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Phase locking in chains of multiple-coupled oscillators

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Abstract

Phase locking in chains of weakly coupled oscillators with coupling beyond nearest neighbors is studied. Starting with a piecewise linear coupling function, a homotopy method is applied to prove the existence of phase locked solutions. Numerical examples are provided to illustrate the existence and the properties of the solutions. Differences between multiple coupling and nearest neighbor coupling are also discussed. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Weakly coupled oscillator arrays arise in many physical and biological systems. In particular, one-dimensional chains of oscillators have been used to model a variety of biological systems such as the swim generator in the lamprey [1] and olfactory waves in the procerebral lobe of the garden slug [5]. These models arise from general systems of coupled oscillators under the assumption that the interactions between oscillators are sufficiently weak. Under this "weak coupling" assumption, each oscillator is reducible to a single variable that describes the phase. The most general form that these phase equations can take is

$$\theta_i' = \omega_i + H_i(\theta_1 - \theta_i, \dots, \theta_n - \theta_i), \quad i = 1, \dots, n,$$

where the functions H_i are 2π-periodic in each of their arguments and the parameters ω_i are the local variations in uncoupled frequency. Typically, we are interested in solutions that are periodic, i.e., $\theta_i(t+T) = \theta_i(t) + 2\pi$. The stability of solutions for general coupling was studied in [4], however, the structure of the solutions is never discussed.

The most comprehensive results concern either globally coupled all-to-all systems of oscillators, e.g., [2] or [3], or chains of oscillators with *nearest-neighbor coupling* [6,7,14]. In the latter papers, phase locked solutions were analyzed which correspond to traveling waves. Such waves have been observed in several central nervous system preparations using imaging of the electrical potentials [9,13]. Recent experimental work, however, indicates that the coupling in the lamprey spinal cord cannot be regarded as nearest neighbor [11]. Similarly, local application of nitric oxide in the slug procerebral lobe indicates that coupling between oscillators extends beyond the nearest

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neighbors [8]. Thus, it is important to determine under what conditions oscillator chains that have coupling beyond nearest neighbors can lead to phase locked solutions such as waves, which is the subject of the present paper.

Weak coupling in a chain of neurons or neural circuits simplifies the general structure of the phase models considerably. Suppose the coupling strength depends only on the distance between two circuits. Since inputs to neurons are treated independently and sum in a linear fashion, the resulting phase models have the general form

$$\theta_i' = \omega_i + \sum_{j=1}^{m} H_j^+(\theta_{i+j} - \theta_i) + \sum_{j=1}^{m} H_j^-(\theta_{i-j} - \theta_i), \tag{1}$$

where $i = 1, \dots, n+1$, θ_i is the phase and ω_i is the frequency of the ith oscillator, and $H_j^{\pm}$ are 2π-periodic functions of their arguments. We delete terms in the sum whenever $i + j > n + 1$ or $i - j < 0$ so that the "boundary conditions" are those of a finite chain. The boundary effects are crucial and they make the analysis of these equations difficult. We are interested in phase locked solutions, i.e., solutions for which θ_i' is independent of i and t. The equivalent equations, with the variables $\{\theta_i\}$ replaced by $\{\phi_i = \theta_{i+1} - \theta_i\}$, are considered. If $\theta_i' = \Omega$ which is the unknown frequency of the phase locked ensemble of the oscillators, then (1) becomes

$$\Omega = \omega_i + \sum_{j=1}^{m} H_j^+\left(\sum_{k=1}^{j}\phi_{i+k-1}\right) + \sum_{j=1}^{m} H_j^-\left(-\sum_{k=1}^{j}\phi_{i-k}\right). \tag{2}$$

It was shown [6,7] that phase locked solutions of chains with nearest neighbor coupling could be approximated, when there is a large number of oscillators, by passing to a continuum limit and analyzing the solutions of the resulting singularly perturbed second-order two-point boundary value problem (BVP). Thus, over much of the chain, the solution behaves like a solution to a first-order "outer equation". The particular "outer equation" is determined by the boundary conditions for the BVP. In [14], we considered chains with finitely many oscillators. It was shown that under weak assumptions on the coupling functions, the phase lags between successive oscillators have the property of monotonicity provided that the frequency difference between any two successive oscillators is a sufficiently small constant along the chain. This implies that most chains of locally coupled oscillators that phase lock will form traveling wave solutions similar to those found in the limit of large n in [6,7].

Kopell et al. [10] considered the problem of chains with m neighbors in the limit as the number of oscillators tends to infinity. In this limit, phase locked solutions of (2) may be viewed as a one-parameter family of $(2m-1)$th-order discrete dynamical systems, where the independent variable is the position along the chain and whose dependent variable is the phase difference between successive oscillators. In [10] it was shown that for each value of the parameter Ω in some range, the $(2m-1)$th-order system has a one-dimensional hyperbolic global center manifold. This was done by using the theory of exponential dichotomies to show the system "shadows" a simple one-dimensional system. For a finite chain, the dynamical system is constrained by manifolds of boundary conditions. It was shown that for open sets of such conditions, the solution to the equation for phase locking in long chains stays close to the center manifold except near the boundaries. These facts were used to show that a multiply coupled system behaves, except near the boundaries, as a modified nearest-neighbor system. The existence of asymptotically stable phase locked solutions was proven provided that the chain is long enough and the frequencies of oscillators are sufficiently close.

In this paper, a special form of Eqs. (1) is considered for chains with finitely many oscillators, i.e., we do not require that the length of the chain to tend to infinity. For simplicity, we assume $n \geq 2m+1$ (as a matter of fact, all the results will also be true as long as $n \geq m+2$). The equations have the following form:

$$\theta_i' = \omega_i + \sum_{j=1}^{m} \alpha_j^+ H^+(\theta_{i+j} - \theta_i) + \sum_{j=1}^{m} \alpha_j^- H^-(\theta_{i-j} - \theta_i), \tag{3}$$

where $\alpha_1^{\pm} \geq \alpha_2^{\pm} \geq \cdots \geq \alpha_m^{\pm} > 0$ and $H^{\pm}$ are 2π-periodic functions of their arguments. This particular form is not unreasonable for neural models. If we assume that each local region oscillates in a similar manner and that the coupling depends on the distance between units, then this form is quite natural. With these assumptions (2) becomes

$$\Omega = \omega_i + \sum_{j=1}^{m} \alpha_j^{+} H^{+} \left(\sum_{k=1}^{j} \phi_{i+k-1} \right) + \sum_{j=1}^{m} \alpha_j^{-} H^{-} \left(-\sum_{k=1}^{j} \phi_{i-k} \right). \tag{4}$$

Note that the terms are omitted from (3) and (4) if $i+j$ or $i-j$ goes beyond $1, 2, \ldots, n+1$. This form will allow us to prove the existence of stable solutions to (4) via a simple constructive method. Our strategy will be to first consider a piecewise linear model for the functions $H^{\pm}$. In this case, the existence of solutions is reduced to finding a solution to a linear matrix equation. We then smoothly move from the piecewise linear version of the functions $H^{\pm}$ to the desired version by using the implicit function theorem.

Crucial to our continuation of argument are certain hypotheses on the functions $H^{\pm}(\phi)$. We define two functions f and g as $f(\phi) + g(\phi) = H^{+}(\phi)$ and $f(\phi) - g(\phi) = H^{-}(-\phi)$. We assume the following hypotheses on f and g in a sufficiently large interval around $\phi = 0$:

(H1) $g'(\phi) > |f'(\phi)|$ for $\phi \in J$.

(H2) There exists a unique solution ϕ_{L} (respectively ϕ_{R}) to $f(\phi) = g(\phi)$ (respectively $f(\phi) = -g(\phi)$) for some $\phi \in J$.

Note that if $H^{+}(\phi) = H^{-}(\phi) = H(\phi)$, i.e., the coupling is isotropic, then $g(\phi)$ is just the odd part of the function $H(\phi)$ and $f(\phi)$ is the even part. This set of conditions is exactly the same as in [7] and is a subset of those in [6]. In addition, $\phi_{\mathrm{L}} \neq \phi_{\mathrm{R}}$ should be imposed. It can be shown that $\phi_{\mathrm{R}} < 0 < \phi_{\mathrm{L}}$ when $f(0) > |g(0)|$ and $\phi_{\mathrm{L}} < 0 < \phi_{\mathrm{R}}$ when $f(0) < -|g(0)|$. We can restate these hypotheses in terms of the functions $H^{\pm}$:

(H1′) $H^{\pm'}(\phi) > 0$ for $\phi \in J$.

(H2′) There exists a unique solution ϕ_{L} (respectively ϕ_{R}) to $H^{-}(-\phi) = 0$ (respectively $H^{+}(\phi) = 0$) for some $\phi \in J$.

Hypothesis (H1′) is analogous to the hypothesis made in [4]. The second hypothesis is required in order to get some bounds on the behavior of the ends of the chain.

