Numerical Algorithm for computing maximal Lyapunov exponent

Suppose we have a linear system

$$\dot{x} = A(t) x$$

where $A(t)$ comes from say a chaotic system

Lets write $A_j = A(t_j)$

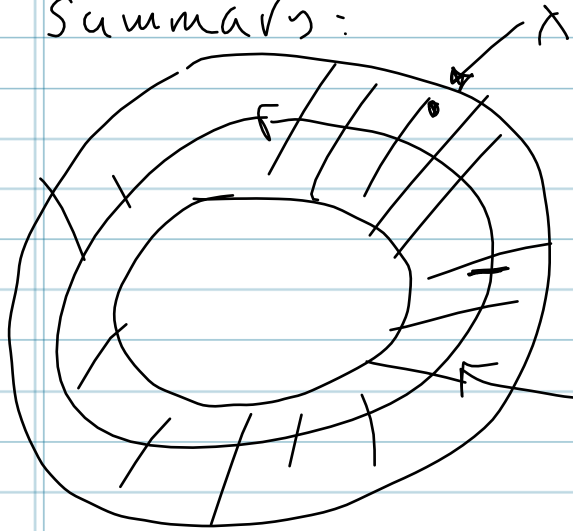
$t_1 < t_2 < \dots < t_p$ are times that you have computed the orbit

$$x(t+\Delta t) = x(t) + \Delta t \; A(t_j \Delta t)$$

$$= (I + \Delta t \; A_j)$$

so apply same ideas as before to this

Summary:



$\Theta(x) = \varphi$ is asymptotic phase

$\theta = 0, T$

$u(t) \equiv \Gamma$ is limit cycle

$\Theta(u(\varphi)) = \varphi \mod T$

$\nabla \Theta(u(\varphi)) \equiv Z(\varphi) \qquad Z(\varphi) \cdot \dot{u}(\varphi) = 1$

$-\frac{dZ}{dt} = A(t)^T Z(t)$ (✱), $\dot{u} = F(u(t))$

where $A(t) = D_x F(u(t))$

$Z(t)$ is the unique periodic solution to (✱) s.t. $Z(t) \cdot \dot{u}(t) = 1$

Scalar equations on circle

$\dot{x} = f(x)_{u(t)} \qquad x \in S^1 \qquad f(x) > 0$

~~u(t)~~ $\int_0 \frac{ds}{f(s)} = t$

$$\boxed{z(t) = 1/\dot{u}(t)}$$

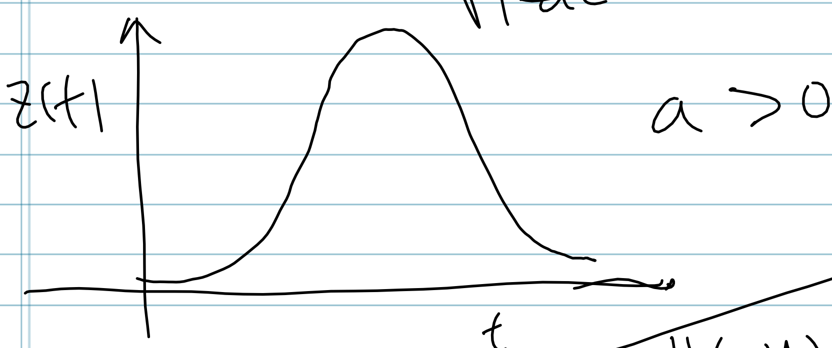
For example:

$$\dot{x} = 1 + a\cos x \qquad a \in (-1, 1)$$

$$u(t) = 2\tan^{-1}\left[\tan\left(\frac{\sqrt{1-a^2}}{2}t\right)\sqrt{\frac{1+a}{1-a}}\right]$$

$$z(t) = \frac{1}{u(t)} = \frac{1 + a - 2a\cos^2\left(\frac{t\sqrt{1-a^2}}{2}\right)}{1-a^2}$$

Note $T = \dfrac{2\pi}{\sqrt{1-a^2}}$



"Construct an adjoint!"

