Toys! Toys! Toys!

The Dynamics of Some Curious Toys

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Introduction

- Many mechanical, magnetic, toys and carnival rides
- Toys often illustrate fundamental physical principles.
- Easy to produce the models – diff eqs
- Can produce very complex dynamics.
- Accessible to undergrads with just a little math

Follow the bouncing ball. I

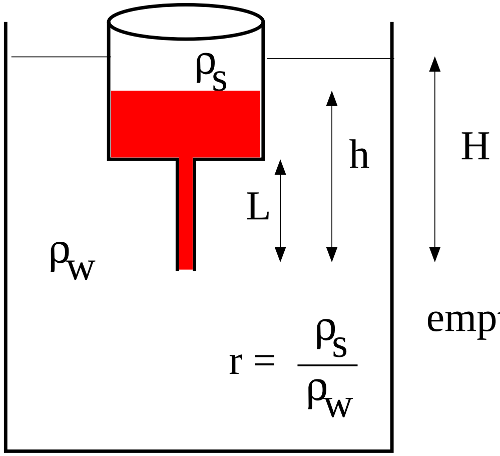
- A ball bouncing on a hard surface suggests a simple model
- Let $-v_n$ be the downward velocity of the n^{th} bounce
- Assume that it hits the surface and reverses course with damping
- The new velocity is $rv_n \ldots$
- $\ldots$ and when it returns, it is $-rv_n$ so we get

$$v_{n+1} = rv_n$$

Example: $v_0 = 5, r = .8$,

$$v_n = 5(0.8)^n$$

Sugar-water oscillator



filled: $\rho_s gh = \rho_w gH$

$$h_{eq} = r H$$

empty: $\rho_s g(h - L) = \rho_w g(H - L)$

$$h_{eq} = r H - (1-r)L$$

How does it work?

- Archimedes principle
- When tube filled with sugar – goes to first equilibrium
- When tube filled with water – goes to second equilibrium
- Both equilibria unstable – Rayleigh instability
- Thus, it oscillates

Equations

$$\frac{dh}{dt} = \mu(h_{eq} - h)$$

- Flowing down:
  - μ is resistance of sugar water
  - h_{eq} is filled
- Flowing up:
  - μ is resistance of water
  - h_{eq} is empty
- Switch when rate, dh/dt becomes small in magnitude
- Simulation

Extensions

- More than one cup?
- Assumption that H is fixed
- Actually changes inversely with h
- Leads to **Coupled SWO** – oscillate out of phase

The Drinking Bird

- A favorite since 1946!
- Another oscillator

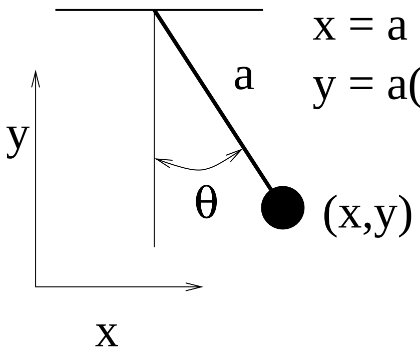
Questions about the bird

1. What causes the fluid to rise up the toy bird's neck?
2. What causes the bird to dip and then erect himself
3. What is the purpose of the fuzzy head of the bird? the water?
4. How do temperature & humidity affect operation?
5. How can he fail?

Explaining the bird

- Evaporative cooling lowers pressure
- Fluid rises to head
- Head drops – wetting head again
- Pressure seal broken, fluid returns to base

Euler-Lagrange Equations



$x = a \sin\theta$

$y = a(1 - \cos\theta)$

K.E. $\frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2} a^2 \dot{\theta}^2$

P.E $mgy = mg\, a(1 - \cos\theta)$

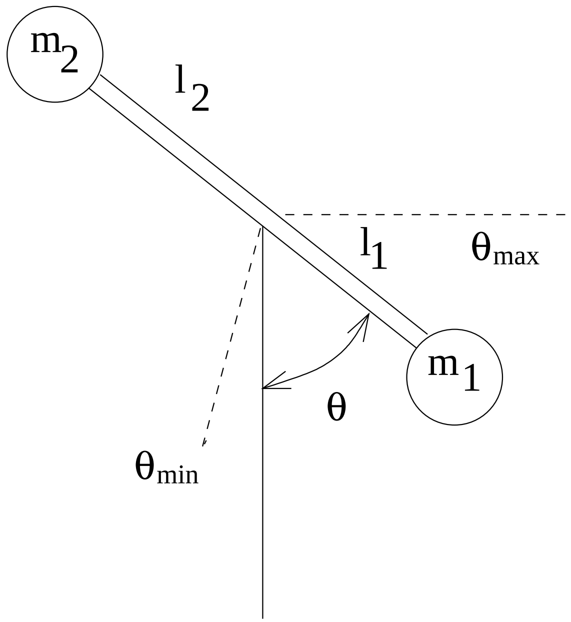
$L = \text{K.E.} - \text{P.E.}$

$$\frac{d}{dt}\frac{\partial L}{\partial v_j} = \frac{\partial L}{\partial x_j}$$