The numbers, $\phi_{\mathrm{L}}, \phi_{\mathrm{R}}$ and the hypotheses on the interaction functions can be understood intuitively by looking at the case of just a pair of mutually coupled oscillators. Consider a pair of coupled oscillators:

$$\theta_1' = \omega + H^{+}(\theta_2 - \theta_1), \qquad \theta_2' = \omega + H^{-}(\theta_1 - \theta_2).$$

The phase difference between them, $\phi = \theta_2 - \theta_1$ satisfies

$$\phi' = H^{-}(-\phi) - H^{+}(\phi) = -2g(\phi).$$

Thus, phase locked solutions are just roots of $g(\phi) = 0$. If the coupling is only forward, i.e., $H^{+} \equiv 0$ then the phase locked solution is $\phi = \phi_{\mathrm{L}}$. Furthermore, it is a stable phase locked solution since we have assumed that $\phi_{\mathrm{L}} \in J$ and that $g'(\phi) > 0$ in the interval J. Thus, ϕ_{L} is the unique stable phase locked solution for a forwardly coupled pair of oscillators. Similarly, ϕ_{R} is the the unique phase locked solution for a pair of backwardly coupled ($H^{-} \equiv 0$) oscillators. For H^{+} and H^{-} nonzero, the unique phase locked solution is between ϕ_{L} and ϕ_{R}. It is stable since both $H^{\pm'} > 0$ in an interval containing $\phi_{\mathrm{L}}, \phi_{\mathrm{R}}$.

A simple example is $H^{\pm} = \alpha^{\pm} H$, $H(\phi) = A\cos\phi + B\sin\phi$ where $B > 0$, $A \neq 0$, $\alpha^{\pm} > 0$. Furthermore, A should not be too large in magnitude.

We now introduce equations for the local phase differences. If we let $\phi_i = \theta_{i+1} - \theta_i, \beta_i = \omega_{i+1} - \omega_i, i = 1, \ldots, n$, then (3) leads to

$$\phi_i' = \beta_i + \sum_{j=1}^{m} \alpha_j^+ [f+g] \left(\sum_{k=1}^{j} \phi_{i+k} \right) + \sum_{j=1}^{m} \alpha_j^- [f-g] \left(\sum_{k=1}^{j} \phi_{i-k+1} \right) - \sum_{j=1}^{m} \alpha_j^+ [f+g] \left(\sum_{k=1}^{j} \phi_{i+k-1} \right) - \sum_{j=1}^{m} \alpha_j^- [f-g] \left(\sum_{k=1}^{j} \phi_{i-k} \right). \quad (5)$$

Again the terms out of index range will be ignored. Through most part of this paper, we study the case of $\beta_i \equiv 0$ (which means that all the oscillators have the same frequency). Then (5) can be rewritten as

$$\phi'_i = \sum_{j=1}^{m} \alpha_j^+ [f+g] \left(\sum_{k=1}^{j} \phi_{i+k} \right) + \sum_{j=1}^{m} \alpha_j^- [f-g] \left(\sum_{k=1}^{j} \phi_{i-k+1} \right) - \sum_{j=1}^{m} \alpha_j^+ [f+g] \left(\sum_{k=1}^{j} \phi_{i+k-1} \right) - \sum_{j=1}^{m} \alpha_j^- [f-g] \left(\sum_{k=1}^{j} \phi_{i-k} \right). \quad (6)$$

For phase locked solutions, we have $\phi_i' = 0$ so that

$$\sum_{j=1}^{m} \alpha_j^+ [f+g] \left(\sum_{k=1}^{j} \phi_{i+k-1} \right) + \sum_{j=1}^{m} \alpha_j^- [g-f] \left(\sum_{k=1}^{j} \phi_{i-k+1} \right) = \sum_{j=1}^{m} \alpha_j^+ [f+g] \left(\sum_{k=1}^{j} \phi_{i+k} \right) + \sum_{j=1}^{m} \alpha_j^- [g-f] \left(\sum_{k=1}^{j} \phi_{i-k} \right), \quad (7)$$

where $i = 1, \dots, n$. Note that the terms containing ϕ_i are placed on the left-hand side and the terms without ϕ_i are put on the right-hand side. This arrangement simplifies the analysis below.

In Section 2, $H^\pm$ are chosen to be piecewise linear functions. The reason for this is that we can explicitly find solutions with these simple functions. Then a "bridge" can be built from the simple to the general case based on the information collected from the simple case.

Section 3 provides a way to construct the "bridge". That is, we set up a homotopy path starting with the solution which we obtain in Section 2. Under very general assumptions, this homotopy path will lead to the solution of (7). The solution is a unique asymptotically stable solution of (6) for a wide range of functions.

Numerical experiments are shown in Section 4. They confirm the results obtained from Section 3.

2. Piecewise linear coupling functions

We consider piecewise linear systems in this section in order to collect the information we need. Two piecewise linear 2π-periodic coupling functions are constructed as $H^\pm(\phi) = H_E^\pm(\phi) + H_O^\pm(\phi)$ with $H_E^\pm$ and $H_O^\pm$ (as even parts and odd parts of $H^\pm$, respectively) are defined as

$$H_E^\pm(\phi) \equiv b^\pm, \qquad H_O^\pm(\phi) = \begin{cases} \phi, & 0 \le \phi \le c, \\ \dfrac{c(\pi - \phi)}{\pi - c}, & c < \phi \le \pi, \\ -H_O^\pm(-\phi), & -\pi \le \phi < 0, \end{cases}$$

where $-\pi < -c < \min(b^-, -b^+) < 0 < \max(b^-, -b^+) < c < \pi$.

Then if we choose c such that $b^{\pm} \in J = (-c, c)$, the hypotheses (H1) and (H2) hold. We can also deduce that $\phi_L = b^-$ and $\phi_R = -b^+$.

Note that if $|\phi_i| \le c/m$ for the solution of the Eqs. (7), we have $f(\phi) = \frac{1}{2}(b^+ + b^-)$ and $g(\phi) = \phi + \frac{1}{2}(b^+ - b^-)$ in Eqs. (7). Then (7) yields

$$\sum_{j=1}^{m}\alpha_j^+\left[b^+ + \sum_{k=1}^{j}\phi_{i+k-1}\right] + \sum_{j=1}^{m}\alpha_j^-\left[\sum_{k=1}^{j}\phi_{i-k+1} - b^-\right] = \sum_{j=1}^{m}\alpha_j^+\left[b^+ + \sum_{k=1}^{j}\phi_{i+k}\right] + \sum_{j=1}^{m}\alpha_j^-\left[\sum_{k=1}^{j}\phi_{i-k} - b^-\right], \tag{8}$$

where $i = 1, \dots, n$ and the out-of-range terms are ignored as before.

More specifically, (8) can be reduced to

$$\left(\sum_{j=1}^{m}\alpha_j^+ + \sum_{j=1}^{i}\alpha_j^-\right)\phi_i = \alpha_i^-\phi_0 + \sum_{j=1}^{i-1}(\alpha_j^- - \alpha_i^-)\phi_{i-j} + \sum_{j=1}^{m}\alpha_j^+\phi_{i+j} \tag{9}$$

for $1 \le i \le m$,

$$\left(\sum_{j=1}^{m}\alpha_j^+ + \sum_{j=1}^{m}\alpha_j^-\right)\phi_i = \sum_{j=1}^{m}\alpha_j^-\phi_{i-j} + \sum_{j=1}^{m}\alpha_j^+\phi_{i+j} \tag{10}$$

for $m + 1 \le i \le n - m$,

$$\left(\sum_{j=1}^{n+1-i}\alpha_j^+ + \sum_{j=1}^{m}\alpha_j^-\right)\phi_i = \sum_{j=1}^{m}\alpha_j^-\phi_{i-j} + \sum_{j=1}^{n-i}(\alpha_j^+ - \alpha_{n+1-i}^+)\phi_{i+j} + \alpha_{n+1-i}^+\phi_{n+1} \tag{11}$$

for $n - m + 1 \le i \le n$, where $\phi_0 = b^-$ and $\phi_{n+1} = -b^+$, i.e., $\phi_0 = \phi_L$ and $\phi_{n+1} = \phi_R$.

From this, (9)–(11) can be written as a matrix equation

$$B\Phi = S, \tag{12}$$

where

$$S = [\alpha_1^-\phi_0, \dots, \alpha_m^-\phi_0, 0, \dots, 0, \alpha_m^+\phi_{n+1}, \dots, \alpha_1^+\phi_{n+1}]^T,$$

$\Phi = (\phi_1, \dots, \phi_n)^T$ and $B = D - L - U$. Here D is a diagonal matrix and L (respectively, U) is lower triangular (respectively, upper triangular) with zero entries on the diagonal. D, L and U are matrices with nonnegative entries.

Lemma 2.1. *Assume that* $\min(\phi_L, \phi_R) < 0 < \max(\phi_L, \phi_R)$, *then Eq.* (12) *has a unique solution* $\bar{\Phi}$. $\bar{\Phi}$ *satisfies* $\min(\phi_L, \phi_R) < \bar{\phi}_i < \max(\phi_L, \phi_R)$, $i = 1, \dots, n$.

Proof. Without loss of generality, we only consider the case when $\phi_R < 0 < \phi_L$, i.e., $\phi_{n+1} < 0 < \phi_0$. To show (12) has a unique solution, we only need to verify that B is nonsingular. By the special form of Eqs. (9)–(11), we have $b_{ii} \ge \sum_{j\ne i}|b_{ij}|$, $i = 1, \dots, n$ and there is at least one ">".

Also it is quite clear that B is irreducible. Thus B is irreducibly diagonally dominant. Any irreducibly diagonally dominant matrix is nonsingular (see [12]). Hence (12) has a unique solution $\bar{\Phi}$.