HW?? Let $z(t)$ satisfy $z(0) = z(T)$
and $z(t) > 0 \quad \forall t \in [0, 1]$

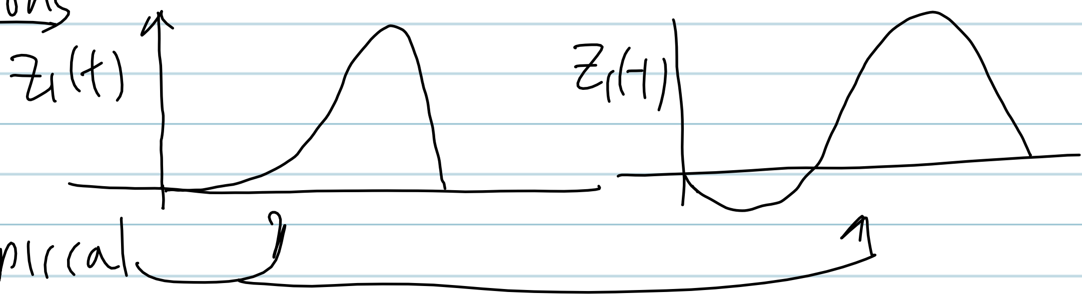
Find a scalar function $f(u) > 0$ such that

$$u = f(u) \qquad u(0) = 0$$

and $z(t) = \dfrac{k}{u(t)} \qquad u(T) = 1$

Phase resetting curves in biology

Neurons



In this case $z_1(t)$ is just the voltage component of the system

That is:

$$\begin{cases}\dot{x}_1 = f_1(x_1, .., x_m) \\ \vdots \\ \dot{x}_m = f_m(x_1, .., x_m)\end{cases}$$

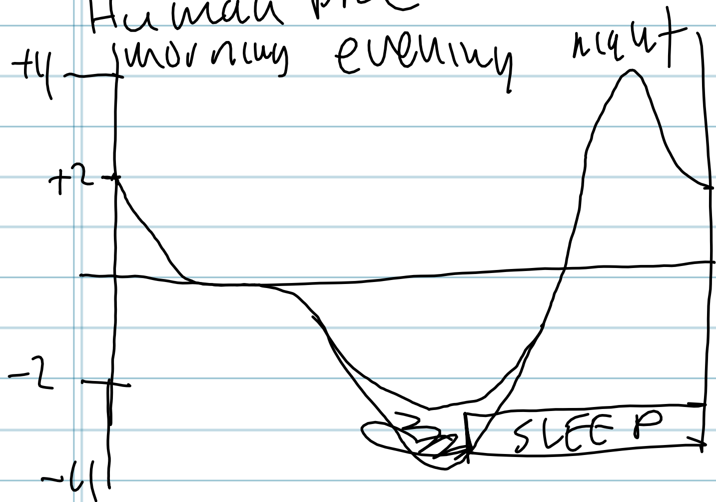
m-dim system

$(x_1, .., x_m) \equiv \vec{X}$

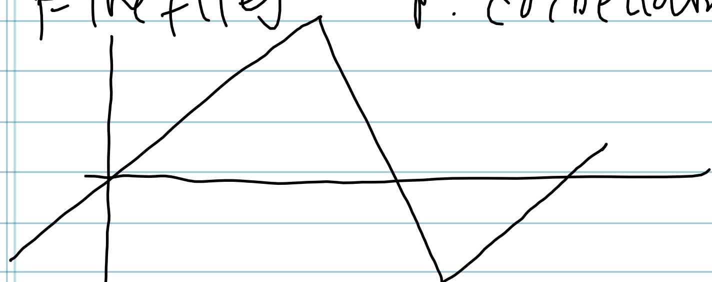
Increment x_1 by small amount at different times during its cycle + measure time of next crossing.

This will be the so-called phase resetting curve or PRC

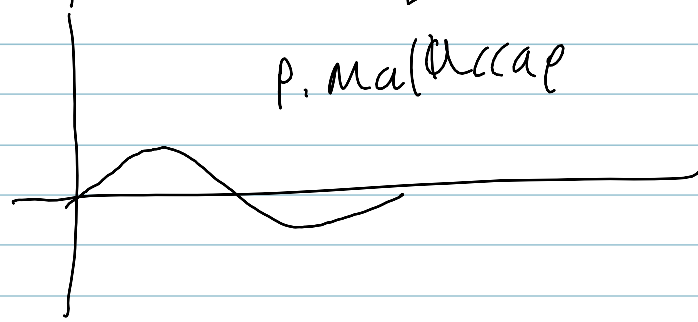
Human PRC



Fireflies P. coribellata



P. malaccae



Most other organisms have circadian PRCs

You can go Pubmed search phase resetting to see others.

