Synchronization -

Spring 2011

While most examples will come from biology, I will start out with some classic ~~nonbiological~~ nonbiological Examples

What is synchronization?

Don't use "synchronicity" - This is a term from psychology & a POLICE Album

Mathematically it is usually associated with repetitive or periodic phenomena & it means that two or more systems are working together in a locked or coherent state. (we will make this more precise later on)

In real world & practical examples, it is a bit harder to determine & to ~~consider~~ establish. In particular, in neuroscience many many pages of papers have been written on synchrony of rhythms in the brain.

I will try to include some of the statistical notions of synchrony later in the class such as coherence, spectral density, covariance & correlation.

Some books worth reading or looking at

"Synchronization: A universal concept in nonlinear science"
A. Pikovsky, M. Rosenblum, J. Kurths
CAMBRIDGE 2001

"Geometry of Biological Time"
A.T. Winfree, Springer 2001 (2^{nd} EDN)

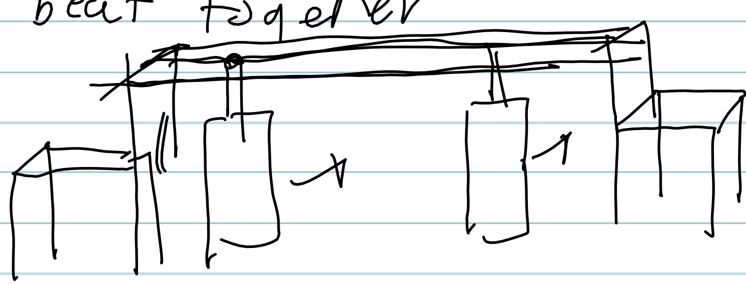
"Sync": Steve Strogatz (popular nonfiction)
Hyperion 2004

"Chemical Oscillations, Waves & Turbulence"
Y. Kuramoto Dover $11.50 on AMAZON!
(2003)

More later .....

(History taken from Pikovsky)
First example –
Christiaan Huygens (1629–1695)

noticed when two pendulums were on same wall, they would synchronize & beat together



Great example on YouTube (link is on web site) & search: synchronize metronom

Jean-Jacques Dortous de Meiran 1729
found evidence for oscillations in
beanleaves (first evidence for circadian
oscillation)

My favorite:
Engelbert Kaempfer (1727) voyaged
to SIAM (Thailand) in 1680
wrote

"The glow worms... represent another shew...
sometimes hide Their Light all at once,
and a moment after make it
appear again with The utmost regularity
+ exactness..."

Synchrony of many fireflies in S.E Asia.
(Link to video)

A very simple example

Earn, Levin, Rohani Science 290 : 1360 – 1365 2000

Consider a simple population model, discrete in time

$x^{t+1} = F(x^t)$

Logistic equation

where $x \in \mathbb{R}$, eg. $x^{t+1} = a x^t (1 - x^t)$

Suppose we consider N such populations, Labeled : X_i

That are distributed in N patches

At each time step t, $F(X_i^t)$ are born on patch i (Assume all patches are the same)

Let M_{ij} = proportion of ~~species~~ patches j

dispersing to patch i so That

$\sum_{i=1}^{n} M_{ij} = 1$ ← # of j dispersing everywhere is 1

$M_{ij} \geq 0$. We call M a probability

matrix since it is also descriptive of changes of states in a random M-state process.

we thus have!

$$x_i^{t+1} = \sum_{j=1}^{N} m_{ij} F(x_j^t) \quad (1)$$

This says # coming into patch i at $t+1$ is total born on any patch j that migrate to patch i.

To make life easy, we will assume symmetric dispersal $m_{ij} = m_{ji}$

Thus $\sum_{j=1}^{N} m_{ij} = 1$ also! (2)

A completely coherent or synchronous state, satisfies $x_i^t = x_j^t$ for all i, j, t

observation 1 The synchronous state exists.

