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Existence and uniqueness of travelling waves for a neural network

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Synopsis

A one-dimensional scalar neural network with two stable steady states is analysed. It is shown that there exists a unique monotone travelling wave front which joins the two stable states. Some additional properties of the wave such as the direction of its velocity are discussed.

1. Introduction

There has been a great deal of recent interest in the behaviour and analysis of nonlinear neural networks. These networks are believed to mimic the behaviour of real masses of neurons. Most of the work on these systems concerns discrete space and time models which are used to devise learning and computational rules [1], and the mathematical analysis of the discrete systems has been limited to steady-state in behaviour of symmetric systems. In this paper, we are interested in the asymptotic behaviour of a network which is continuously distributed in space and varies continuously in time. Previous results on networks of this type have dealt with bifurcation from trivial equilibria [2] and consequently with small amplitude solutions. We analyse a fully nonlinear system far from any small amplitude branches of solutions. Our goal is to understand the propagation of excitation through an autonomous isolated piece of tissue. We assume that the network can operate stably at a high rate of activity and a low rate. We then study the transition between these two states as a travelling wave.

There are several motivations for analysing such a network. The model is a good first approximation for the spread of excitation through a more general network where the inhibition is slow (see [3] for a discussion of the relationship between a scalar reaction-diffusion equation and the full model for a spiral wave propagation). There is good evidence that the scotomas associated with migraine headaches travel across the cortex at a uniform velocity. The scotoma consists of a radially progressing wave which leaves in its wake a strongly inhibited region that persists for many minutes after the wave has passed [4]. The onset of Jacksonian seizures is marked by a spread of rhythmicity across regions of the motor cortex: the main mechanism underlying the spread is excitatory synaptic interactions within the cortex. Recently, Smith and Bullock [5] proposed a model for the spread of activity across the skin of a sea urchin. This model consists of a two-dimensional array of neurons with excitatory connections and two stable

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modes of activity–high and low. Our model can be regarded as a one-dimensional analogue of this phenomenon.

In Section 2, we derive the model equations and set up the problem as an integrodifferential equation. In Sections 3 and 4, the main existence and uniqueness results are proved. In the last section, we consider an exactly solvable model and also provide some numerical simulations of the travelling wave.

2. Derivation of the model

We consider a single-layer neural network distributed over the real line. Let $u(x, t)$ denote the mean membrane potential of a patch of tissue at position x and time t. We assume the connections between neurons are excitatory and that the strengths of connections between a neuron at x and one at x' fall off with distance at a rate given by $k(|x-x'|)$. The output of a neuron is a spike train, the firing rate, $F(x, t)$, of which is a nonlinear function of the mean membrane potential: $F(x, t)=S(u(x, t))$. A single spike arriving at a neuron at time t' results at time $t>t'$ in a small change in the membrane potential of the receiving neuron, the post-synaptic potential (PSP) $h(t-t')$. All inputs to a given cell from the other cells and prior times are integrated to yield the total membrane potential:

$$u(x, t)=\int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} dx' h(t-t') k(|x-x'|) S(u(x', t')). \tag{2.1}$$

This model assumes that there are no inputs to a cell other than the recurrent excitatory connections. If we consider the particular case where the postsynaptic potential h has an instantaneous rise-time and an exponential fall-off, then $h(t)=e^{-t}(t>0)$, and differentiation of (2.1) yields

$$\partial u(x, t)/\partial t+u(x, t)=\int_{-\infty}^{\infty} dx' k(|x-x'|) S(u(x', t')), \tag{2.2}$$

but we shall not restrict attention to this case.

We are interested in travelling wave solutions of (2.1), i.e. solutions of the form $u(x, t)\equiv u(x-ct)$, and we now set out the nature of the solutions in which we are interested and the conditions on the given functions h, k, S. We want there to exist two stable rest states, which we take to be $u\equiv 0$ and $u\equiv 1$, and we are interested in travelling waves which connect these states monotonically. Hence we are interested in solutions $u(z)$, $z=x-ct$, where

$$u \text{ is monotonic,}$$
$$0\leqq u\leqq 1.$$
$$u(-\infty)=0, \quad u(\infty)=1.$$

The function S, defined on $[0, 1]$, is such that

(a) S is continuously differentiable with $S'>0$,

(b) $f(u)\equiv -u+S(u)$ has precisely three zeros, at $u=0$, a, 1, with $0<a<1$, and

(c) $S'(0)<1$, $S'(1)<1$.

The assumption (a), essentially that S is an increasing function of u, is natural. The assumption (b), along with the conditions on h and k, allows $u \equiv 0$ and $u \equiv 1$ (and also $u \equiv a$) to be solutions of (2.1). The assumption (c) corresponds to the setting in which the states 0 and 1 are stable. (If, for example, we take (2.2) with k the Dirac δ-function, we see that for a solution u that is small, we have, at least formally,

$$\frac{\partial u}{\partial t} + u = S'(0)u,$$

so that stability corresponds to $S'(0) < 1$.)

The assumption that there is just one other rest state (unstable) between 0 and 1 is a convenient rather than a crucial assumption. The analysis can still be made to work without it, but the travelling wave connecting 0 and 1 may be replaced by a "stack" of travelling waves connecting 0 to a_1, a_1 to $a_2, \ldots, a_n$ to 1, where $a_1, a_2, \ldots, a_n$ are stable rest states with $0 < a_i < 1$. This is comparable to the result in **[6, 7]** for a single reaction-diffusion equation, and we intend to consider such generalisations, and also the question of stability of the travelling waves, in a later paper.

The function k is defined on $(-\infty, \infty)$, such that

(a) k is absolutely continuous, with $k' \in L^1(-\infty, \infty)$.

(b) k is even,

(c) $k \geqq 0$,

(d) $\int_{-\infty}^{\infty} k(t)\, dt = 1$.