In order to show $\phi_{\mathrm{R}} < \bar{\phi}_i < \phi_{\mathrm{L}}$, $i = 1, \dots, n$, we need to construct an iterative process. That is

$$\Phi^{(0)} = (0, \dots, 0)^{\mathrm{T}}, \qquad \Phi^{(l+1)} = D^{-1}S + D^{-1}(L+U)\Phi^{(l)}, \tag{13}$$

where $l = 0, 1, \dots$.

Let $A = D^{-1}(L+U)$ and $Q = D^{-1}S$, then $\Phi^{(l+1)} = A\Phi^{(l)} + Q$. Thus,

$$\Phi_l = A^l \Phi_0 + \sum_{k=0}^{l-1} A^k Q. \tag{14}$$

It can be shown [12] that the spectral radius, $\rho(A)$, is less than 1. Thus, the sums in (14) converge and the iteration (13) converges. That is $\Phi^{(l)} \to \bar{\Phi} = (I-A)^{-1}Q$ as $l \to \infty$.

We claim that for $i = 1, \dots, n$, we have

$$\phi_{n+1} < \phi_i^{(l)} < \phi_0. \tag{15}$$

By referring to (9)–(11), the iteration (13) can be written as

$$\left(\sum_{j=1}^{m}\alpha_j^+ + \sum_{j=1}^{i}\alpha_j^-\right)\phi_i^{(l+1)} = \alpha_i^-\phi_0 + \sum_{j=1}^{i-1}(\alpha_j^- - \alpha_i^-)\phi_{i-j}^{(l)} + \sum_{j=1}^{m}\alpha_j^+\phi_{i+j}^{(l)},$$

for $1 \le i \le m$,

$$\left(\sum_{j=1}^{m}\alpha_j^+ + \sum_{j=1}^{m}\alpha_j^-\right)\phi_i^{(l+1)} = \sum_{j=1}^{m}\alpha_j^-\phi_{i-j}^{(l)} + \sum_{j=1}^{m}\alpha_j^+\phi_{i+j}^{(l)},$$

for $m+1 \le i \le n-m$,

$$\left(\sum_{j=1}^{n+1-i}\alpha_j^+ + \sum_{j=1}^{m}\alpha_j^-\right)\phi_i^{(l+1)} = \sum_{j=1}^{m}\alpha_j^-\phi_{i-j}^{(l)} + \sum_{j=1}^{n-i}(\alpha_j^+ - \alpha_{n+1-i}^+)\phi_{i+j}^{(l)} + \alpha_{n+1-i}^+\phi_{n+1},$$

for $n-m+1 \le i \le n$.

We prove (15) by induction on l. For $l = 0$, (15) holds. Suppose (15) holds for l, then

$$\phi_i^{(l+1)} < \frac{\alpha_i^-\phi_0 + \sum_{j=1}^{i-1}(\alpha_j^- - \alpha_i^-)\phi_0 + \sum_{j=1}^{m}\alpha_j^+\phi_0}{\sum_{j=1}^{m}\alpha_j^+ + \sum_{j=1}^{i}\alpha_j^-} < \phi_0,$$

where $1 \le i \le m$. Similarly, we can get $\phi_i^{(l+1)} < \phi_0$ for $m+1 \le j \le n$. So $\phi_i^{(l+1)} < \phi_0, i = 1, \dots, n$. By similar arguments, we have $\phi_{n+1} < \phi_i^{(l+1)}, i = 1, \dots, n$. Hence (15) holds for any $l \in \mathcal{N}$. Then we must have $\phi_{n+1} \le \bar{\phi}_i \le \phi_0$, since $\Phi^{(l)} \to \bar{\Phi}$ as $l \to \infty$.

We know that $(\phi_0, \dots, \phi_0)$ is not the solution, so there is at least an index i_0 such that $\bar{\phi}_{i_0} < \phi_0$. Then by (9)–(11) and $\phi_{n+1} \le \bar{\phi}_i \le \phi_0$, we can get $\bar{\phi}_i < \phi_0$ for all i. Similarly, we have $\phi_{n+1} < \bar{\phi}_i$ for all i. Hence $\phi_{n+1} < \bar{\phi}_i < \phi_0$, i.e., $\phi_{\mathrm{R}} < \bar{\phi}_i < \phi_{\mathrm{L}}$ for $i = 1, \dots, n$. □

Theorem 2.1. *Assume that*

$$-\frac{c}{m} \le \min(\phi_{\mathrm{L}}, \phi_{\mathrm{R}}) < 0 < \max(\phi_{\mathrm{L}}, \phi_{\mathrm{R}}) \le \frac{c}{m} \tag{16}$$

for the piecewise linear functions f and g. Then the system (6) *has an asymptotically stable equilibrium* $\bar{\Phi} = (\bar{\phi}_1, \dots, \bar{\phi}_n)$ *such that* $\min(\phi_L, \phi_R) < \bar{\phi}_i < \max(\phi_L, \phi_R)$.

Proof. The existence and boundedness of $\bar{\Phi}$ have been proven. The linearized system of (6) around $\bar{\Phi}$ is $\Phi' = B\Phi$. It was shown in Lemma 2.1 that B is nonsingular so that B has no zero eigenvalue. For each i, $b_{ii} \geq \sum_{j \neq i} |b_{ij}|$. If we apply the Gerschgorin disk theorem, all the eigenvalues of B stay in $Re(z) < 0$, i.e., all the eigenvalues have negative real parts such that $\bar{\Phi}$ is asymptotically stable. □

The condition (16) in Theorem 2.1 will be violated for large m. We would like to modify it since most of $\bar{\phi}_i$ are not necessarily close to ϕ_L or ϕ_R (only those which are near the two ends might be close to ϕ_L and ϕ_R). The key point that guarantees that we can stably continue the solution is that the phase differences between any two oscillators that are connected should lie in a region such that $H^{\pm}$ is increasing (i.e., within the interval $(-c, c)$). For then, we can apply the results in [4]. The theorem gives sufficient conditions which guarantee all these phase differences lie in the interval $(-c, c)$ but they are rather stringent. Thus, we can more directly give conditions looking at the total phase lag between any two connected oscillators. Note that since $\phi_i = \theta_{i+1} - \theta_i$, the total phase lag between oscillators i and $i + l$ is just the sum of the local phase differences. Hence we have the following theorem.

Theorem 2.2. *Assume that the solution* $\bar{\Phi}$ *in Lemma* 2.1 *satisfies the following conditions*:

$$-c \leq \sum_{j=0}^{l} \bar{\phi}_{i+j} \leq c, \quad l = 0, 1, \dots, m-1, \tag{17}$$

for $i = 1, \dots, n$ (*note that if* $i + j$ *is out of range of* $\{1, \dots, n\}$, *the term* $\bar{\phi}_{i+j}$ *is ignored in the sum*), *then* $\bar{\Phi}$ *is an asymptotically stable equilibrium of* (6). *Also* $\min(\phi_L, \phi_R) < \bar{\phi}_i < \max(\phi_L, \phi_R)$.

Remarks.
1. *As noted above, the sums in* (17) *are nothing more than the total phase lags* $\theta_i - \theta_{i \pm l}$ *so that this condition is an assertion that the maximal phase lag between any pair of oscillators that are coupled lies in the interval* $J = (-c, c)$.
2. *From* (9)–(11), *each* $\bar{\phi}_i$ *seems to be the average of its* $2m$ *neighbors in some sense. For* $m + 1 \leq i \leq n - m$, *i.e. in the middle of the chain, the average is the weighted average. But on the two ends, the averages have some portions lost* (*or gained*). *This is the boundary effect and the reason why there exists nonzero values of* ϕ_i *in the chain.*

3. General coupling functions

In this section, we assume that $H^{\pm}$ satisfy (H1) and (H2). In addition, we assume that either $\phi_R < 0 < \phi_L$ or $\phi_L < 0 < \phi_R$.

Let $b^- = \phi_L$ and $b^+ = -\phi_R$. We choose $c \in (0, \pi)$ such that $J \subset [-c, c]$. Then the piecewise linear functions in Section 2 can be constructed. We denote them as H_0^+, H_0^-, f_0 and g_0, respectively.

With these preliminaries, we can construct two homotopy coupling functions $H_\lambda^{\pm}(\phi)$ as $H_\lambda^{\pm}(\phi) = (1-\lambda)H_0^{\pm}(\phi) + \lambda H^{\pm}(\phi)$, $0 \leq \lambda \leq 1$. Then $H_\lambda^{\pm}(\phi) = H_0^{\pm}$ when $\lambda = 0$ and $H_\lambda^{\pm}(\phi) = H^{\pm}(\phi)$ when $\lambda = 1$. Accordingly, we have the corresponding f_λ and g_λ. They are $f_\lambda(\phi) = (1-\lambda) f_0(\phi) + \lambda f(\phi)$ and $g_\lambda(\phi) = (1-\lambda) g_0(\phi) + \lambda g(\phi)$. As we can see, the corresponding two numbers are $\phi_L(\lambda)$ and $\phi_R(\lambda)$. Luckily, we have $\phi_L(\lambda) = \phi_L$ and $\phi_R(\lambda) = \phi_R$ for $0 \leq \lambda \leq 1$.