Let $x_i^t = u^t$ where $u^{t+1} = F(u^t)$

Claim $x_i^t = u^t$ solves (1)

Proof $u^{t+1} = \sum_{j=1}^{N} m_{ij} F(u^t) = \left(\sum_{j=1}^{N} m_{ij}\right) F(u^t)$

$= F(u^t)$

by equation (2).

Is $x_i^t = u^t$ stable?

What do we mean by stable? If we start with $x_i^t = u^t + y_i^t$ where y_i^t is small then will $x_i^t \to u^t$ as $t \to \infty$?

Write ~~F(u)~~

$$F(u+y) = F(u) + F'(u)y + \frac{F''(\xi)}{2} y^2 \quad (3)$$

by Taylors remainder theorem.

Then write $x_j^t = u^t + y_j^t$

$$x_i^{t+1} = u^{t+1} + y_i^{t+1} = \sum_{j=1}^{N} m_{ij} F(u^t + y_j^t)$$

$$= \sum_{j=1}^{N} m_{ij} \left[F(u^t) + F'(u^t) y_j^t + \frac{F''(\xi_j^t)}{2} y_j^{t\,2} \right]$$

If y_j^t ~ small then $(y_j^t)^2$ is smaller so we ignore it to get

$$u^{t+1} + y_i^{t+1} \simeq \sum_{j=1}^{N} m_{ij} F(u^t) + \sum_{j=1}^{N} m_{ij} F'(u^t) y_j$$

$$\simeq F(u^t) + F'(u^t) \sum_{j=1}^{N} m_{ij} y_j^t$$

$$\Rightarrow y_j^{t+1} \simeq F'(u^t) \sum_{j=1}^{N} m_{ij} y_j^t \quad (4)$$

<u>Aside</u> Stability + Lyapunov exponents for maps. (scalar)

consider:

$$y^{t+1} = a^{t+1} y^t \qquad a^t \in \mathbb{R}$$

what is the dynamics of this as $t \to \infty$?

~~yt~~ $y^1 = a^1 y^0, \quad y^2 = a^2 a^1 y^0, \ldots, \quad y^t = a^t a^{t-1} \cdots a^1 y^0$

$$|y^t| = \prod_{k=1}^{t} |a^k| \, |y^0|$$

This will grow or decay to zero according as to whether or not $\prod_{k=1}^{t} |a^k|$ stays above or below 1 in magnitude

We write
$$\prod_{k=1}^{t} |a^k| \equiv \beta_t^t$$

$$\Rightarrow \beta_t = \left(\prod_{k=1}^{t} |a^k| \right)^{1/t}$$ is average growth per iteration.

$$\lambda_t \equiv \ln(\beta_t) = \frac{1}{t} \sum_{k=1}^{t} \ln|a^k|$$

$\lim_{t \to \infty} \lambda_t \equiv \lambda$ is called the Lyapunov exponent.

If $\lambda < 0$ then $\lambda_t < 0$ & $\beta_t = e^{\lambda_t} < 1$

so that $\beta_t^t \to 0$ as $t \to \infty \Rightarrow y^t \to 0$

Example:

~~u_{t+1}~~ $u^{t+1} = a u^t (1 - u^t)$

consider $a = 3.1$

$u^t \to .558, .7645, \dots$

$|F'(u^t)| \to (.3596, 1.64), \dots$

$[(.3596)(1.64)]^{1/2} \simeq .7679$ $\lambda = \ln(.7679) = -.25$

$a = 3.5$

$u^t \to .5, .875, .383, .826, .5, \dots$

$|F'(u^t) F'(u^{t+1}) F'(u^{t+2}) F'(u^{t+3})|^{1/4} = 0.416$

$\lambda = -.8375 < 0$

$a = 3.9$

u^t doesn't repeat

$\lambda = .492 > 0$ so unstable - in

This case, chaotic!

(This means if we start near some value of u^t, we diverge fast at a rate $e^{.492}$ per iterate!)