The assumption (a) is merely technical, and the assumptions (b) and (c) are natural. The assumption (d) is an expression of the fact that $k(t)$ dies away as $|t| \to \infty$. (Since it is only the product hkS which appears in (2.1), we have some choice in the way in which we separately normalise h, k, S.)

The function h, defined on $[0, \infty)$, is such that

(a) $h \geqq 0$,

(b) h is monotonic decreasing,

(c) $\int_0^{\infty} h(t)\, dt = 1$.

(d) $\int_0^{\infty} th(t)\, dt < \infty$.

Our object is to prove the following theorem:

THEOREM 2.1. *Under the above conditions on h, k, S, there exists one and (modulo translation) only one monotonic travelling wave solution to* (2.1), *with* $u(-\infty) = 0$, $u(\infty) = 1$.

(This is proved as Theorem 4.5 in Section 4.)

The method of proof is to use a homotopy argument to move from the general problem to one where everything is known. If in (2.2) we suppose that the speed $c = 0$, so that $\partial u/\partial t = 0$ (and we shall see that we can ensure this by insisting that

$$\int_0^1 \{-u + S(u)\}\, du = 0),$$

and if we take $k(t) = \frac{1}{2}e^{-|t|}$, then (2.2) after differentiation yields

$$u'' + f(u) = 0,$$

so that the problem in this particular case is that of looking at steady solutions of

$$u_t = u_{xx} + f(u). \tag{2.3}$$

This problem, for the type of f under discussion, has been dealt with in [**6, 7**].

There does not appear to have been much prior work on such integral equation models. Diekmann and Kaper [**8**] discuss travelling waves which connect one stable equilibrium and one unstable. Both their results and their methods are totally different from ours. Lui [**9**] analysed the discrete-time analogue of (2.2) as a model for the spread of diseases. Again, his techniques are substantially different.

We remark also that there are some cases other than $k(t) = \frac{1}{2}e^{-|t|}$ where the problem reduces to that of a partial differential equation. If we take (2.2) where k is such that its Fourier transform $\hat{k}$ has the form

$$\hat{k}(s) = \frac{1}{\sqrt{2\pi}} \frac{1}{1 + P(s)},$$

where P is a (necessarily even) polynomial

$$P(s) = a_1 s^2 + a_2 s^4 + \dots,$$

then the transform of (2.2) gives

$$\frac{\partial \hat{u}(s, t)}{\partial t} = \frac{1}{1 + P(s)} \hat{S} - \hat{u}.$$

Thus

$$\{1 + P(s)\}\left(\frac{\partial \hat{u}}{\partial t} + \hat{u}\right) = \hat{S},$$

and, transforming back, we have

$$u_t - a_1 u_{xxt} + a_2 u_{xxxxt} + \dots = S - u.$$

Finally, we are interested in travelling wave solutions of (2.1), so that we set $u = u(x - ct)$ and have

$$u(x - ct) = \int_{-\infty}^{t} dt' \int_{-\infty}^{\infty} dx'\, h(t - t')k(x - x')S(u(x' - ct')). \tag{2.4}$$

If we then change to variables

$$\xi = x - ct, \quad \xi' = x' - ct', \quad s = t - t',$$

we have

$$u(\xi) = \int_0^{\infty} h(s) \int_{-\infty}^{\infty} k(\xi + cs - \xi')S(u(\xi'))\, d\xi'\, ds.$$

A trivial change of notation yields

$$u(x) = \int_0^{\infty} h(s) \int_{-\infty}^{\infty} k(x + cs - y)S(u(y))\, dy\, ds, \tag{2.5}$$

and this is the form we shall normally consider.

3. Some remarks on monotonicity and the wave speed

THEOREM 3.1. *If we have a travelling wave solution of* (2.1), *with* $0 \leqq u \leqq 1$, $u(-\infty)=0$, $u(+\infty)=1$, *and if* $h(s)=be^{-bs}$, *for some* $b>0$, *and* k, S *satisfy the conditions in Section* 2, *then the wave speed* c *has the same sign as*

$$\int_0^1 \{u - S(u)\}\, du.$$

Proof. From the assumptions on S, we know that S is differentiable almost everywhere, with $S'>0$. Also, with $h(s)=be^{-bs}$, if we differentiate (2.4) with respect to t, we obtain, with $\xi = x - ct$, $\eta = x' - ct$,

$$-cu'(\xi) = -bu(\xi) + b\int_{-\infty}^{\infty} k(\xi-\eta)S(u(\eta))\, d\eta. \tag{3.1}$$

Thus

$$-b^{-1}cu'^2S'(u) = \{-u+S(u)\}S'(u)u' + \int_{-\infty}^{\infty} k(\xi-\eta)\{S(u(\eta)) - S(u(\xi))\}S'(u(\xi))u'(\xi)\, d\eta,$$

so that

$$-b^{-1}c\int_{-\infty}^{\infty} u'^2S'(u)\, d\xi = \int_0^1 \{-u+S(u)\}S'(u)\, du + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(\xi-\eta)\{S(u(\eta)) - S(u(\xi))\}S'(u(\xi))u'(\xi)\, d\xi\, d\eta.$$

If in the last integral we interchange the dummy variables ξ and η, and then add, we see that the integral is

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(\xi-\eta)\{S(u(\eta)) - S(u(\xi))\}\{S'(u(\xi))u'(\xi) - S'(u(\eta))u'(\eta)\}\, d\xi\, d\eta.$$

Now set $\xi - \eta = t$, and express the double integral in variables ξ, t. It becomes

$$\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} k(t)\{S(u(\xi-t)) - S(u(\xi))\}\{S'(u(\xi))u'(\xi) - S'(u(\xi-t))u'(\xi-t)\}\, d\xi\, dt,$$

and if we integrate first with respect to ξ, we see that the result is 0. Thus

$$-b^{-1}c\int_{-\infty}^{\infty} u'^2S'(u)\, d\xi = \int_0^1 \{-u+S(u)\}S'(u)\, du.$$

But

$$\int_0^1 \{-u+S(u)\}\{S'(u)-1\}\, du = \tfrac{1}{2}\{-u+S(u)\}^2|_{u=0}^1 = 0,$$

so that

$$-b^{-1}c\int_{-\infty}^{\infty} u'^2S'(u)\, d\xi = \int_0^1 \{-u+S(u)\}\, du.$$

The result is then immediate. □

We would like to extend Theorem 3.1 to cases where h satisfies only the conditions of Section 2, but this does not seem to be entirely straightforward. In fact, we are able to so extend it only when we assume the travelling wave to be monotonic and even then only by using the full power of Theorem 2.1. However, there is an easy first result.