For the newly constructed coupling functions $H_\lambda^\pm$, we have new versions of (3), (6) and (7), respectively, i.e.,

$$\theta_i' = \omega + \sum_{j=1}^{m} \alpha_j^+ H_\lambda^+(\theta_{i+j} - \theta_i) + \sum_{j=1}^{m} \alpha_j^- H_\lambda^-(\theta_{i-j} - \theta_i), \tag{18}$$

$$\begin{aligned} \phi'_i = {}& \sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \phi_{i+k} \right) + \sum_{j=1}^{m} \alpha_j^- [f_\lambda - g_\lambda] \left(\sum_{k=1}^{j} \phi_{i-k+1} \right) \\ & - \sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \phi_{i+k-1} \right) - \sum_{j=1}^{m} \alpha_j^- [f_\lambda - g_\lambda] \left(\sum_{k=1}^{j} \phi_{i-k} \right), \end{aligned} \tag{19}$$

$$\begin{aligned} & \sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \phi_{i+k-1} \right) + \sum_{j=1}^{m} \alpha_j^- [g_\lambda - f_\lambda] \left(\sum_{k=1}^{j} \phi_{i-k+1} \right) \\ & \quad = \sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \phi_{i+k} \right) + \sum_{j=1}^{m} \alpha_j^- [g_\lambda - f_\lambda] \left(\sum_{k=1}^{j} \phi_{i-k} \right). \end{aligned} \tag{20}$$

We first prove a useful lemma.

Lemma 3.1. *In Eq.*(19), *if all the sums of ϕ_i in the form of $\sum_{k=1}^{j}$ are in J, then the Jacobian matrix of the right-hand side has only eigenvalues with negative real parts*.

The proof is to apply Lemma 3.1 and Lemma 3.2 in [4] to the system (18).

Theorem 3.1. *Assume* $\min(\phi_L, \phi_R) < 0 < \max(\phi_L, \phi_R)$. *If* $m\phi_L$ *and* $m\phi_R \in J$, *then the system* (6) *has asymptotically stable equilibrium* $\bar{\Phi} = (\bar{\phi}_1, \dots, \bar{\phi}_n)$ *and* $\min(\phi_L, \phi_R) < \bar{\phi}_i < \max(\phi_L, \phi_R), i = 1, \dots, n$. *Also* $\bar{\Phi}$ *is the unique equilibrium of* (6) *in the n-dimensional box* $I \times I \times \cdots \times I$ *where the interval* $I = [\min(\phi_L, \phi_R), \max(\phi_L, \phi_R)]$.

Proof. Without loss of generality, we assume $\phi_R < 0 < \phi_L$. For convenience, we denote the right-hand sides of (19) and (6) by $F_\lambda(\Phi)$ and $F(\Phi)$, respectively, where $F_\lambda, F : R^n \to R^n$. Then $F_\lambda(\Phi) = (1-\lambda)(B\Phi - S) + \lambda F(\Phi)$. Hence B and S are as in (12). The idea of the proof is to trace the homotopy path $\bar{\Phi}(\lambda)$, where $\bar{\Phi}(\lambda)$ is the solution of $F_\lambda(\Phi) = 0$, as λ varies from 0 to 1.

At $\lambda = 0$, $F_\lambda(\Phi) = B\Phi - S$. By Lemma 2.1, $F_\lambda(\Phi) = 0$ has a unique solution $\bar{\Phi}(\lambda) = \bar{\Phi}(0)$ such that $\phi_R < \bar{\phi}_i(0) < \phi_L$. Then the eigenvalues of the Jacobian matrix $DF_\lambda(0) = B$ have negative real parts by Lemma 3.1. So DF_λ is nonsingular. By the implicit function theorem, there exists $\lambda_0 \in (0, 1]$ such that $F_\lambda(\Phi) = 0$ has a solution $\bar{\Phi}(\lambda)$ with $\phi_R < \bar{\phi}_i(\lambda) < \phi_L$ for each $\lambda \in [0, \lambda_0]$. And $DF_\lambda(\bar{\Phi}(\lambda))$ has only eigenvalues with negative real parts by using Lemma 3.1 again.

Starting with λ_0, there exists $\lambda_1 \in (\lambda_0, 1]$ such that for each $\lambda \in (\lambda_0, \lambda_1]$, $F_\lambda(\Phi) = 0$ has a solution $\bar{\Phi}(\lambda)$ with $\phi_R < \bar{\phi}_i(\lambda) < \phi_L$. $DF_\lambda(\bar{\phi}(\lambda))$ has only eigenvalues with negative real parts. Keep iterating this process until the extension cannot be continued. Then we get $0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots$. The properties above hold for all λ_k. Since $\{\lambda_k\}$ is monotonically increasing and bounded above by 1, there is $\lambda^* \in [0, 1]$ such that $\lambda_k \to \lambda^*$ as $k \to \infty$.

We claim $\lambda^* = 1$. Suppose $\lambda^* < 1$ by contradiction. Then continuity tells us that $F_\lambda(\Phi) = 0$ has a solution $\bar{\Phi}(\lambda^*)$ such that $\phi_R \leq \bar{\phi}_i \leq \phi_L$. Then $DF_\lambda(\bar{\Phi}(\lambda^*))$ has only eigenvalues with negative real parts from Lemma 3.1 once more.

It can be verified that both $(\phi_R, \dots, \phi_R)$ and $(\phi_L, \dots, \phi_L)$ are not solutions of $F_{\lambda^*}(\Phi) = 0$. Otherwise we would have a contradiction.

We claim $\phi_R < \bar{\phi}_i(\lambda^*) < \phi_L, i = 1, \dots, n$. Suppose that there is $i_0 \in \{1, \dots, n\}$ such that either $\bar{\phi}_{i_0}(\lambda^*) = \phi_L > \bar{\phi}_{i_0-1}(\lambda^*)$ or $\bar{\phi}_{i_0}(\lambda^*) = \phi_L > \bar{\phi}_{i_0+1}(\lambda^*)$. If $m+1 \le i_0 \le n-m$, noting that

$$\sum_{k=1}^{j} \bar{\phi}_{i_0+k}(\lambda^*) \le \sum_{k=1}^{j} \bar{\phi}_{i_0+k-1}(\lambda^*),$$

$$\sum_{k=1}^{j} \bar{\phi}_{i_0-k}(\lambda^*) \le \sum_{k=1}^{j} \bar{\phi}_{i_0-k+1}(\lambda^*) \quad \text{for } j = 1, \dots, m.$$

At least one inequality is strict and $g'_\lambda \pm f'_\lambda > 0$ in J. Then by (20), we have

$$\begin{aligned} &\sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \phi_{i_0+k-1}(\lambda^*) \right) + \sum_{j=1}^{m} \alpha_j^- [g_\lambda - f_\lambda] \left(\sum_{k=1}^{j} \phi_{i_0-k+1}(\lambda^*) \right) \\ &\quad = \sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \phi_{i_0+k}(\lambda^*) \right) + \sum_{j=1}^{m} \alpha_j^- [g_\lambda - f_\lambda] \left(\sum_{k=1}^{j} \phi_{i_0-k}(\lambda^*) \right) \\ &\quad < \sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \phi_{i_0+k-1}(\lambda^*) \right) + \sum_{j=1}^{m} \alpha_j^- [g_\lambda - f_\lambda] \left(\sum_{k=1}^{j} \phi_{i_0-k+1}(\lambda^*) \right), \end{aligned}$$

which is a contradiction since the first and third lines are the same.

If $i_0 = 1$, then

$$\begin{aligned} &\sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \bar{\phi}_k(\lambda^*) \right) + \alpha_1^- [g_\lambda - f_\lambda](\bar{\phi}_1(\lambda^*)) \\ &= \sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \bar{\phi}_{k+1}(\lambda^*) \right) < \sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \bar{\phi}_k(\lambda^*) \right) \end{aligned}$$

such that $[g_\lambda - f_\lambda](\bar{\phi}_1(\lambda^*)) < 0$. Then $g_\lambda(\phi_L) < f_\lambda(\phi_L)$ since $\bar{\phi}_1(\lambda^*) = \phi_L$. This is a contradiction.

If $2 \le i_0 \le m$, then

$$\begin{aligned} &\sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \bar{\phi}_{i_0+k-1}(\lambda^*) \right) + \sum_{j=1}^{i_0} \alpha_j^- [g_\lambda - f_\lambda] \left(\sum_{k=1}^{j} \bar{\phi}_{i_0-k+1}(\lambda^*) \right) \\ &\quad = \sum_{j=1}^{m} \alpha_j^+ [f_\lambda + g_\lambda] \left(\sum_{k=1}^{j} \bar{\phi}_{i_0+k}(\lambda^*) \right) + \sum_{j=1}^{i_0-1} \alpha_j^- [g_\lambda - f_\lambda] \left(\sum_{k=1}^{j} \bar{\phi}_{i_0-k}(\lambda^*) \right), \end{aligned}$$

such that

$$\sum_{j=1}^{m}\alpha_j^+[f_\lambda+g_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0+k-1}(\lambda^*)\right)+\sum_{j=1}^{i_0}\alpha_j^-[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k+1}(\lambda^*)\right)$$
$$-\sum_{j=1}^{i_0-1}\alpha_{i_0}^-[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k}(\lambda^*)\right)$$
$$=\sum_{j=1}^{m}\alpha_j^+[f_\lambda+g_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0+k}(\lambda^*)\right)+\sum_{j=1}^{i_0-1}(\alpha_j^- -\alpha_{i_0}^-)[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k}(\lambda^*)\right)$$
$$<\sum_{j=1}^{m}\alpha_j^+[f_\lambda+g_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0+k-1}(\lambda^*)\right)+\sum_{j=1}^{i_0-1}(\alpha_j^- -\alpha_{i_0}^-)[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k+1}(\lambda^*)\right).$$