For the pendulum, $v = \dot{\theta}$, $x = \theta$, so

$$ma^2 \frac{d^2}{d\theta^2} = -mga \sin\theta$$

A little model



- P.E.: $(m_2 g l_2 - m_1 g l_1)\cos\theta$
- K.E.: $\frac{1}{2}(m_1^2 l_1^2 + m_2^2 l_2^2)\dot{\theta}^2$
- Evaporative cooling: $\dot{m_2} = r|\dot{\theta}|$; Upper mass rises proportional to swinging speed; $M = m_1 + m_2$.
- Dip: When θ hits a critical value, m_2 reset and velocity set to zero
- Simulation

Chaos & beyond

- Many toys and carnival rides are in fact chaotic
- Some are intrinsically chaotic, but
- Most depend on being periodically driven
- Others are “quasi-periodically” driven

Follow the bouncing ball. II

- What if the ball is bouncing on an oscillating platform?
- Simulation – Try $\omega = 2, 3$; look at sensitive dependence!
- Can study a simplifi ed model on a pocket calculator!

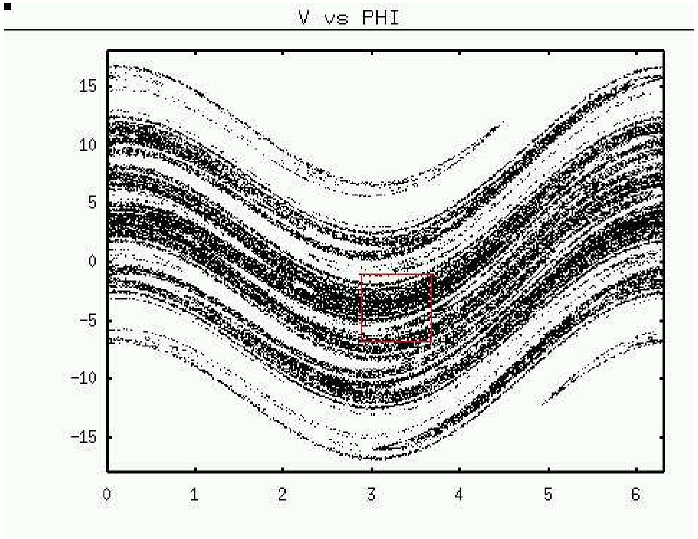
Reduction to a map

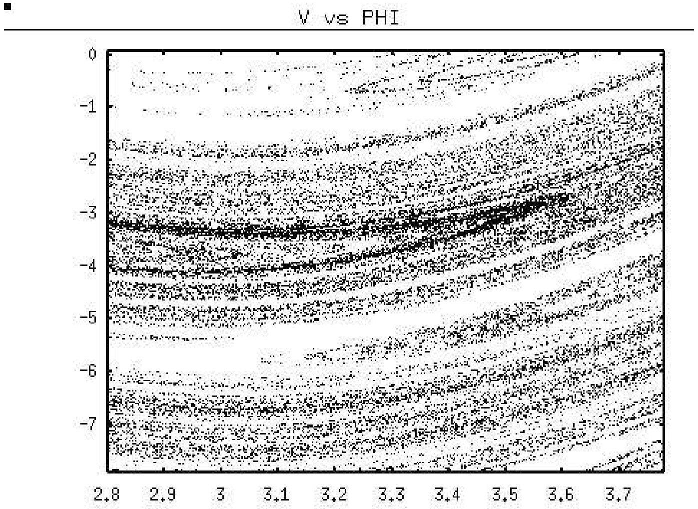
- New velocity is old velocity damped and augmented by the platform: $a \cos f t_n$
- t_n is the time it hits the platform which is prior time plus time in the air, $T = 2v/g$
- Thus we get an iteration:

$$\begin{aligned} t_{n+1} &= t_n + 2v_n/g \\ v_{n+1} &= r v_n + a\cos(f t_{n+1}) = r v_n + a \cos[f(t_n + 2v_n/g)] \end{aligned}$$