THEOREM 3.2. *If we have a steady solution of* (2.1), *i.e.* $c=0$, *under the same conditions as in Theorem* 3.1 *except that now* h *satisfies only the conditions of Section* 2, *then*

$$\int_0^1 \{u-S(u)\}\,du=0.$$

Proof. If $c=0$, then (2.5) shows that, since

$$\int_0^\infty h(s)\,ds=1,$$

we have

$$u(x)=\int_{-\infty}^{\infty} k(x-y)S(u(y))\,dy,$$

which is (3.1) with $c=0$. The proof then follows the proof of Theorem 3.1. □

THEOREM 3.3. *If we have a monotonic travelling wave solution of* (2.1), *with* $0\leqq u\leqq 1$, $u(-\infty)=0$, $u(+\infty)=1$, *and if* h, k, S *satisfy the conditions in Section* 2, *then the wave speed* c *has the same sign as*

$$\int_0^1 \{u-S(u)\}\,du.$$

Proof. We use Theorem 2.1, which we have not yet proved but whose proof does not assume the present theorem. According to Theorem 2.1, for each (h, k, S) there is a unique speed c. If

$$\int_0^1 \{u-S(u)\}\,du=0, \tag{3.2}$$

and h is an exponential, then by Theorem 3.1 we have $c=0$, with $u=u_0(\xi)$, say. But since (2.5) does not involve h if $c=0$, the same pair $(u, c)=(u_0, 0)$ will satisfy (2.5) even when h is not an exponential. Thus (3.2) implies $c=0$ no matter what h is.

If now we have S such that

$$\int_0^1 \{u-S(u)\}\,du>0,$$

and h is exponential, then by Theorem 3.1 we have $c>0$. If we change h continuously, the proof of Theorem 2.1 shows that we change c continuously. But we can never have $c=0$, since this implies (3.2), which is now not true. Hence we always have $c>0$, and the theorem is proved. □

THEOREM 3.4. *We can without loss of generality assume* $c \geqq 0$.

Proof. For if we set $z=-x$, $t=-y$, $w(z)=1-u(x)$, $c^*=-c$, then (2.5) becomes, because of the evenness of k,

$$1-w(z)=\int_0^\infty h(s)\int_{-\infty}^\infty k(z+c^*s-t)S(1-w(t))\,dt\,ds,$$

so that

$$w(z)=\int_0^\infty h(s)\int_{-\infty}^\infty k(z+c^*s-t)\{1-S(1-w(t)\}\,dt\,ds,$$

and it is readily checked that $1-S(1-w)$ satisfies the same conditions as S. Thus the problem has been transformed into an equivalent problem with c changed in sign. □

Finally in this section it would be satisfying to be able to assert that any solution with $0\leqq u\leqq 1$, $u(-\infty)=0$, $u(+\infty)=1$ is necessarily monotone, as is certainly the case for travelling wave solutions of (2.3), but this seems to be difficult (and very likely untrue) under the general conditions on h, k, S in Section 2. In the particular case where $h(s)=be^{-bs}$, $k(t)=(2\lambda)^{-1}e^{-|t|/\lambda}$, we can, however, prove monotonicity.

THEOREM 3.5. *If* $h(s)=be^{-bs}$, $k(t)=(2\lambda)^{-1}e^{-|t|/\lambda}$, $b>0$, $\lambda>0$, *then any travelling wave solution satisfying* $0\leqq u\leqq 1$, $u(-\infty)=0$, $u(+\infty)=1$ *is necessarily monotone.*

Proof. With $h(s)=be^{-bs}$, the problem reduces to (3.1), and after two further differentiations we obtain

$$-b^{-1}cu'''=-u''+\frac{1}{\lambda^2}(-b^{-1}cu'+u)-\frac{1}{\lambda^2}S(u),$$

or

$$-\lambda^2b^{-1}cu'''+b^{-1}cu'+\lambda^2u''=u-S(u).$$

Multiplying by $-u'$ and integrating, we have

$$-b^{-1}c\int_{-\infty}^x u'^2\,dt-b^{-1}c\lambda^2\int_{-\infty}^x u''^2\,dt+b^{-1}c\lambda^2u'u''-\tfrac{1}{2}\lambda^2u'^2$$
$$=\int_0^u\{-v+S(v)\}\,dv. \tag{3.3}$$

(We note that (3.1) certainly implies that u' is bounded, and indeed $u'\to 0$ at $\pm\infty$. Thus $u''\to 0$ at $-\infty$, at least through some sequence of values, and this allows us to omit the integrated terms at $-\infty$ in (3.3).)

Now suppose for contradiction that we do not have $u'\geqq 0$. Thus we certainly have two values x_1, x_2 such that $u'(x_1)=0$, $u'(x_2)=0$, $u'<0$ in (x_1, x_2). By Theorem 3.4, we may suppose without loss of generality that $c \leqq 0$. Since $u - S(u) > 0$ for $0 < u < a$ and $u - S(u) < 0$ for $a < u < 1$, we see that

$$\int_0^u \{v - S(v)\}\, dv > 0 \quad \text{for} \quad 0 < u \leqq a,$$

and from (3.3) it follows that, if $u' = 0$ when $0 < u \leqq a$, then the left-hand side of (3.3) is non-negative while the right-hand side is strictly negative, which is impossible. Thus $u(x_1) > a$, $u(x_2) > a$.