Lets study stability of synchrony again.

But first lets remind ourselves of how to solve linear systems.

Let X, Y be a vector spaces

$L: X \to Y$

Recall that $\boxed{R_L = N_{L^*}^{\perp}}$

The range of L is identical to the orthogonal complement of the ~~adjoint of~~ nullspace of the adjoint.

Put another way

$Lu = f$ has a solution iff

$\langle v^*, f \rangle = 0$ for all v^* st $L^* v^* = 0$

In applied math, this is called the Fredholm alternative Theorem, but it is really just the fundamental Theorem of linear algebra.

Lets apply this to study eigenvalues of a perturbed matrix

$$(A_0 + \varepsilon A_1) u = \lambda u$$

$u = u_0 + \varepsilon u_1 + \cdots \quad \lambda = \lambda_0 + \varepsilon \lambda_1 + \cdots$

$$(A_0+\varepsilon A_1)(u_0+\varepsilon u_1) = (\lambda_0+\varepsilon\lambda_1)[u_0+\varepsilon u_1]$$

1. $\boxed{A_0 u_0 = \lambda_0 u_0}$ $\Rightarrow$ λ_0 is eigenvalue & u_0 is eigenvector

$$A_0 u_1 + A_1 u_0 = \lambda_0 u_1 + \lambda_1 u_0$$

$\Rightarrow$

$$(A_0 - \lambda_0) u_1 = \lambda_1 u_0 - A_1 u_0 \quad (\star)$$

NOT invertible! so nullspace of $A_0 - \lambda_0$ is nontrivial.

Let $(\bar{A}_0 - \bar{\lambda}_0)^T$ be adjoint & Let v^* be eigenvector for this. Then

$\exists$ solution u_1 to $\star$ iff

$$\langle v^*, \lambda_1 u_0 - A_1 u_0 \rangle = 0$$

so this defines λ_1 uniquely

$$\lambda_1 = \frac{\langle v^*, A_1 u_0\rangle}{\langle v^*, u_0\rangle} \quad \star$$

Let us use this to determine stability of synchrony for weak coupling

Recall that for general diffusive coupling we were left with

$$\frac{dW}{dt} = A(t)W + \gamma K W$$

Where γ is a scalar + K is coupling matrix. Suppose $K = \varepsilon \hat{K}$ and for simplicity suppose γ is real (although not necessary)

We want to look at periodic case where $A(t+T) = A(t)$.

Recall that for a stable limit cycle all floquet multipliers save 1 are in unit circle + there is one that is exactly equal to 1. This corresponds to $W_0 = \dot{U}_0(t)$ the derivative of limit cycle

$$LW := \left[\frac{d}{dt} - A(t)\right]W$$

$L\dot{U}_0(t) = 0$ $\quad L^* = -\frac{d}{dt} - A^T(t)$

has 1-dim nullspace, $Z(t)$

If $\varepsilon \ll 1$ then the ~~functions~~ solutions st

$\dot{W} = A(t) W$

decay to zero (the ones corresponding to floquet multipliers inside unit circle) will still perturb to solutions that decay since $\rho = \rho_0 + \varepsilon \rho_1 + O(\varepsilon^2)$ and if $|\rho_0| < 1$ then $|\rho(\varepsilon)| < 1$ for ε small enough by continuity

Thus the only "dangerous" one is $\rho_0 = 1$ corresponding to $W_0 = \dot{U}_0(t)$

So consider:

$$\frac{dW}{dt} = A(t) W + \varepsilon \gamma \hat{K} W$$

$$W = U_0(t) e^{\varepsilon \lambda_1 t} + \varepsilon W_1(t) e^{\varepsilon \lambda_1 t} + \cdots$$

Look for $W_1(t)$ periodic. Note that Floquet theory says solutions have form $p(t) e^{\lambda t}$