Then

$$\alpha_{i_0}^-[g_\lambda-f_\lambda]\left(\sum_{k+1}^{i_0}\bar\phi_{i_0-k+1}(\lambda^*)\right)-\sum_{j=1}^{i_0-1}\alpha_{i_0}^-[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k}(\lambda^*)\right)$$
$$\le -\alpha_{i_0}^-\sum_{j=1}^{i_0-1}[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k+1}(\lambda^*)\right),$$

i.e.,

$$\sum_{j=1}^{i_0}[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k+1}(\lambda^*)\right)\le\sum_{j=1}^{i_0-1}[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k}(\lambda^*)\right),$$

i.e.,

$$[g_\lambda-f_\lambda](\bar\phi_{i_0}(\lambda^*))+\sum_{j=2}^{i_0}[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k+1}(\lambda^*)\right)\le\sum_{j=1}^{i_0-1}[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k}(\lambda^*)\right),$$

i.e.

$$[g_\lambda-f_\lambda](\bar\phi_{i_0}(\lambda^*))+\sum_{j=1}^{i_0-1}[g_\lambda-f_\lambda]\left(\bar\phi_{i_0}(\lambda^*)+\sum_{k=1}^{j}\bar\phi_{i_0-k}(\lambda^*)\right)\le\sum_{j=1}^{i_0-1}[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k}(\lambda^*)\right).\qquad(21)$$

Since $\bar\phi_{i_0}(\lambda^*)=\phi_L>0$, then $[g_\lambda-f_\lambda](\bar\phi_{i_0}(\lambda^*))=0$ and $[g_\lambda-f_\lambda](\bar\phi_{i_0}(\lambda^*)+\sum_{k=1}^{j}\bar\phi_{i_0-k}(\lambda^*))>[g_\lambda-f_\lambda]\left(\sum_{k=1}^{j}\bar\phi_{i_0-k}(\lambda^*)\right)$, $j=1,\dots,i_0-1$. So (21) is not possible. We get a contradiction.

Similarly, if $m+1\le i_0\le n-m$, we get a contradiction. Therefore, there is no i_0 such that $\bar\phi_{i_0}(\lambda^*)=\phi_L$. This leads to $\bar\phi_i(\lambda^*)<\phi_L$, $i=1,\dots,n$.

By similar arguments, $\bar\phi_i(\lambda^*)>\phi_R$, $i=1,\dots,n$. So $\phi_R<\bar\phi_i(\lambda^*)<\phi_L$, $i=1,\dots,n$. Thus we can extend λ beyond λ^*. This is a contradiction. So $\lambda^*=1$. And $\bar\Phi(1)$ is the solution to $F(\Phi)=0$, which satisfies $\phi_R<\phi_i(1)<\phi_L$.

Now we need to prove uniqueness. Suppose $\bar\Phi$ and $\hat\Phi$ are two solutions to $F(\Phi)=0$ in $I^n=[\phi_R,\phi_L]^n$. Then by the mean value theorem, $0=F(\bar\Phi)-F(\hat\Phi)=\int_0^1 DF(\hat\Phi+\tau(\bar\Phi-\hat\Phi))(\bar\Phi-\hat\Phi)\,d\tau=[\int_0^1 DF(\hat\Phi+\tau(\bar\Phi-\hat\Phi))\,d\tau](\bar\Phi-\hat\Phi)$.

But $\int_0^1 DF(\hat{\Phi}+\tau(\bar{\Phi}-\hat{\Phi}))\,d\tau$ has only eigenvalues with negative real parts . It is nonsingular implying that $\bar{\Phi}-\hat{\Phi}=0$, i.e., $\bar{\Phi}=\hat{\Phi}$. □

Remarks.

1. *For nearest neighbor coupling (i.e.,* $m = 1$*), we proved in* [14] *the existence of an asymptotically stable equilibrium. We did not prove the uniqueness there. The uniqueness is automatically obtained from Theorem* 3.1.
2. *We note that along the homotopy path, we can prove*

$$\min_{1\le j\le n}(\bar{\phi}_j(\lambda)) < \bar{\phi}_i(\lambda) < \max_{1\le j\le n}(\bar{\phi}_j(\lambda)), \tag{22}$$

where $i = m+1,\dots,n-m$*. Furthermore numerical results show that* $\min_{1\le j\le n}(\bar{\phi}_j(\lambda))$ *only occurs at* $i = 1$ *or* n*. So does* $\max_{1\le j\le n}(\bar{\phi}_j(\lambda))$.

The condition

(C) $m\phi_{\rm L} \in J$ and $m\phi_{\rm R} \in J$

could be weakened for the same reason as in Section 2. Thus we introduce the following condition set:

(C1) $\phi_{\rm R} + \sum_{j=1}^{l}\bar{\phi}_{i+j}(\lambda) \in J$, $l = 1,\dots,\min(m-1, n-i)$ *for* $i = 1,\dots,n$.

(C2) $\phi_{\rm L} + \sum_{j=1}^{l}\bar{\phi}_{i+j}(\lambda) \in J$, $l = 1,\dots,\min(m-1, n-i)$ *for* $i = 1,\dots,n$.

(C3) $\sum_{j=0}^{l-1}\bar{\phi}_{i+j}(\lambda) + \phi_{\rm R} \in J$, $l = 0,\dots,\min(m-1, n-i)$ *for* $i = 1,\dots,n$.

(C4) $\sum_{j=0}^{l-1}\bar{\phi}_{i+j}(\lambda) + \phi_{\rm L} \in J$, $l = 0,\dots,\min(m-1, n-i)$ *for* $i = 1,\dots,n$.

As we see from the proof of Theorem 3.1, if the condition (C) holds, we have $\min(\phi_{\rm R},\phi_{\rm L}) < \bar{\phi}_i(\lambda) < \max(\phi_{\rm R},\phi_{\rm L})$ for each $\lambda \in [0,1]$. Hence the condition set (C1)–(C4) is satisfied for each $\lambda \in [0,1]$ along the homotopy path. We thus have the extension of Theorem 3.1.

Theorem 3.2. *Assume* $\min(\phi_{\rm R},\phi_{\rm L}) < 0 < \max(\phi_{\rm R},\phi_{\rm L})$*. If the solution in Lemma* 2.1*, which is* $\bar{\Phi}(\lambda)$ *at* $\lambda = 0$*, satisfies* (C1)–(C4)*, then there is a maximal* $\lambda^* \in (0,1]$ *such that for each* $\lambda \in [0,\lambda^*)$ *the solution* $\bar{\Phi}(\lambda)$ *satisfies* (C1)–(C4) *and* $\min(\phi_{\rm R},\phi_{\rm L}) < \bar{\phi}_i(\lambda) < \max(\phi_{\rm R},\phi_{\rm L})$*. If* $\lambda^* = 1$*, then* $\bar{\Phi}(1)$ *is an asymptotically stable equilibrium of* (6)*. It is unique in the region* $G = \{\Phi|\sum_{j=0}^{l}\phi_{i+j} \in J, l = 0,1,\dots,\min(m-1,n-i) \text{ for each } i = 1,\dots,n\} \subset R^n$*. Furthermore, G is a convex set.*

Remarks.

1. *The proof of Theorem* 3.2 *is to mimic each step in Theorem* 3.1*. The conditions* (C1)–(C4) *guarantee that all the summation terms of* $\bar{\phi}_i(\lambda)$ *in* (20) *stay in J such that* $g'_\lambda \pm f'_\lambda > 0$ *is insured.*
2. *For* $\lambda = \lambda^*$, $\bar{\Phi}(\lambda^*)$ *might not satisfy* (C1)–(C4)*. But by continuity, if we substitute J by* $\bar{J}$ *(i.e., the closed interval of the open interval J) in* (C1)–(C4)*,* $\bar{\Phi}(\lambda^*)$ *satisfies the modified* (C1)–(C4) *and* $\min(\phi_{\rm R},\phi_{\rm L}) < \bar{\phi}_i(\lambda^*) < \max(\phi_{\rm R},\phi_{\rm L})$*. So all the summation terms of* $\bar{\phi}_i(\lambda^*)$ *stay in J. Then the asymptotic stability of* $\bar{\Phi}(\lambda^*)$ *is also obtained by Lemma* 3.1*. Also* $DF(\bar{\Phi}(\lambda^*))$ *has only eigenvalues with negative real parts. Thus if* $\lambda^* < 1$*,* λ *still could be extended to an open neighbor* $(\lambda^*,\lambda^*+\delta)$ *such that* $\bar{\Phi}(\lambda)$ *is an asymptotically stable equilibrium of* (19) *for each* $\lambda \in (\lambda^*,\lambda^*+\delta)$*. This is done by applying the implicit function theorem.*
3. *Theorem* 3.2 *provides us a way to verify* (*even though the conditions are only sufficient ones*) *whether there is an asymptotically stable equilibrium of* (6) *in the convex domain G. This can be done using a numerical approach. We can partition the interval* $[0,1]$ *into L subintervals* $0 = \lambda_0 < \lambda_1 < \cdots < \lambda_{\rm L} = 1$ *such that* $\lambda_l = lh$ *where* $h = 1/L$*. It can be shown that if* $\lambda^* = 1$ *and h is small, then* $\bar{\Phi}(\lambda_l)$ *will be in the asymptotically convergent range of system* (19) *for* $\lambda = \lambda_{l+1}$*. Then we could take* $\bar{\Phi}(\lambda_l)$ *as an initial vector to solve the IVP* (19)*. In such a way, we can get* $\bar{\Phi}(\lambda_{l+1})$*. If* $\bar{\Phi}(\lambda_l)$ *does not satisfy* (C1)–(C4) *somewhere, we stop. Otherwise we continue until*

$\lambda_l = 1$. *One important thing is how to get* $\bar{\Phi}(0)$. *This is done by the iteration* (13) *which is convergent as we saw in Section* 2.