Now evaluate (3.3) at $x = x_1$, $x = x_2$, and take the difference. We have

$$-b^{-1}\int_{x_1}^{x_2} u'^2\, dt - b^{-1}c\lambda^2 \int_{x_1}^{x_2} u''^2\, dt = \int_{u_1}^{u_1} \{-v + S(v)\}\, dv,$$

where $u_1 = u(x_1)$, $u_2 = u(x_2)$. Again, the left-hand side is non-negative while the right-hand side is strictly negative since $u_1 > u_2 > a$. This final contradiction completes the proof of the theorem. □

4. The continuation argument

In order to carry out the continuation argument mentioned in Section 2, we have to study the linear operator that arises when we perturb a solution u of (2.5). The linearisation operator L is given by

$$(L\phi)(x) = \int_0^\infty h(s) \int_{-\infty}^\infty k(x + cs - y) S'(u(y))\phi(y)\, dy\, ds, \tag{4.1}$$

and we discuss this operator in the Banach space $C_0(-\infty, \infty)$, consisting of continuous functions ϕ with $\phi(\pm\infty) = 0$.

THEOREM 4.1. *If h, k, S satisfy the conditions of Section 2, and if $u(-\infty) = 0$, $u(+\infty) = 1$, and u monotonic, then we can express L in the form*

$$L = L_1 + L_2,$$

where $\|L_1\| < 1$ and L_2 is compact.

Proof. We write

$$(L\phi)(x) = \int_0^\infty h(s)\, ds \left\{\int_{-\infty}^{-A} + \int_A^\infty + \int_{-A}^A\right\} k(x + cs - y) S'(u(y))\phi(y)\, dy\, ds,$$

where A is chosen so that, in $(-\infty, -A]$ and $[A, \infty)$, we have $S'(u) \leqq 1 - \delta$, say, $\delta > 0$. Then if L_1 is the operator arising from the integrals over $(-\infty, -A)$ and (A, ∞), and L_2 the remainder, we clearly have $\|L_1\| < 1$ and L_2 compact. (For a bounded sequence $\{\phi_n\}$, the functions $L_2\phi_n$ are uniformly bounded and equicontinuous on compact sets, and are uniformly small at $\pm\infty$.) □

THEOREM 4.2. *If in Theorem 4.1 the function u is a monotonic solution of (2.5) satisfying $u(-\infty) = 0$, $u(+\infty) = 1$, then the operator L has an eigenvalue 1, with* *non-negative eigenfunction. There are no other eigenfunctions corresponding to the eigenvalue* 1.

Proof. Differentiating (2.5) under the integral, which is permissible under the assumptions on k, we have

$$u' = \int_0^\infty h(s)\,ds \int_{-\infty}^\infty k'(x+cs-y)S(u(y))\,dy \tag{4.2}$$

$$= \int_0^\infty h(s)\,ds \int_{-\infty}^\infty k(x+cs-y)S'(u(y))u'(y)\,dy, \tag{4.3}$$

by integration by parts. Thus from (4.2) or (4.3) we see easily that u' is bounded and indeed continuous; we show below that $u'(\pm\infty)=0$. Hence u' is a non-negative eigenfunction of L corresponding to the eigenvalue 1.

To complete the proof of the theorem, we could appeal to the theory of positive operators, but for completeness we give an *ad hoc* proof. Suppose for contradiction that there is a second eigenfunction ϕ. Then for any constants λ, μ, we have

$$\lambda u' + \mu\phi = \int_0^\infty h(s)\,ds \int_{-\infty}^\infty k(x+cs-y)S'(u(y))(\lambda u' + \mu\phi)\,dy.$$

We assert first that $u'(\pm\infty)=0$, $\phi(\pm\infty)=0$, and it will be sufficient to give the argument for ϕ. If the result is not true, as $x\to-\infty$, say, we have a sequence $\{x_n\}$ such that $\phi(x_n)\to l>0$, where

$$l = \lim_{x\to-\infty} \sup \phi(x).$$

If $\phi(x_n)=l_n$, we have

$$l_n = \int_0^\infty h(s)\,ds \int_{-\infty}^\infty k(t)S'(u(x_n+cs-t))\phi(x_n+cs-t)\,dt. \tag{4.4}$$

From the integrability of h and k, we can neglect the contributions from $s \geqq S_0$ and $|t| \geqq T_0$, say. Hence as $n\to\infty$ we are interested only in $x_n+cs-t\to-\infty$, so that $0 \leqq S' \leqq 1-\delta$, say, for some $\delta>0$, and $\phi \leqq l+\varepsilon$, for any given $\varepsilon>0$. This clearly contradicts (4.4).

Since u is monotonic, we have $u' \geqq 0$. But in fact $u'>0$. For from (4.3) we have

$$u'(x) = \int_0^\infty h(s)\,ds \int_0^\infty h(\sigma)\,d\sigma \int_{-\infty}^\infty k(x+cs-y)S'(u(y))\,dy$$

$$\times \int_{-\infty}^\infty k(y+c\sigma-t)S'(u(t))u'(t)\,dt.$$

Since u' is not identically zero, suppose for contradiction that $u'>0$ in some interval (x_0, x_1), with $u'(x_1)=0$. Since k is not identically zero, we can choose y_0 so that $k(y_0-t)\neq 0$ at $t=x_1$, and so for t sufficiently close to x_1. Then from s, σ close to zero, and y close to y_0, and t close to (and less than) x_1, we get a strictly positive contribution to the integral for $u'(x_1)$. This gives the required contradiction.