Since we are perturbing from a solution with $\rho_0 = 1 \Rightarrow \lambda_0 = 0$ we expect λ to be small. Since $\ln(\rho(\varepsilon))$ will also be small as $\rho(\varepsilon) \sim 1$

Lets plug this in:

$$\boxed{\frac{dW_1(t)}{dt} + \lambda_1 \dot{W}_0 = A(t) W_1 + \gamma \hat{k} u_0(t)}$$

$$\Rightarrow \left[\frac{d}{dt} - A(t)\right] W_1(t) = -\lambda_1 \dot{W}_0 + \gamma \hat{k} \dot{u}_0(t)$$

Has 1 dim nullspace so must be in orthogonal complement of N.S. of L^*

That $0 = \int_0^T Z(t) \cdot \left[-\lambda_1 \dot{u}_0(t) + \gamma \hat{k} \dot{u}_0(t)\right] dt$

Since $Z(t) \cdot \dot{u}_0(t) = 1$ by normalization:

$$\boxed{\lambda_1 = \frac{\gamma}{T} \int_0^T Z(t)\, \hat{k}\, \dot{u}_0(t)\, dt}$$

If $\hat{k} = kI$ then $\lambda_1 = \gamma k$!!

For weak coupling, synchronous stable if

$$\mathrm{Re}\left[\frac{\gamma}{T}\int_0^T Z(t)\cdot \hat{k}\dot{u}_0(t)\,dt\right] < 0$$

This is a "local" condition & easily computed.

Lets use the PRC to study periodic forcing of an oscillator.

$$\dot{X} = F(X) + \vec{V}\sum_{j=1}^{\infty}\delta(t - jT_f)$$

without forcing

$$\dot{\theta} = 1$$

so what is the dynamics with forcing

(A) ~~(X)~~

Perturbed Oscillators

$$\frac{dx}{dt} = F(x) + \varepsilon p(x,t) \qquad \varepsilon \ll 1 \text{ small}$$

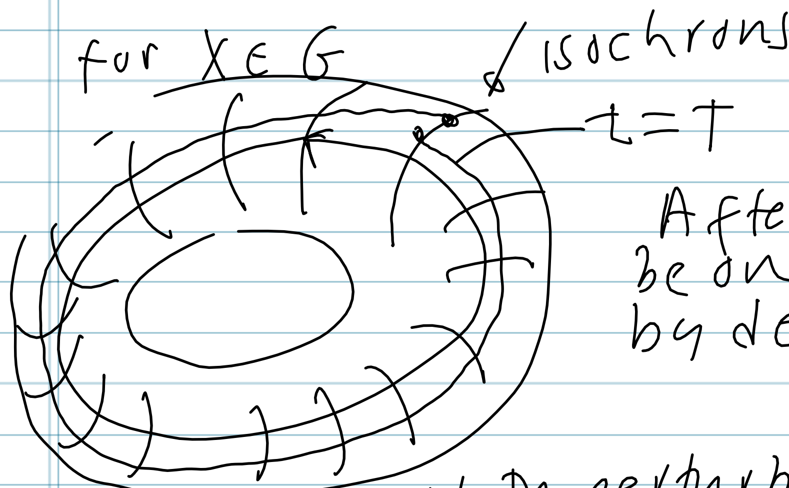
Let ϕ be the phase of the oscillator as defined by the isochrons

$\phi = \Theta(x)$ for x in an open neighborhood of the limit cycle Γ. Call G the neighborhood

With no coupling perturbation

$$\frac{d\Theta(x)}{dt} = \nabla\Theta(x) \cdot \frac{dx}{dt} = 1$$

for $x \in G$



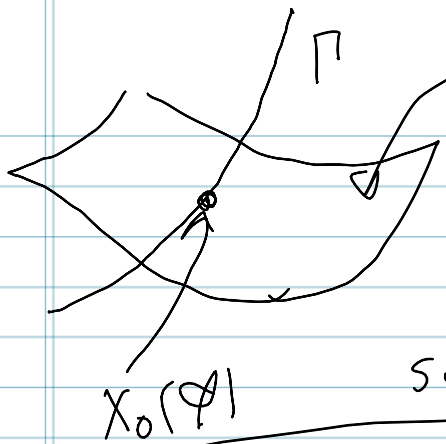
After one period must be on same isochron by definition