- Simulation

Fractal attractors



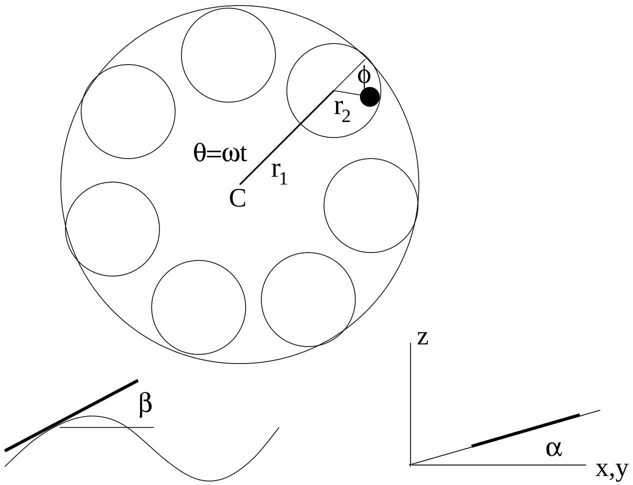


Successive snapshots are “self-similar” another hallmark of chaos

The Tilt-a-whirl

It is the random tipping and spinning of the Tilt-A-Whirl's seven cars that make this ride so much fun. The Tilt-A-Whirl is a platform type ride that is based on a circular track of bridge type construction. The seven Tilt-A-Whirl cars are fixed to a pivot pin on each platform. The revolving platforms roll over hills and valleys causing centrifugal and gravitational forces upon the cars causing them to tip and spin randomly.

A model tilt-a-whirl

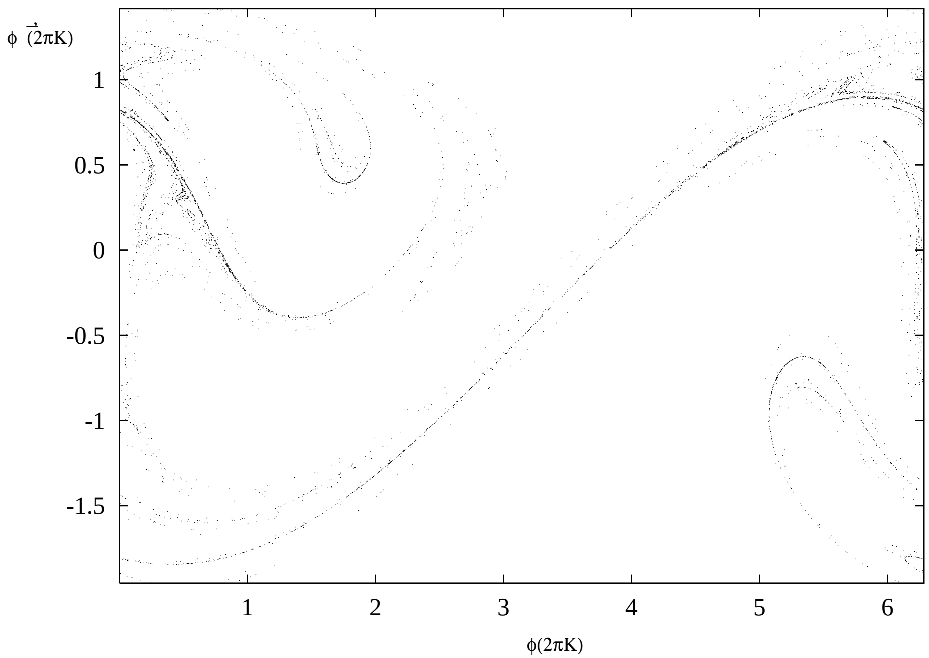


- P.E.: mgz
- K.E.: $\frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$
- Angles α, β are small so

$$
\begin{aligned}
mr_2^2 \frac{d^2\phi}{dt^2} &= -mr_1 r_2 \omega^2 \sin\phi - \rho \frac{d\phi}{dt} \\
&+ mgr_2(\alpha \sin\phi - \beta\cos\phi) \\
\alpha &= \alpha_0 - \alpha_1 \cos 3\theta \\
\beta &= 3\alpha_1 \sin 3\theta
\end{aligned}
$$

Simulation & Explanation

- Simulation
- This, like the ball problem, is a periodically forced, damped oscillator
- At each turn of the big wheel, we plot the angular velocity and angle: Poincare Map



Other toys

- Dolphins! – Like the bouncing ball & Tilt-a-Whirl
- Magnetron – Autonomous transient chaos
- Zipper – Quasiperiodically forced damped pendulum

Conclusions

- Lots of simple toys have very interesting dynamics.
- Try: pendula driven to sit upside down, multiple pendula, lava lamps, and so on ...
- Most of all, have fun!