All the results we have obtained are for the identical oscillators, i.e., $\omega_i \equiv \omega$ (i.e., $\beta_i \equiv 0$). If ω_i are sufficiently close to each other, i.e., β_i is close to zero for each $i = 1, \dots, n$, we can apply the implicit function theorem to get asymptotically stable equilibria for the system (5).

Theorem 3.3. *If the conditions in Theorem* 3.2 *hold,* $\lambda^* = 1$ *and* β_i *is sufficiently close to zero, then the system* (5) *has an asymptotically stable equilibrium.*

Remark. *Theorem* 3.3 *is obtained from perturbing* β_i *from zero. It is reasonable to assume that the coupling strength between two oscillators far away is sufficiently small. Let* $m = m_1 + m_2$. *We assume* $\alpha_j^{\pm}$ *are very small for* $m_1 + 1 \le j \le m_1 + m_2$. *Then if the system* (5) *has an asymptotically stable equilibrium for the case when* $\alpha_j^{\pm} = 0$, $j \ge m_1 + 1$, *then* (5) *with sufficiently small* $\alpha_j^{\pm}$ $(j \ge m_1 + 1)$ *still has an asymptotically stable equilibrium. We will see this in our numerical results.*

4. Numerical results

In this section, we choose $H^{\pm}(\phi) = H(\phi) = 0.5\cos\phi + \sin\phi$. Then $f(\phi) = 0.5\cos\phi$ and $g(\phi) = \sin\phi$. Hence $\phi_{\mathrm{L}}, \phi_{\mathrm{R}}$ and the interval J can be determined. And $\phi_{\mathrm{L}} = \arctan(0.5)$, $\phi_{\mathrm{R}} = -\arctan(0.5)$ and $J = (-\arctan 2, \arctan 2)$.

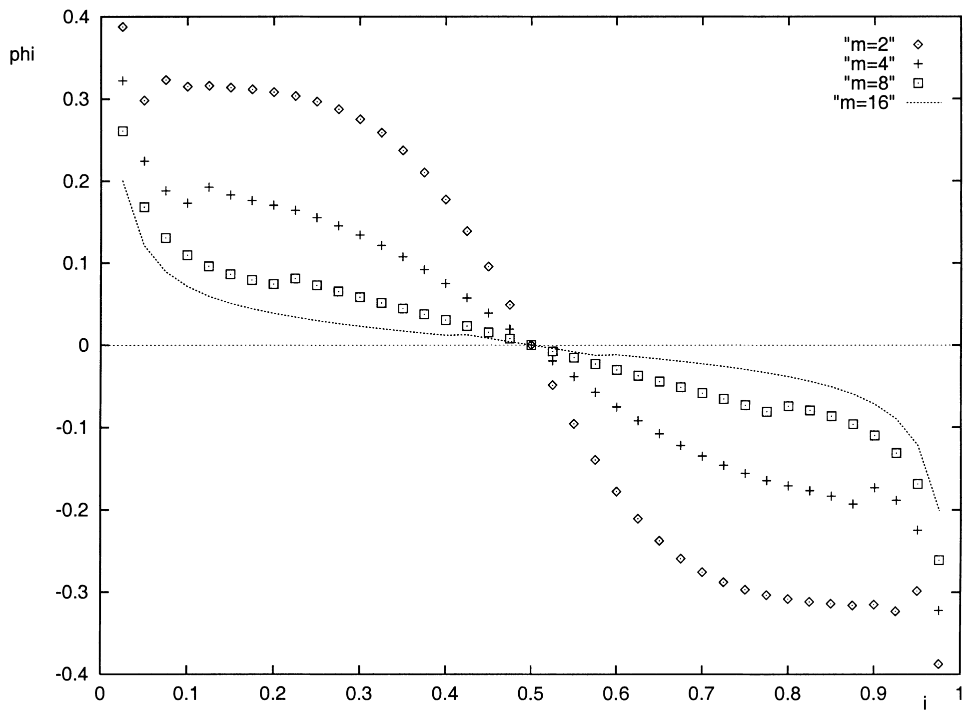

Fig. 1. $n + 1 = 40$, $\beta_i = 0$, the coupling strength sets are C_m and $m = 2, 4, 8$ and 16, respectively.

We choose two sets of coupling strengths for our numerical experiments. They are $C_m = \{\alpha_j^\pm = 1/j, j = 1, \dots, m\}$ and $E_m = \{\alpha_j^\pm = \exp(-j + 1), j = 1, \dots, m\}$ where $m = 1, 2, \dots$.

For both C_m and E_m, we always have $\alpha_1^\pm = 1 > \alpha_2^\pm > \cdots > \alpha_m^\pm > 0$. $\alpha_j^\pm$ are very small for $\alpha_j^\pm \in E_m$ if j is large, e.g. $j \geq 4$.

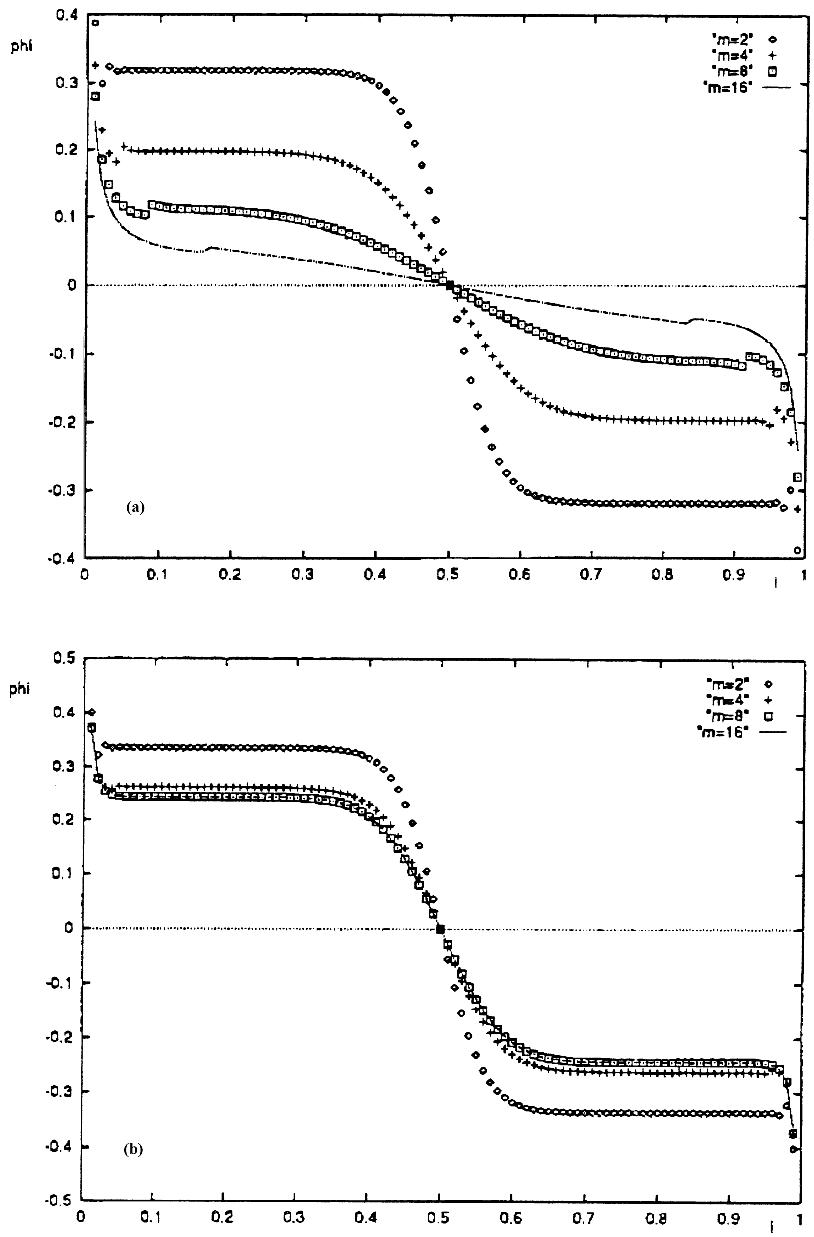

Fig. 2. $n + 1 = 100$, $\beta_i = 0$, the coupling strength sets are (a) C_m and (b) E_m with $m = 2, 4, 8$ and 16.

Remarks.

1. $\arctan(0.5) \approx 0.464$ *and* $\arctan 2 \approx 1.107$. *Then if* $m \geq 3$, *the condition* (C) *in Section* 3 *is violated. But as we see in the numerical results,* (C1)–(C4) *are still fine.*
2. *If we choose* $H = a\cos\phi + \sin\phi$, *then the smaller* $|a|$ *is, the larger m we can get to satisfy* (C). *For example, let* $a = 0.1$, *then m can be as large as* $m = 14$.