With perturbation:

$$\frac{d\phi}{dt} = \frac{d\Theta(x)}{dt} = \nabla\Theta \cdot [F(x) + \varepsilon p(x,t)]$$

$$= 1 + \varepsilon \nabla\Theta(x) \cdot p(x,t)$$

This is not a "closed" system in that the dynamics of ϕ still depends on the full dynamics of $\dot{x} = F + \varepsilon p$



If ε is small then
$X(\phi)$ is close to $X_0(\phi)$
so we replace $X(\phi)$ with $X_0(\phi)$

$$\boxed{\frac{d\phi}{dt} = 1 + \varepsilon\,\Omega(\phi,t)} \quad \star$$

Where $\Omega(\phi) = Z(\phi)\,P(\phi,t)$
with $Z(\phi) = \nabla\Theta(X_0(\phi))$

$$P(\phi,t) = p(X_0(\phi),t)$$

This is the moral equivalent of averaging

If $\varepsilon \ll 1$ and $\Omega(\phi,t)$ is T-periodic in t, then we can write

$$\phi = t + \psi$$

$$\frac{d\psi}{dt} = \varepsilon\,\Omega(t+\psi,t)$$

which is approximately the same as

$$\frac{d\bar{\psi}}{dt} = \varepsilon\,\bar{\Omega}(\bar{\psi})$$

$$\bar{\Omega}(\bar{\psi}) = \frac{1}{T}\int_0^T \Omega(t+\bar{\psi},t)\,dt$$

Example of stability

$PRC_x = -\sin t + q\cos t$

~~$u(t) = (\sin t, \cos t)$~~

~~$\dot{u}(t) = (\cos t, -\sin t)$~~

$u(t) = (\cos t, \sin t)$

$\dot{u}(t) = (-\sin(t), \cos t)$

$K = \begin{bmatrix} 1 & \beta \\ 0 & 0 \end{bmatrix}$

$\int_0^{2\pi} Z(t) \cdot K \dot{u} = \Big\{ -\sin t + q\cos t$

$\lambda \frac{1}{2\pi} \int_0^{2\pi} (-\sin t + q\cos t)\,[-\sin t + \beta\cos t]$

$= \lambda \left[\frac{1}{2} + \frac{\beta q}{2} \right] \Rightarrow$ if $\beta q < -1$ Then unstable.

Periodic Forcing using the PRC
(MAPS revisited)

Consider:

$$\frac{dX}{dt} = F(X) + \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \sum_{j=1}^{\infty} \delta(t - \tau_j)$$

$\dot{\varphi} = 1$ for $j\tau < t < (j+1)\tau$

Let φ be phase right before pulse hits

then right after $\varphi = \varphi + \Delta(\varphi) \leftarrow$ PRC

τ later it is $\varphi + \tau + \Delta(\varphi)$

So

$$\varphi_{New} = \varphi_{OLD} + \tau + \Delta(\varphi_{OLD}) \mod T$$

where T is period.

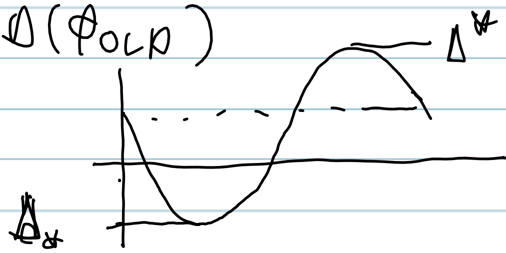
Assume that $F(\varphi) = \varphi + \Delta(\varphi)$ is invertible, that is $F'(\varphi) > 0$

$\Delta(\varphi + T) = \Delta(\varphi)$.