Lets consider 2 patches

$x_1^{t+1} = (1-m) F(x_1^t) + m F(\cancel{y}x_2^t)$

$\cancel{y}x_2^{t+1} = (1-m) F(\cancel{y}x_2^t) + m F(x_1^t)$

$m = 0$ No mixing $m = \frac{1}{2}$ 50%

$x_1^t = x_2^t = u^t$ is synchronous solution

$x_1^t = u^t + y_1^t \quad x_2^t = u^t + y_2^t$

As above:

$y_1^{t+1} = (1-m) F'(u^t) y_1^t + m F'(u^t) y_2^t$

$= F'(u^t) [(1-m) y_1^t + m y_2^t]$

$y_2^{t+1} = F'(u^t) [(1-m) y_2^t + m y_1^t]$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^{t+1} = F'(u^t) \begin{bmatrix} (1-m) & m \\ m & 1-m \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^t$$

in matrix form. Let $z^t = \frac{y_1^t + y_2^t}{2}$

$w^t = \frac{y_1^t - y_2^t}{2}$

$y_2^{t+1} + y_1^{t+1} = F'(u^t) [1-m+m] [y_2^t + y_1^t]$

$= F'(u^t) [y_2^t + y_1^t]$

$\Rightarrow \boxed{z^{t+1} = F'(u^t) z^t} \quad (5)$

$$w^{t+1} = \frac{y_1^{t+1} - y_2^{t+1}}{2} = F'(u^t)[1-2m]\frac{y_1^t - y_2^t}{2}$$
$$= F'(u^t)(1-2m)\, w^t$$

So, instead of solving 1 2X2 system, we solve 2 1X1 system.

z^t system is the same as a single patch + has same stability, λ

w^t system

$$\left(\prod_{t=1}^{T} |F'(u^t)||1-2m|\right)^{1/T} = |1-2m|\left(\prod_{t=1}^{T}|F'(u^t)|\right)^{1/T}$$

so $\beta_w = |1-2m|\,\beta$ (6)

$\Rightarrow \lambda_w = \lambda + \ln(|1-2m|)$ ☆

as $m \to \frac{1}{2}$ $\lambda_w \to -\infty$ so if $m = \frac{1}{2}$ then $w^t \to 0$ really Fast

$\Rightarrow y_1^t - y_2^t \to 0 \Rightarrow y_1^t \to y_2^t$

$\Rightarrow x_1^t \to x_2^t$

$\Rightarrow$ synchrony!

since $\ln|1-2m| < 0$ for $0 < m < 1$

$\Rightarrow$ if $\lambda < 0 \Rightarrow \lambda_w < 0 \Rightarrow$ sync

~~What~~ What if $\lambda > 0$? Then we see that we must choose m so that

$$\lambda_w < 0 \Rightarrow (1-2m)e^{\lambda} < 1$$

$$\Rightarrow 1-2m < e^{-\lambda}$$

$$\Rightarrow m > \frac{1-e^{-\lambda}}{2}$$

and of course $m < \frac{1}{2}$ since we want migration into other patches less than staying.

This says that for coupling ~~close~~ large enough, we can stabilize synchrony!

We can learn much from this example.

Suppose we consider the original problem

$$x_i^{t+1} = \sum_{j=1}^{N} m_{ij} F(x_j^t)$$

is synchrony stable?

For our simple example, we had the matrix

$$M = (m_{ij}) = \begin{pmatrix} 1-m & m \\ m & 1-m \end{pmatrix}$$

"1" is an eigenvalue of M since $M\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$

but so is $1-2m$ $\quad M\begin{pmatrix}1\\-1\end{pmatrix} = 1-2m\begin{pmatrix}1\\-1\end{pmatrix}$

Thus, we studied

$$z^{t+1} = 1 \cdot F'(u^t)\, z^t$$

and

$$w^{t+1} = (1-2m)\, F'(u^t)\, w^t$$

Consider now the stability problem.