Fig. 1 shows the numerical results for the cases when $n+1 = 40$, $\beta_i = 0$ and C_m is the coupling strength set. Here we take $m = 2, 4, 8$ and 16 respectively. We plot $\bar{\phi}_i$ versus $i/(n+1)$. As we can see, $\bar{\phi}_i$ lie in the interval $(\phi_{\rm R}, \phi_{\rm L})$ and (C1)–(C4) ($\lambda = 1$) hold. These guarantee asymptotic stability. The figure shows that the inequalities (22) hold for $i = m+1, \dots, n-m$ and $\bar{\phi}_1 = \max_{i=1,\dots,n}\bar{\phi}_i$ and $\bar{\phi}_n = \min_{i=1,\dots,n}\bar{\phi}_i$. Also we find that $\bar{\phi}_i$ is monotonically decreasing when $i = m+1, \dots, n-m$. But on the two boundaries, i.e., $i \leq m$ or $i \geq n-m+1$, the monotonicity can be destroyed. This observation matches our comment that at the ends of the chain, the nonlinear averages have some portions lost (or gained). So $\bar{\phi}_i$ can fall below the nonlinear averages at the left end (except at $i = 1$) and $\bar{\phi}_i$ could jump above the nonlinear averages at the right end in this example.

In Fig. 2(a), $n+1 = 100$, $\beta_i = 0$ and C_m is the coupling strength set for $m = 2, 4, 8, 16$. In Fig. 2(b) $n+1 = 100$, $\beta_i = 0$ and E_m is the coupling strength set for $m = 2, 4, 8, 16$.

We can see from Figs. 1 and 2(a) and (b) that coupling with more oscillators will reduce the phase lags $\bar{\phi}_i$. That is the observation in [10] and it was explained in the case of piecewise linear coupling functions in Section 2.

In Fig. 2(b), the conditions (C1)–(C4) for $m = 8$ and $m = 16$ at $\lambda = 1$ are violated as we can see. But since $\alpha_j^{\pm}$ is small for $j \geq 5$, the comments in the end of Section 3 tell us that we still expect the existence of a stable solution. This is confirmed by the numerical results in Fig. 2(b).

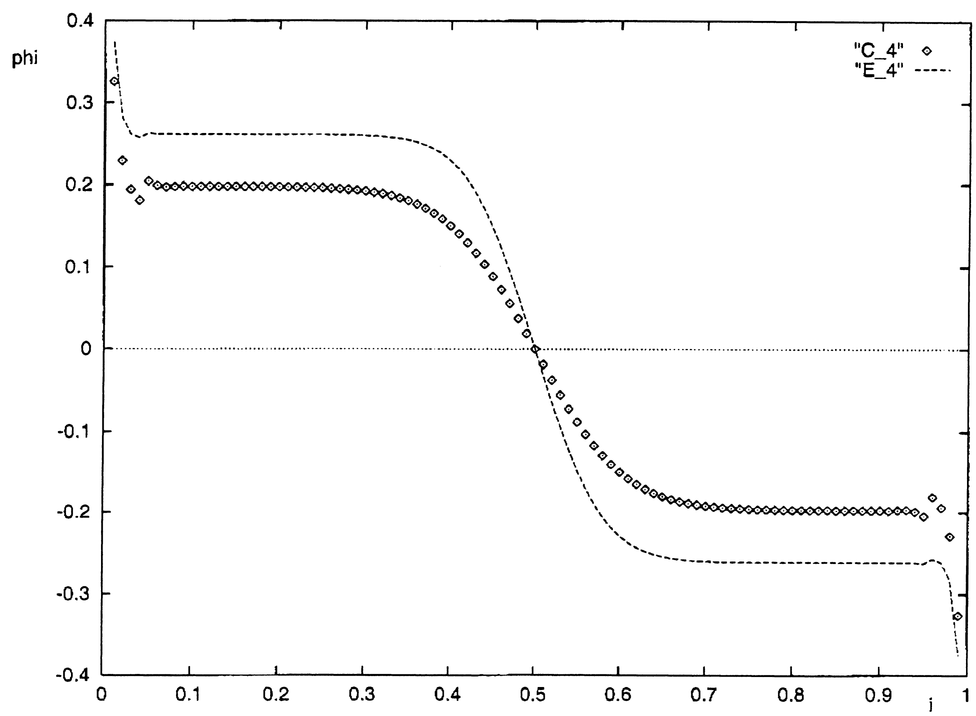

Fig. 3. $n+1 = 100$, $\beta_i = 0$, the coupling strength sets are C_m and E_m with $m = 4$.

We now show how different coupling strengths will affect the phase lags. This is done by comparing the results of E_m and C_m. Fig. 3 shows this for $m = 4$. It shows that strong coupling will reduce the phase lags $\bar{\phi}_i$. Note that $\exp(-j+1) < 1/j$ for $j = 2, \ldots, m$. This means the C_m type coupling is "stronger" than the E_m type.

In the following numerical experiments, we consider the non-isotropic cases when $\alpha_j^+ \in C_m$ and $\alpha_j^- \in E_m$, i.e., $\alpha_j^+ = 1/j$ and $\alpha_j^- = \exp(-j+1)$, $j = 1, \ldots, m$.

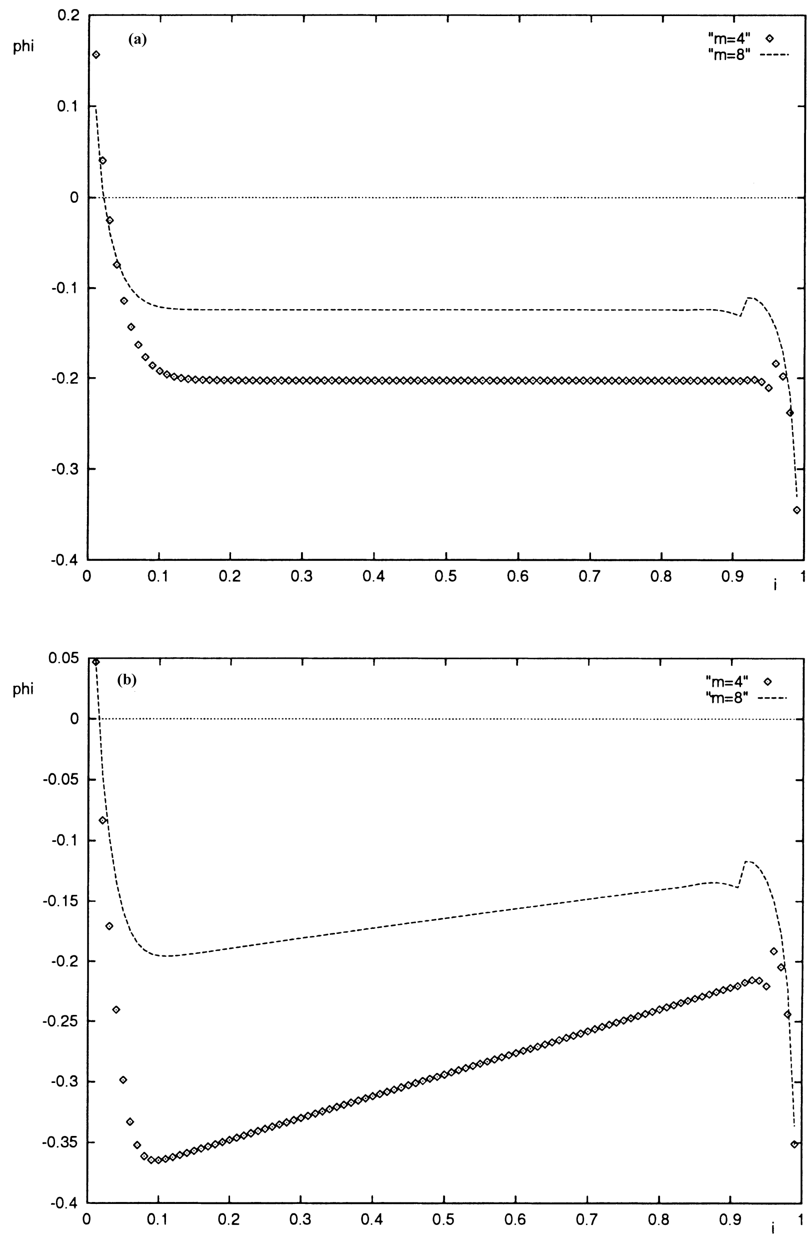

Fig. 4. $\alpha_j^+ = 1/j$ and $\alpha_j^- = \exp(-j+1)$, $n+1 = 100$, (a) $\beta_i = 0$, and (b) $\beta_i = -0.005$, and $m = 4, 8$.

In Fig. 4(a) $n + 1 = 100$ and $\beta_i = 0$ with $m = 4$ and 8. It confirms the results of Theorem 3.2. In Fig. 4(b) $n+1 = 100$ and $\omega_i = 0.5(n+1-i)/(n+1)+\omega$ (i.e., $\beta_i = -0.005$) where ω can be any positive constant. $m = 4$ and 8 are chosen.

In Fig. 5 $n + 1 = 100$ and $\omega_i = \omega + \delta_i$ where δ_i are randomly chosen from the interval $(0, 0.5)$. Thus β_i are chosen from $(-0.5, 0.5)$ randomly.