Now consider $u' + \mu\phi$, and let us suppose without loss of generality that ϕ has a negative minimum. (If ϕ does not have a negative minimum, use $-\phi$ in place of ϕ.) For $\mu = 0$, $u' + \mu\phi > 0$, and it has a negative minimum if μ is sufficiently large and positive. Thus there exists a value μ_0 such that $u' + \mu_0\phi \geqq 0$ but $u' + \mu\phi$ has negative values if $\mu > \mu_0$. As $\mu \downarrow \mu_0$, let $x^*(\mu)$ be a point where $u' + \mu\phi < 0$. If x^* remains finite as $\mu \downarrow \mu_0$, then $u' + \mu_0\phi$ has a minimum value of zero attained at a finite point, and this is impossible. (For, as we have already seen, a non-negative eigenfunction is in fact strictly positive.) Thus, as $\mu \downarrow \mu_0$, any points where $u' + \mu\phi < 0$ are large in modulus. Suppose that the negative minimum is attained at $x^*(\mu)$. Then

$$(u' + \mu\phi)(x^*) = \int_0^\infty h(s)\, ds \int_{-\infty}^\infty k(x^* + cs - y) S'(u(y))(u' + \mu\phi)\, dy. \tag{4.5}$$

If y is such that $(u' + \mu\phi)(y) < 0$, we can now assume that $S'(u(y)) \leqq 1 - \delta$, say, $\delta > 0$, and

$$|(u' + \mu\phi)(y)| \leqq |(u' + \mu\phi)(x^*)|.$$

Thus the negative contribution from the right-hand side of (3.5) does not exceed $(1 - \delta)(u' + \mu\phi)(x^*)$, which contradicts (4.5). This final contradiction completes the proof of the theorem. □

THEOREM 4.3. *The adjoint operator* L^* *has a simple eigenvalue* 1, *with non-negative eigenfunction,* ψ_1, *say.*

Proof. The solution of

$$L^*\psi = \psi$$

is equivalent to the solution of

$$(L_1^* + L_2^*)\psi = \psi,$$

where $\|L_1^*\| < 1$, L_2^* is compact. Thus it is equivalent to the solution of

$$(I - L_1^*)\psi = L_2^*\psi,$$

or, with

$$(I - L_1^*)\psi = \xi,$$

of

$$\xi = L_2^*(I - L_1^*)^{-1}\xi.$$

But the operator on the right is compact, and its adjoint, $(I - L_1)^{-1}L_2$, has a simple eigenvalue at 1. Hence the operator itself has a simple eigenvalue at 1, and the theorem is proved except for showing that the corresponding eigenfunction ψ_1 is of one sign, which we may take to be positive.

Define

$$\psi^+ = \begin{cases} \psi_1, & \psi_1 \geqq 0, \\ 0, & \psi_1 < 0. \end{cases}$$

Since

$$(L^*\psi)(x) = S'(u(x)) \int_0^\infty h(s)\, ds \int_{-\infty}^\infty k(x - cs - y)\psi(y)\, dy,$$

it is clear that

$$L^*\psi^+ \geqq L^*\psi_1 = \psi_1,$$

and so, by considering separately the points where $\psi^+ = \psi_1$ and $\psi^+ = 0$, we see that almost everywhere

$$L^*\psi^+ \geqq \psi^+.$$

Thus, recalling that $u' > 0$ and that $(L - I)u' = 0$, we have

$$0 = \langle (L - I)u', \psi^+ \rangle = \langle u', (L^* - I)\psi^+ \rangle \geqq 0,$$

and we avoid a contradiction only if $(L^* - I)\psi^+ = 0$. This completes the theorem. □

THEOREM 4.4. *If u_0 is a monotone solution of* (2.5) *satisfying $u(-\infty) = 0$, $u(+\infty) = 1$ and corresponding to functions h_0, k_0, S_0, and if the associated speed is c_0, then, given $\varepsilon > 0$, there exists $\delta(u_0, h_0, k_0, S_0, c_0, \varepsilon)$ such that, if h, k, S satisfy the conditions of Section* 2 *and also*

$$\int_0^\infty (s+1)\,|h - h_0|\, ds < \delta, \quad \int_{-\infty}^\infty |k - k_0|\, ds < \delta,$$

$$\int_{-\infty}^\infty |(k - k_0)'|\, ds < \delta. \quad |(S - S_0)'| < \delta, \qquad (4.6)$$

then there exists, modulo translation of u, a unique pair (u, c) in $\|u - u_0\| < \varepsilon$, $|c - c_0| < \varepsilon$ which satisfies (2.5) *and $u' < 0$, $u(-\infty) = 0$, $u(+\infty) = 1$.*

Proof. We first establish that there is a solution pair (u, c), and indeed a unique pair (u, c), if we make the additional demand that

$$\int_{-\infty}^\infty u\psi_1\, dx = \int_{-\infty}^\infty u_0\psi_1\, dx, \qquad (4.7)$$

where ψ_1 is the eigenfunction introduced in Theorem 4.3. To do this, we write the equation (2.5) in the form

$$u - T(u, c) = 0, \qquad (4.8)$$

where

$$T(u, c) = \int_0^\infty h(s)\, ds \int_{-\infty}^\infty k(x + cs - y)S(u(y))\, dy.$$

Since (4.8) is satisfied by (u_0, c_0), with $T = T_0$, we can subtract and write (4.8) in the form

$$u - u_0 - L(u - u_0) - (c - c_0)\int_0^\infty sh_0(s)\,ds\int_{-\infty}^\infty k_0'(x + c_0s - y)S_0(u_0(y))\,dy$$
$$= T(u, c) - T_0(u_0, c_0) - L(u - u_0)$$
$$- (c - c_0)\int_0^\infty sh_0(s)\,ds\int_{-\infty}^\infty k_0'(x + c_0s - y)S_0(u_0(y))\,dy. \quad (4.9)$$

Now $T(u, c)$ is a continuously differentiable function of u, c, and the Fréchet derivatives are continuous in h, k, S, at least in a neighbourhood of $(u_0, h_0, k_0, S_0, c_0)$, neighbourhood being defined in terms of the norms implied by (4.6); indeed L is the Fréchet derivative with respect to u at $(u_0, h_0, k_0, S_0, c_0)$, and

$$\int_0^\infty sh_0(s)\,ds\int_{-\infty}^\infty k_0'(x + c_0s - y)S_0(u_0(y))\,dy \quad (4.10)$$

is the derivative with respect to c. Further, by an integration by parts, (4.10) can be written in the form

$$\int_0^\infty sh_0(s)\,ds\int_{-\infty}^\infty k_0(x + c_0s - y)S_0'(u_0)u_0'(y)\,dy,$$

which is clearly strictly positive.

