Pecora - Carroll Master stability equation.

$$\dot{Z} = A(t) Z + (\alpha + i\beta) K Z$$

Find the regions in (α, β) where $Z(t) \to 0$ as $t \to \infty$

If we write $Z = R + iS$ then

$$\frac{dR}{dt} = A(t) R + \alpha K R - \beta K S$$

$$\frac{dS}{dt} = A(t) S + \alpha K S + \beta K R$$

Can just integrate this or ~~assume~~ for periodic $A(t)$ you can compute the monodromy matrix as a function of (α, β)

Surprisingly, even for oscillators it is possible that some kind of coupling lead to instabilities

For Homework, I will give you the following

$$\dot{z}_j = z_j (1 - (1 + iq) z_j \bar{z}_j) + (\beta + i\gamma)(z_k - z_j)$$

q, γ are real parameters

In the mean time, consider the following example

Coupled Brusselators:

$$\dot{X}_j = A - (B+1) X_j + X_j^2 Y_j + D_x (X_n - X_j)$$

$$\dot{Y}_j = B X_j - X_j^2 Y_j + D_y (Y_n - Y_j)$$

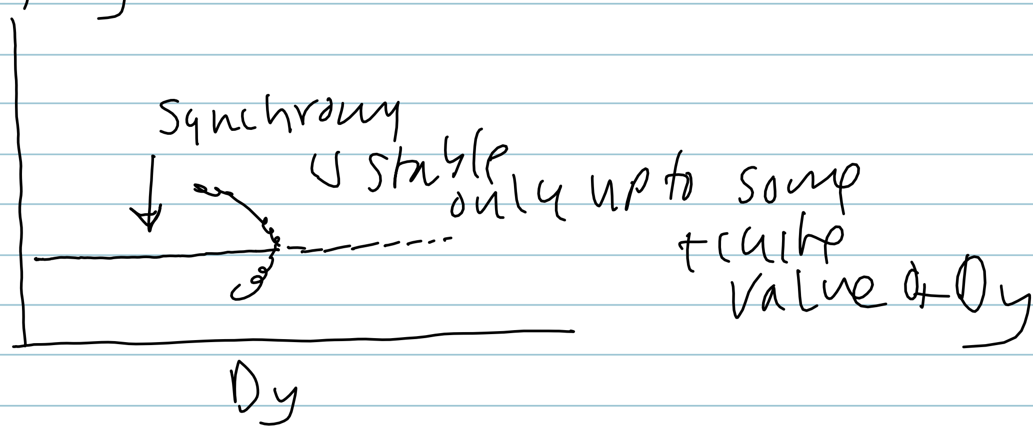
I will use the commutator form [1].

Fix $A =$

$B =$

$D_x = 0.05$

Vary D_y



General diffusive coupling

$$\dot{X}_i = F(X_i) + \sum_{j=1}^{N} \hat{C}_{ij}\, K(X_j - X_i) \qquad i = 1, \dots, N \qquad (\hat{\star})$$

$\hat{C}_{ii} = 0$

Where $F : \mathbb{R}^m \to \mathbb{R}^m$ $\quad K \in \mathbb{R}^{m \times m}$

$(\hat{C}_{ij}) \equiv \hat{C} \in \mathbb{R}^{N \times N}$

Let $X_i = u(t)$ be a synchronous solution.

That is

$$\dot{u}(t) = F(u(t)) \qquad (S)$$

Define $c_{ij} = \begin{cases} \hat{C}_{ij} & i \neq j \\ -\sum_{j=1}^{N} c_{ij} & i = j \end{cases}$

Let $C = (c_{ij})$ Note $C \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = 0$

We can rewrite $(\hat{\star})$ as

$$\dot{X}_i = F(X_i) + \sum_{j=1}^{N} c_{ij} K X_j \qquad (\star)$$

since $-\sum_{\substack{j=1 \\ j \neq i}}^{N} \hat{C}_{ij} = c_{ii}$

To check stability of $X_i = u(t)$ write

$x_i(t) = u(t) + Y_i \implies$ (L)

$$\dot{Y}_i = A(t) Y_i + \sum_{j=1}^{N} c_{ij} K Y_j \quad \text{where } A(t) \equiv D_X F(u(t))$$

We can write this as

$$\dot{Y} = [I \otimes A(t) + C \otimes K]Y$$

But, lets just stick with (L) for now.

Let $C\vec{v} = \lambda \vec{v}$ be eigenvalue-eigenvector pair for C. Note

$$C \in \mathbb{R}^{N \times N} \quad \& \quad \vec{v} \in \mathbb{C}^N$$

Write $Y_i(t) = v_i W(t)$ where $(v_1, \dots, v_N)^T \equiv \vec{v}$

Note $W(t) \in \mathbb{C}^m$ for each t.
(keep this straight!) . Plug this into (L) to get

$$v_i \dot{W}(t) = v_i A(t) W(t) + \sum C_{ij} v_j K W(t)$$
$$= v_i [A(t) W(t) + \lambda K W(t)]$$

multiply by $\bar{v}_i$ & sum up to get

$$\|\vec{v}\|^2 \dot{W}(t) = \|\vec{v}\|^2 [A(t) W(t) + \lambda K W(t)]$$

Since $\|v\| \neq 0 \Rightarrow$

$$\dot{W}(t) = A(t) W(t) + \lambda K W(t) \quad (A)$$

Thus we've reduced a $N \times m$ dim system to N m-dim systems!

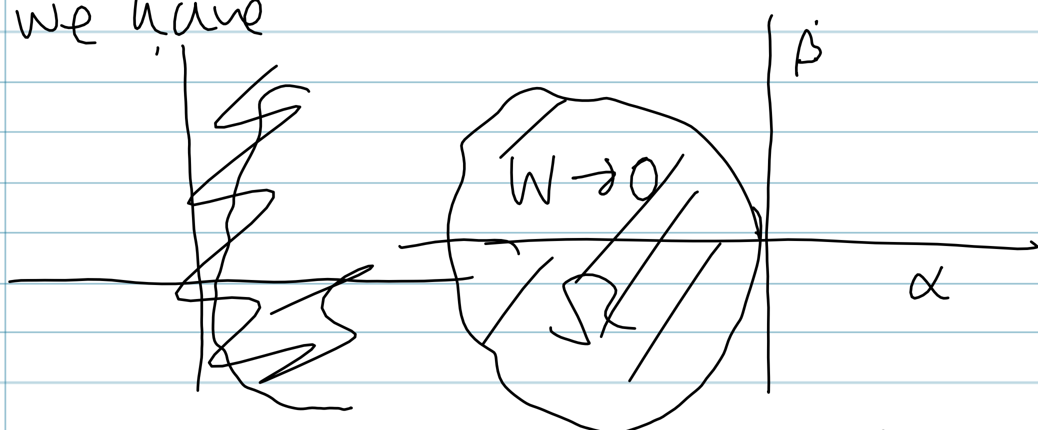
since $\lambda \in \mathbb{C}$ we need only look at

$$\dot{w}(t) = A(t)\,w(t) + (\alpha + i\beta)\,\Gamma\,w(t)$$

as α, β range around complex plane

~~Then for given value~~

suppose for some ~~set of param~~ choice Γ we have



Then if $\lambda \in \Omega \Rightarrow w(t) \to 0$

so we can pick coupling matrix C so that all eigenvalues of C are in Ω to guarantee that synchrony is stable!

This is the essence of the Pecora Master Stability Theorem

Unfortunately, you still have to compute!