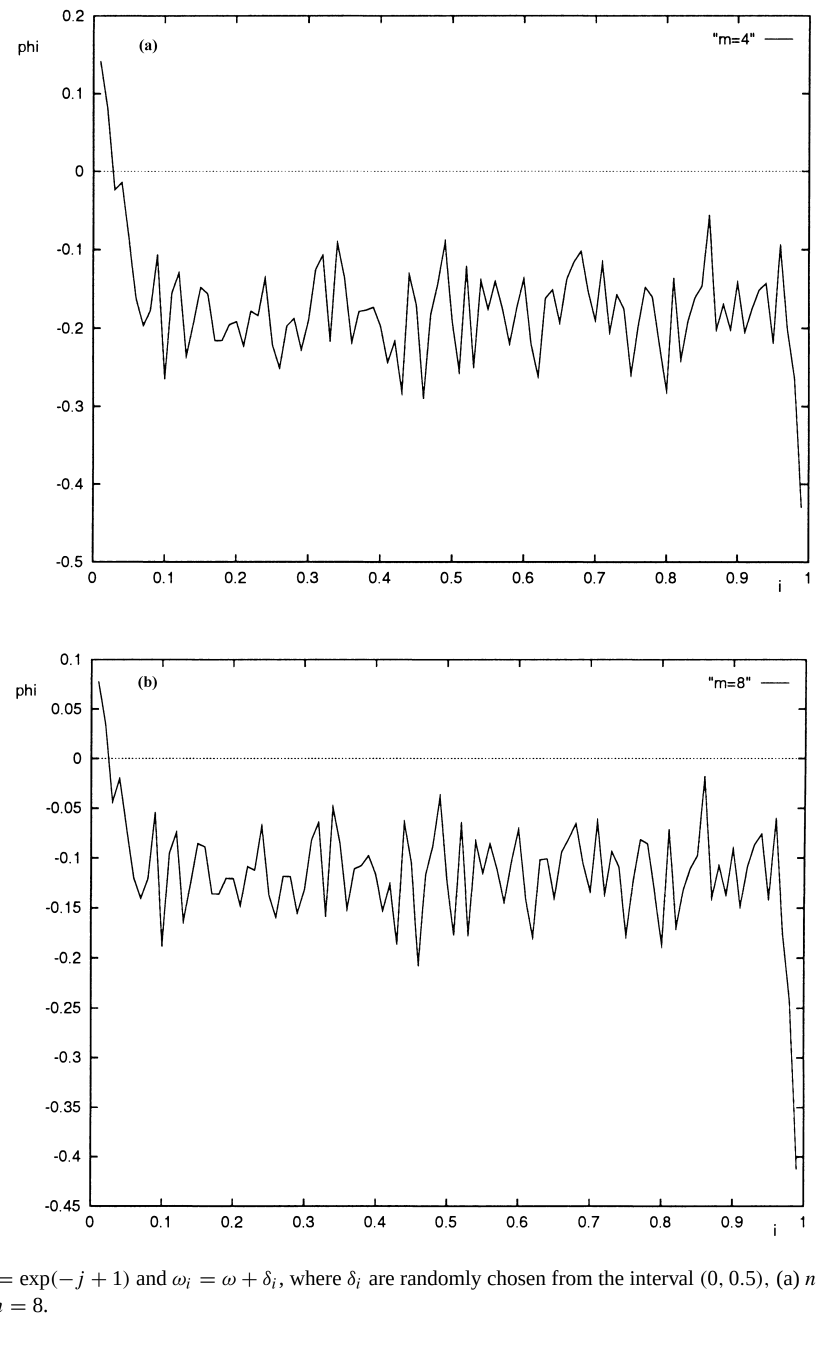

Fig. 5. $\alpha_j^+ = 1/j$, $\alpha_j^- = \exp(-j+1)$ and $\omega_i = \omega + \delta_i$, where δ_i are randomly chosen from the interval $(0, 0.5)$, (a) $n + 1 = 100$ and $m = 4$, (b) $n + 1 = 100$ and $m = 8$.

Figs. 4(b), 5(a) and (b) verify the results of Theorem 3.3. In the case of Fig. 4(b), there is a frequency gradient. This gradient is small so that ω_i stay close to each other. In the cases of Fig. 5(a) and (b), ω_i are chosen randomly and close to each other. Once more we mention that coupling with more oscillators will reduce the phase lags [10].

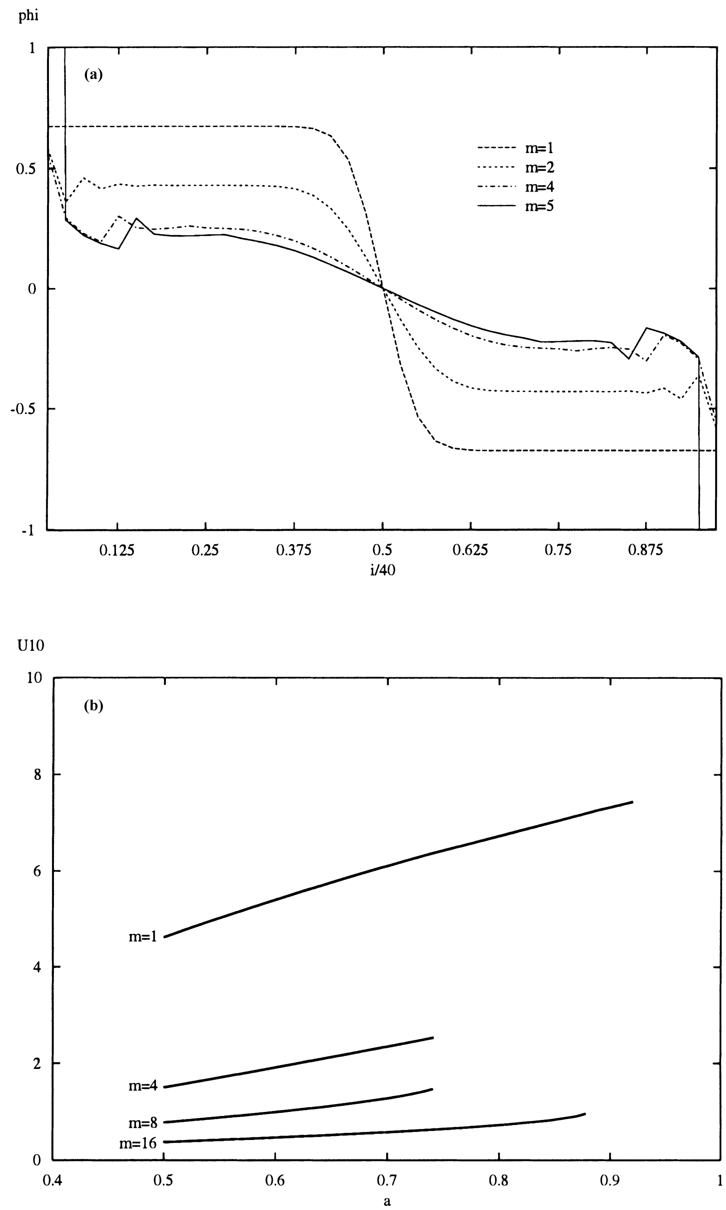

Fig. 6. (a) $H^{\pm}(\phi) = 0.8\cos\phi + \sin\phi$, coupling is $\alpha_j^{\pm} = 1/(2m+1)$, $n+1 = 40$ and $m = 1, 2, 4, 5$. For $m < 5$ there is stable phase locking. However, note the nonmonotonicity near the boundaries. For $m > 4$ oscillators at the edges "break away" and phase locking is lost. (b) Bifurcation diagram showing the range of existence of phase locking for $H^{\pm}(\phi) = a\cos\phi + \sin\phi$ as a function of the parameter a. Right endpoint is the maximum value of a for stable phase locking.

In the previous numerical simulations, it was shown that one of the effects of multiple coupling is to reduce the phase lags between successive oscillators. Thus, it would seem that increasing the coupling length encourages tighter phase locking. However, if the interval J becomes too short relative to the roots ϕ_L, ϕ_R, then it may be possible to achieve phase locking with short range coupling, but lose it with longer range coupling. Fig. 6 illustrates this. We first choose, $H^{\pm}(\phi) = 0.8\cos\phi + \sin\phi$. The interval J is now $(-0.896, 0.896)$ and $\phi_L = 0.674$, $\phi_R = -0.674$. For $m = 1$ this is still in the range for which we expect phase locking with monotonically varying phase shifts, ϕ_i, However, if m gets larger, there is no guarantee that there will be a locked solutions. In the figure, $m = 1, 2, 4$ all lead to phase locked solutions. Note the phase differences away from the edges are compressed, but the behavior near the edges oscillates. When $m = 5$, the oscillators at the ends "drift" away; they are no longer able to phase lock with the interior oscillators. However, as m continues to increase, stable locking can occur again. This is shown Fig. 6(b). Each curve represents the total phase lag, $\theta_{10} - \theta_1$ as a function of the parameter a where $H^{\pm}(\phi) = a\cos\phi + \sin\phi$. Note that for $m = 1$, the existence of phase locking extends to $a = 1$, while for $m = 4, 8$ it is considerably shortened. However, for $m = 16$ the range is again quite large. If $m = 40$ then coupling is "all-to-all" and synchrony is stable for any value of a thus we expect that the range of phase locking should be a nonmonotonic function of the connectivity, m. What is somewhat surprising is that the "worst" case for locking occurs at about $m = 10$ or connectivity over a quarter of the chain. The investigation of these phenomena remains an open problem.

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