With this in mind, we consider (4.9) as a mapping from a pair (u, c) on the right-hand side, where u is in the subspace defined by (4.7), to a unique pair $(\bar{u}, \bar{c})$ on the left-hand side, with again $\bar{u}$ satisfying (4.7). To see that this mapping is indeed well-defined, we first multiply (4.9) by ψ_1 and integrate. Since

$$\langle (I - L)(\bar{u} - u_0), \psi_1\rangle = \langle (\bar{u} - u_0), (I - L^*)\psi_1\rangle = 0,$$

this defines $\bar{c} - c_0$. With $\bar{c} - c_0$ defined, we can then solve

$$(I - L)(\bar{u} - u_0) = \ldots$$

uniquely for $\bar{u}$ satisfying (4.7), by Theorem 4.2.

Thus the mapping in (4.9) is well-defined. Further it is a contraction mapping. For if we have $(\bar{u}_i, \bar{c}_i)$ on the left corresponding to (u_i, c_i) on the right, $i = 1,2$, we obtain

$$(I - L)(\bar{u}_2 - \bar{u}_1) - (\bar{c}_2 - \bar{c}_1)\int_0^\infty sh_0(s)\,ds\int_{-\infty}^\infty k_0'(x + c_0s - y)S_0(u_0(y))\,dy$$
$$= T(u_2, c_2) - T(u_1, c_1) - L(u_2 - u_1)$$
$$- (c_2 - c_1)\int_0^\infty sh_0(s)\,ds\int_{-\infty}^\infty k_0'(x + c_0s - y)S_0(u_0(y))\,dy$$
$$= L(u_1, c_1)(u_2 - u_1) + (c_2 - c_1)\int_0^\infty sh(s)\,ds\int_{-\infty}^\infty k'(x + c_1s - y)S(u_1(y))\,dy$$
$$- L(u_2 - u_1) - (c_2 - c_1)\int_0^\infty sh_0(s)\,ds\int_{-\infty}^\infty k_0'(x + c_0s - y)S_0(u_0(y))\,dy$$
$$+ o(\|u_2 - u_1\|) + o(|c_2 - c_1|),$$

where $L(u_1, c_1)$ is the Fréchet derivative of T with respect to u at (u_1, c_1). In view of the continuity of the Fréchet derivatives, we see that the right-hand side can be reduced to

$$o(\|u_2 - u_1\|) + o(|c_2 - c_1|),$$

and then the same argument as that which established that the mapping is well-defined establishes that

$$\|\bar{u}_2 - \bar{u}_1\| + |\bar{c}_2 - \bar{c}_1\| = o(\|u_2 - u_1\| + |c_2 - c_1|),$$

so that the mapping is contractive. Further, the mapping maps a small ball

$$\|u - u_0\| < \varepsilon, \quad |c - c_0| < \varepsilon$$

into itself if δ in (4.6) is sufficiently small, and so by the contraction mapping theorem we have the existence of a solution pair (u, c), continuously dependent on h, k, S, and unique if (4.7) is demanded.

We now show that, given any solution pair (u, c) in a neighbourhood of (u_0, c_0), we can arrange by a translation of u that (4.7) is satisfied (and so any solution pair is unique modulo translation). For if (u, c) is a solution of (4.8) with $\|u - u_0\|$, $|c - c_0|$ small, then also $|u' - u_0'|$ is small, as we see by differentiating (4.8). It immediately follows that $u'(t) > 0$ and $0 \leqq u(t) \leqq 1$ except possibly for $|t|$ large. Also, if we suppose for contradiction, for example, that $u(t)$ has a negative minimum for some large negative t, then we can obtain a contradiction by evaluating (4.8) at the negative minimum and noticing (as we did, for example, in the proof of Theorem 4.2) that the major contribution from the integral is positive.

Thus $u' > 0$. We want to show finally that we can find a translation $u(x + x_0)$ such that

$$\int_{-\infty}^{\infty} u(x + x_0)\psi_1 \, dx = \int_{-\infty}^{\infty} u_0\psi_1 \, dx.$$

Then

$$\begin{aligned}\int_{-\infty}^{\infty} u(x + x_0)\psi \, dx &= \int_{-\infty}^{\infty} \{u(x + x_0) - u(x)\}\psi_1 \, dx + \int_{-\infty}^{\infty} \{u(x) - u_0(x)\}\psi_1 \, dx \\ &\quad + \int_{-\infty}^{\infty} u_0\psi_1 \, dx \\ &= x_0 \int_{-\infty}^{\infty} u'(x + \theta x_0)\psi_1 \, dx \\ &\quad + \int_{-\infty}^{\infty} \{u(x) - u_0(x)\}\psi_1 \, dx + \int_{-\infty}^{\infty} u_0\psi_1 \, dx, \qquad (4.11)\end{aligned}$$

where $0 < \theta < 1$. Since $u' > 0$, there is no difficulty in choosing x_0 so that the first two terms in (4.11) cancel. Indeed, to a first approximation,

$$x_0 = -\left\{\int_{-\infty}^{\infty} (u - u_0)\psi_1 \, dx\right\} \Big/ \int_{-\infty}^{\infty} u_0'\psi_1 \, dx.$$

The proof of the theorem is thus completed. □

THEOREM 4.5. *If h, k, S satisfy the conditions of Section 2, then there exists a unique pair (u, c) (modulo translation of u) which satisfies (2.5) and $u' > 0$, $u(-\infty) = 0$, $u(+\infty) = 1$.*

Proof. We begin with $h(t) = e^{-t}$, $k(t) = \frac{1}{2}e^{-|t|}$, and S chosen so that

$$\int_0^1 (u - S(u)) \, du = 0. \tag{4.12}$$

In this case, as we saw in Section 2, the problem reduces to the equation

$$u'' + (S(u) - u) = 0. \tag{4.13}$$

For any solution of this problem we necessarily have $c = 0$, and it is an elementary phase-plane exercise (which is carried out in [**6**, **7**]) to check that there is (modulo translation) a unique solution u to (4.13) with $u(-\infty) = 0$, $u(+\infty) = 1$, and this solution is necessarily monotonic.