$$y_i^{t+1} = F'(u^t) \sum_{j=1}^{N} M_{ij}\, y_j^t$$

in vector form

$$\vec{y}^{\,t+1} = F'(u^t)\, M\, \vec{y}^{\,t} \quad (6)$$

Let $(\vec{\xi}, \nu)$ be eigenvector-eigenvalue pair for M; that is $M\vec{\xi} = \nu\vec{\xi}$

since M is symmetric, ν is real

Let us write

$$\vec{y}^{\,t} = w_1^t \vec{\xi}_1 + w_2^t \vec{\xi}_2 + \cdots + w_N^t \vec{\xi}_N$$

Where $\vec{\xi}_1, \vec{\xi}_2, \ldots, \vec{\xi}_N$ are the N eigenvectors associated with M. Since M is symmetric, these form a basis for $\mathbb{R}^N$. Furthermore, the basis is orthonormal

$$\vec{\xi}_i \cdot \vec{\xi}_j = \delta_{i,j}$$

plug this into (6) to see

$$W_1^{t+1}\vec{\xi}_1 + \cdots + W_N^{t+1}\vec{\xi}_N = F'(u^t)\, M\left[W_1^t\vec{\xi}_1 + \cdots + W_N^t\vec{\xi}_N\right]$$
$$= F'(u^t)\left[\nu_1\vec{\xi}_1 W_1^t + \cdots + \nu_N\vec{\xi}_N W_N^t\right]$$

where $\nu_1, \ldots, \nu_N$ are the eigenvalues corresponding to $\vec{\xi}_1, \ldots, \vec{\xi}_N$

Since these are orthogonal, we get

$$W_1^{t+1} = F'(u^t)\,\nu_1\, W_1^t$$
$$\vdots$$
$$W_N^{t+1} = F'(u^t)\,\nu_N\, W_N^t$$

So, again, we must solve N 1×1 systems instead of 1 $N\times N$. We have diagonalized the system.

ν_j depend on M. Since $M\cdot\mathbb{1} = \mathbb{1}$, we know that $\nu_1 = 1$ is always an eigenvalue. The other eigenvalues correspond to perturbations orthogonal to $\mathbb{1}$

We have proven the following Theorem:

Theorem Let $\lambda = \lim_{T\to\infty} \frac{1}{T}\sum_{t=1}^{T} \log|F'(u^t)|$

Suppose $\lambda + \ln(|\nu_i|) < 0$ for $i = 2, .., N$

Then synchrony is stable

If $\lambda + \ln(\nu_i) > 0$ for some i, Then synchrony is unstable

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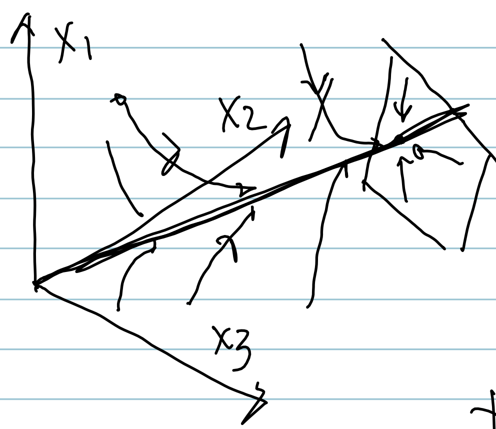
Geometric interpretation



$(x_1 = x_2 = x_3)$

in the plane orthogonal to $x_1 = x_2 = x_3$

all solutions decay to $x_1 = x_2 = x_3$

Return to our matrix $M = (M_{ij})$

$M_{ij} \geq 0 \quad \sum_{j=1}^{N} M_{ij} = 1 \quad (\text{since } M_{ij} = M_{ji})$

The Perron Theorem says the following

Assume No patch is isolated

Then $\nu_1 = 1$ is the maximal eigenvalue & all other eigenvalues $|\nu_i| < 1$

So we are very good for this since $\ln|\nu_i| < 0$ for $i = 2, .., N$ & thus we have here that $\lambda + \ln|\nu_i| < 0$ even if $\lambda > 0$.