambda - \frac{\lambda}{2}\left[e^{-d-x} + e^{d+x}\right] + A\,e^{-x-d} + B\,e^{x-d}$$

$\Rightarrow A = B = \frac{\lambda}{2} \Rightarrow$

$\rho^* = \frac{\lambda}{2}\left[1 + \delta(x-d) + \delta(x+d)\right]$ $\quad \int_{-d}^{d}\rho^*(y)\,dy = M \Rightarrow M = d\lambda + \lambda \Rightarrow \lambda = \frac{M}{1+d}$

$\rho^* = \frac{M}{1+d}\left[1 + \frac{1}{2}\delta(x-d) + \frac{1}{2}(x+d)\right]$



Gravity swarm example

Domain is $x \geq 0$, Assume repulsive swarm but gravity pulls down

$$\rho^*(x) = \frac{\lambda}{2} - \frac{g x}{2} \quad \left(\frac{\lambda}{2} - \frac{1}{2}(F - F'')\right) \qquad F = g x$$

Need $\underset{0}{\vdash}\!\!-\!\!\underset{\beta}{+}$ for compact swarm (Note $-g x$ trumps $\int Q(y-x)\rho(y)dy$ since integral is bded

As above, we expect the swarm to pile up at the boundary $x = 0$

so choose $\rho(x) = \frac{\lambda}{2} - \frac{g x}{2} + A\delta(x)$

Plug into equation $\int_0^\beta e^{-|x-y|}\rho^*(y)dy$

$$\int_0^\beta e^{-|x-y|}\rho^*(y)dy = \lambda - g x + A e^{-x} + \frac{e^{-x}}{2}\left[\quad -g - \lambda\right] + \frac{e^{x-\beta}}{2}\left[g\beta + g - \lambda\right]$$

As usual, must kill exponentials

$$A - \frac{g}{2} - \frac{\lambda}{2} = 0 \qquad g\beta + g - \lambda = 0 \qquad \int_0^\beta (\lambda - g y + A\delta(y))dy = M$$

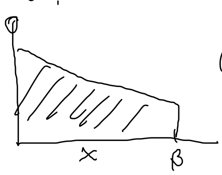
$$\lambda\beta - g\frac{\beta^2}{2} + A = M$$

$A = \frac{g + \lambda}{2}$, $\lambda = (\beta + 1)g$

$A = g(2 + \beta)$

$$\beta(\beta+1)g - \frac{g\beta^2}{2} + g(2+\beta) = M$$

$\Rightarrow \beta = 2\sqrt{M/g} - 2 \quad (M > g)$ if $M < g$ then what happens? $M < g$ then $\rho = M\delta(x)$ + they all lie down on the ground



compact swarm again!