Now change h, k, S continuously. By Theorem 4.4, there continues to exist (modulo translation) a unique solution pair (u, c), at least for (u, h, k, S, c) sufficiently close to the initial pentad $(u_0, h_0, k_0, S_0, c_0)$. As we continue to change h, k, S, the continuation process cannot terminate unless, as $(h, k, S) \to (h^*, k^*, S^*)$, say, we have $c \to \pm\infty$ or (in the limit) u fails to satisfy $u' > 0$, $u(-\infty) = 0$, $u(+\infty) = 1$.

We now rule out these possibilities. If $c \to \pm\infty$, we may without loss of generality suppose $c \to +\infty$, by Theorem 3.4. We have

$$\begin{aligned} u(x) - S(u(x)) &= \int_0^{\infty} h(s) \, ds \int_{-\infty}^{\infty} k(x + cs - y)\{S(u(y)) - S(u(x))\} \, dy \\ &= \int_A^B h(s) \, ds \int_{x+cs-B}^{x+cs+B} k(x + cs - y)\{S(u(y)) - S(u(x))\} \, dy + o(1), \end{aligned} \tag{4.14}$$

where $o(1)$ denotes a term not exceeding any given $\varepsilon > 0$ by a choice of $A(\varepsilon)$ sufficiently small and $B(\varepsilon)$ sufficiently large, this choice being uniform in $(h, k, S) \to (h^*, k^*, S^*)$. If $c \to \infty$, then $y > x$ in the range of integration in (4.14), and so $S(u(y)) > S(u(x))$ and

$$u(x) - S(u(x)) > o(1).$$

But $u - S(u)$ takes strictly negative values and this provides the necessary contradiction.

We may now suppose that, as $(h, k, S) \to (h^*, k^*, S^*)$, we have (at least by a subsequence) $c \to c^*$ and $u \to u^*$ (uniformly on compact sets), and

$$u^* = \int_0^\infty h^*(s)\, ds \int_{-\infty}^\infty k^*(x + c^*s - y) S^*(u^*(y))\, dy. \tag{4.15}$$

Certainly we must have $u^{*\prime} \geqq 0$, and by differentiating (4.15) we see that either $u^{*\prime} > 0$ or $u^{*\prime} \equiv 0$. Also, we can, by translation, arrange the continuation process so that the minimum value of $u'(x)$ in the u-interval $[a - u_0, a + u_0]$ is taken at $x = 0$. (Here u_0 is a fixed number chosen so that $S'(u) > 1$ in $[a - u_0, a + u_0]$, say

$$S'(u) > 1 + \eta, \quad \text{for some } \eta > 0.)$$

If $u^{*\prime} > 0$, then u^* is a solution of (4.15) with $u^*(-\infty) \neq u^*(+\infty)$, and the three possibilities are:

(i) $u^*(-\infty) = 0$, $u^*(+\infty) = 1$,
(ii) $u^*(-\infty) = 0$, $u^*(+\infty) = a$,
(iii) $u^*(-\infty) = a$, $u^*(+\infty) = 1$.

The first is what we want, and we can rule out (ii) and (iii) because $u^{*\prime}(x)$ clearly does not have its minimum at $x = 0$, contradicting the manner in which we defined our translations.

The remaining possibility is that $u' \equiv 0$, $u^* \equiv a$. Given any number B, we know, since $u \to u^*$ on compact sets, that ultimately (as $u \to u^*$)

$$u(B) < a + u_0, \quad u(-B) > a - u_0.$$

Then

$$u'(0) = \int_0^\infty h(s)\, ds \int_{-\infty}^\infty k(t) S'(u(cs - t)) u'(cs - t)\, dt$$
$$\geqq \int_0^A h(s)\, ds \int_{-A}^A k(t) S'(u(cs - t)) u'(cs - t)\, dt,$$

where we choose A, as we may, so that both

$$\int_0^A h(s)\, ds \int_{-A}^A k(t)\, dt > \frac{1}{1 + \eta}$$

and $|c|\, A + A < B$. Then, recalling that $u'(0)$ is the minimum value of u', we have $u'(0) > u'(0)$, a final contradiction which allows us to assert that, as $(h, k, S) \to (h^*, k^*, S^*)$, the solution pair $(u, c) \to (u^*, c^*)$, where (u^*, c^*) is again a solution pair of the type required.

Thus the process of continuing solutions can be carried out for all (h, k, S). Nor can there ever be two solutions corresponding to a particular set (h, k, S). For if there were, then we could continue back from such a set (h, k, S) to (h_0, k_0, S_0). Since there exists a unique solution for (h_0, k_0, S_0), somewhere in the backwards continuation there must be a set (h_1, k_1, S_1) at which the two solutions merge, which contradicts Theorem 4.4 at (h_1, k_1, S_1) and completes the proof of the theorem. □

We remark that it is only in Theorem 4.5 that we require the assumption that $-u + S(u)$ has only one zero for u in $(0, 1)$, and we require it only to rule out solutions of type (ii) or (iii) or $u^* \equiv a$. If $-u + S(u)$ is allowed more internal zeros, then, as in [**6, 7**], there will arise the possibility of "stacks" of travelling waves.