1:1 locking $\Rightarrow \varphi_{New} = \varphi_{OLD} + T$ (one cycle)

$\Rightarrow \varphi_{OLD} + T = \varphi_{OLD} + \tau + \Delta(\varphi_{OLD})$

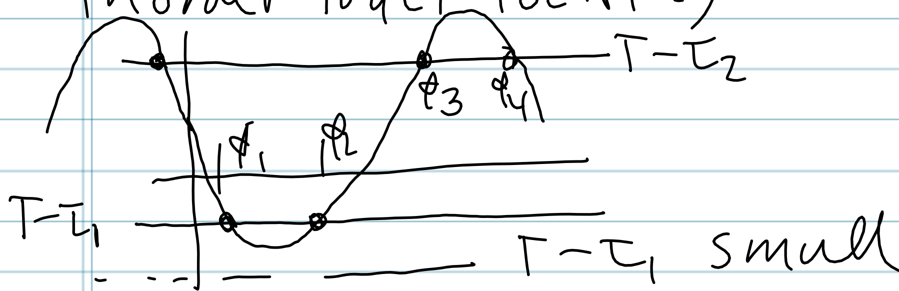
$\Rightarrow T - \tau = \Delta(\varphi_{OLD})$



$\Delta_* < T - \tau < \Delta^*$

$\Rightarrow \tau < T - \Delta_* \quad \tau > T - \Delta^*$

in order to get locking



stability:

$$y_{n+1} = y_n + \Delta'(\bar{\varphi})\, y_n = (1 + \Delta'(\bar{\varphi}))\, y_n$$

$$-1 < 1 + \Delta'(\bar{\varphi}) < 1$$

$$\Rightarrow -2 < \Delta'(\bar{\varphi}) < 0$$

so only φ_1, φ_4 can be stable

$T - \tau_1 < 0 \Rightarrow T < \tau_1$ means oscillator is faster than forcing so $\bar{\varphi} = \varphi_1$ is advanced

$T - \tau_2 > 0 \Rightarrow T > \tau_2 \Rightarrow$ oscillator is slower $\Rightarrow \bar{\varphi} = \varphi_4$ is delayed

is $\Delta' > -2$? since $F' = 1 + \Delta'(\varphi) > 0$

$\Rightarrow \Delta' > -1$ so No problem!

~~Coupling with pulses~~

~~Each time~~ oscill

1:2 + 2:1 forcing:

Suppose that ~~Φ~~ ~~Φ~~ $T-\tau$ is too small for locking eg $T=1, \tau=2$ Δ = small

perhaps oscillator will fire twice ~~to~~ before stimulus comes

$\varphi_{New} - \varphi_{OLD} = 2T$ 2:1 locking

$\Rightarrow 2T = \tau + \Delta(\varphi_{OLD})$ 2 oscillator to 1 stimulus

~~Can~~ is $2T - \tau \in (\Delta_*, \Delta^*)$

Same idea as 1:1

However 1:2 is more complicated since oscillator receives two kicks.

~~$\varphi_1 = \varphi_{New} = \varphi_{OLD} + \tau + \Delta(\varphi_{OLD}) = \tau + F(\varphi_{OLD})$~~

~~$\varphi_2 = \varphi_1 + \tau + \Delta(\varphi_1) = \tau + F(\varphi_0)$~~

φ_2 ~~$= \quad = 2\tau + \Delta(\varphi_0) + \Delta(\varphi_0 + \tau + \Delta(\varphi_0))$~~

~~Let~~ $\varphi_1 = \varphi_0 + \tau + \Delta(\varphi_0)$ phase after 1 cycle

$$\varphi_2 = \varphi_1 + \tau + \Delta(\varphi_1) = \varphi_0 + 2\tau + \Delta(\varphi_0) + \Delta(\varphi_0 + \tau + \Delta(\varphi_0))$$

$\varphi_2 = \varphi_0 + T \Rightarrow$

$$T = 2\tau + \Delta(\varphi_0) + \Delta(\varphi_0 + \tau + \Delta(\varphi_0))$$

$|T-2\tau|$ smaller than $|T-\tau|$

Stability:

$$\phi_{n+1} = 2\tau + \Delta(\phi_n) + \Delta(\phi_n + \tau + \Delta(\phi_n))$$

$$y_{n+1} = \underbrace{\left[\Delta'(\bar{\phi}) + \Delta'(\bar{\phi} + \tau + \Delta(\bar{\phi}))\left[1 + \Delta'(\bar{\phi})\right]\right]}_{\circledast} y_n$$