5. An exactly solvable model and numerical solutions

By choosing the function S to be the Heaviside step function, we can explicitly compute a travelling wave for (2.2). (We note that, since S is then not continuously differentiable, this example does not strictly fall under the hypotheses of Theorem 2.1, although a theorem to cover it could be obtained as a limiting case.) Let $S(u) \equiv H(u-\theta)$ where $0<\theta<\frac{1}{2}$. The parameter θ represents the threshold for excitation of the network. Since the wave is translation invariant, we require that $u(0)=\theta$ as a normalisation. Because the threshold is less than $\frac{1}{2}$, the wave travels to the right, if we take it to be monotonic decreasing, with $u(-\infty)=1$, $u(+\infty)=0$, so that $c>0$, $u(\xi)<\theta$ for $\xi>0$ and $u(\xi)>\theta$ for $\xi<0$. Thus we must solve:

$$-cu'(\xi)+u(\xi)=\int_{-\infty}^{0} k(|\xi-\xi'|)\,d\xi'=\int_{\xi}^{\infty} k(y)\,dy \equiv G(\xi) \tag{5.1}$$

subject to the condition that $u(0)\equiv\theta$. Here

$$G(\xi)\equiv\frac{1}{2}-\int_{0}^{\xi} k(y)\,dy. \tag{5.2}$$

Note that $G(-\infty)=1$ and $G(\infty)=0$ by our normalisation of k. We can solve (5.1) for $u(\xi)$:

$$u(\xi)=\exp(\xi/c)\left(\theta-\frac{1}{c}\int_{0}^{\xi}\exp(-s/c)G(s)\,ds\right). \tag{5.3}$$

We must choose c so that as $\xi\to\infty$, the expressions remain bounded. Clearly, the minimum requirement is that the term within the large parentheses tends to zero as $\xi\to\infty$, i.e.

$$\theta=\frac{1}{c}\int_{0}^{\infty}\exp(-s/c)G(s)\,ds=\int_{0}^{\infty}\exp(-s)G(sc)\,ds. \tag{5.4}$$

We immediately notice a few properties of the relationship between θ and c. As $c\to 0$, $\theta\to\frac{1}{2}$ and as $c\to\infty$, $\theta\to 0$. Furthermore $d\theta/dc<0$ since G is monotone decreasing. Thus, for each $0<\theta<\frac{1}{2}$ there is a unique positive c which solves (5.4). Substitution of (5.4) into (5.3) yields:

$$u(\xi)=\frac{1}{c}\int_{\xi}^{\infty}\exp((\xi-s)/c)G(s)\,ds.$$

An application of L'Hôpital's rule shows that $u(\xi)$ satisfies the correct conditions at $\pm\infty$ (noting that $G(\infty)=0$ and $G(-\infty)=1$).

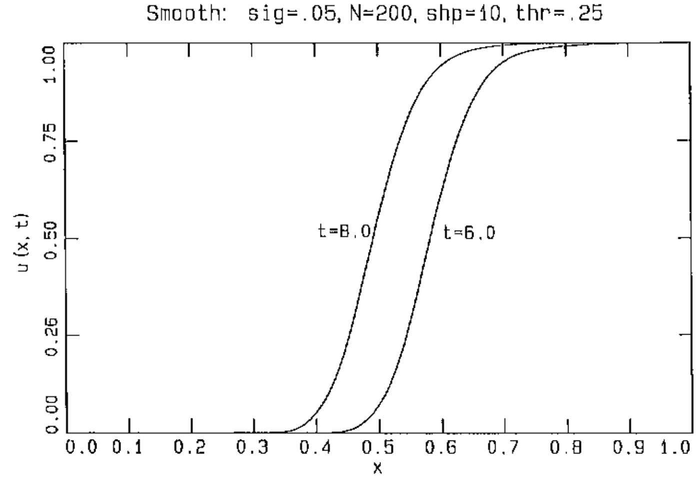

Figure 5.1 Spatial profiles at two different times for (2.2). We have chosen $k(x) = 10\exp(-20|x|)$ and $S(u) = 0{\cdot}5(1 + \tanh(10(u - 0{\cdot}25)))$. There are 200 spatial points in our grid and we use Euler's method with a timestep of 0·1.

We have numerically solved (2.2) and compared it to the solutions to (5.1). In Figure 5.1, the spatial profile of the solution to the smooth equation is depicted at two different times. For comparison, in Figure 5.2 we show the same simulation with the smooth function replaced by the Heaviside function. The parameters are as given in the figure legends. While the present results concern scalar neural networks in one spatial dimension, they can be used to obtain formally results for much more general neural networks. The key result is that a wavefront exists which connects the two stable states. The formal asymptotic results of Tyson and Keener show how once a front is constructed, many other solutions can be constructed in systems of reaction-diffusion equations. The analogues of these

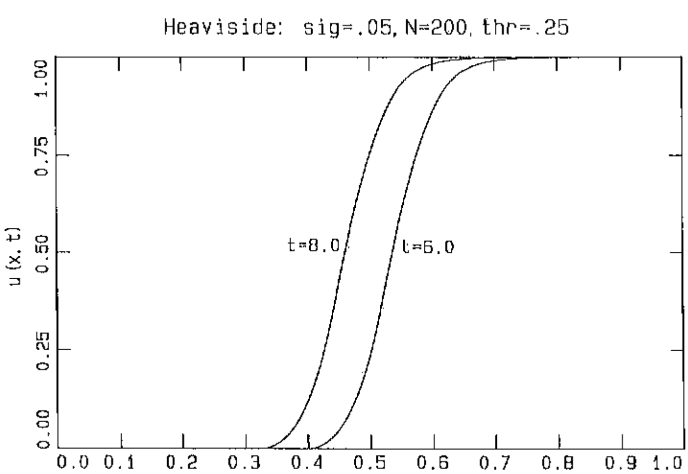

Figure 5.2. Same as Figure 5.1 with $S(u)$ replaced by $H(u - 0{\cdot}25)$ and H the Heaviside function.

results are easily constructed for neural networks now that wave fronts have been proved. In a later paper, we shall describe some of these calculations.

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