In general

Let $f(\phi, \tau) = \phi + \tau + \Delta(\phi)$

Then for 1:2 locking need

$$\boxed{\phi + T = f(f(\phi,\tau),\tau)} \quad \bigstar$$

<u>HW</u> Let $\Delta(\phi) = -0.5\sin\phi$, $T = 2\pi$

Find the range of values of τ such that

($\bigstar$) has a solution

Hint: you can plot

$$f(f(\phi,\tau),\tau) - \phi - T$$

& find where this is tangent to the ϕ-axis

Suppose you wanted to find, say 3:2 locking. That is oscillator goes 3 times when stimulus goes twice:

$$\boxed{\phi + 3T = f(f(\phi,\tau),\tau)}$$

$$\boxed{\phi + nT = (f \circ \cdots \circ f)^m(\phi)} \qquad n:m$$

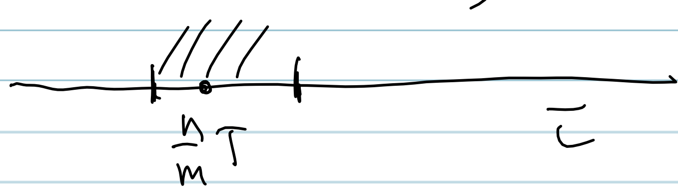
Note that if $\Delta(\phi) = 0$ then this is the same as

$$\phi + nT = \phi + m\tau$$

so $\tau = \frac{n}{m} T$

Hence, we expect that if $\tau \sim \frac{n}{m}T$ and $\Delta(\phi)$ is small, then there should be a range $|\tau - \frac{n}{m}T|$ small st there is locking

n:m Locking

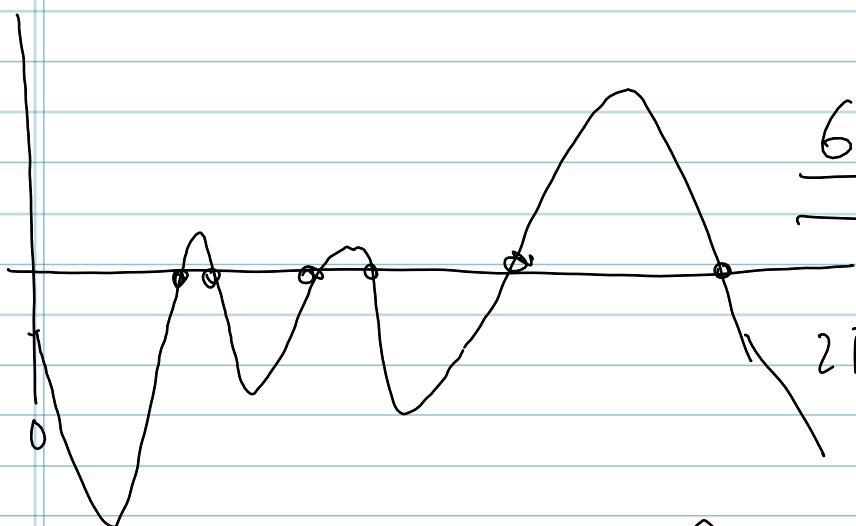


Example $f(x,\tau)$ $\Delta(X) = -.75 \sin X$

2:3 locking $\quad f(f(f(\phi,\tau),\tau),\tau) - 4\pi - \phi$

$\tau = 4.1$

6 roots (2m)



1:3 locking $\quad \tau = 2.2$

$2.124 < \tau < 2.21$

very narrow window

1:1 $\quad \phi + 2\pi = \tau - .75 \sin\phi + \phi$

$|2\pi - \tau| < .75$ wide window!

Consider the unwrapped map

$$\varphi_{n+1} = F(\varphi_n)$$

on the circle (without the mod T)

define $\rho = \lim_{n \to \infty} \frac{1}{T} \frac{\varphi_n}{n}$

ρ is called the rotation number

<u>Theorem</u> (Denjoy) The rotation number is well-defined, that is, the limit exists and is independent of the initial condition. If f is C^2 in φ then:

1. ρ is rational if and only if $F(\varphi, \tau)$ has a periodic orbit of some period ~~that~~ $\varphi_{n+m} = \varphi_n \bmod T$

2. ρ is irrational iff $\{\varphi_n\}$ densely fills the circle

3. ρ is a cts function of any parameters

This leads to the so called devil's staircase