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# A heuristic description of spiral wave instability in discrete media

Bard Ermentrout *

*Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260, USA*

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**Abstract**

A set of simple low-dimensional differential equations representing a chain of coupled oscillators is suggested as a model for explaining the instability of a rigidly rotating spiral wave that has been observed in spatially discrete and continuous models. Bifurcation to an invariant circle on a torus appears to be the mechanism responsible for this instability in the discrete case and is thus in agreement with Barkley's assertion of an instability due to a Hopf bifurcation.

## 1. Introduction

Spiral waves in active media have been the focus of a great deal of recent attention primarily due to their widespread occurrence in biology, chemistry, and physics [13]. Most of the current analysis deals with spatially continuous models which model chemical reactions and cardiac muscle ([8,9,14]). Barkley has numerically studied the stability of these spiral waves as various intrinsic parameters are varied [2]. He and others are interested in the behavior of the "core" which in some cases does not remain rigidly rotating but rather meanders (often chaotically) about the medium. He believes (as do most researchers of spiral waves) that the mechanism that leads to meander is a Hopf bifurcation of the rgidly rotating solution. Barkley's numerical analysis has led him to propose a simple normal form for this bifurcation. The normal form does an excellent job of explaining the behavior of the core as two distinguished parameters vary [3].

In this note, we wish to consider a simplified model for spiral waves that is spatially discrete but has many of the properties of the continuous model. In fact, it can be derived in some asymptotic limits from a spatial discretization of the continuous spiral wave models. The model is based on the idea that each "cell" in the medium is capable of undergoing a spontaneous oscillation and that these cells are coupled to their 4 neighbors on a two-dimensional lattice. This model was first proposed by Kuramoto [10] and later studied numerically

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|---|---|---|---|---|---|---|---|
| 0 | $Y_{11}$ | ..... | $Y_{1n}$ | $\frac{\pi}{2}-Y_{1n}$ | ..... | $\frac{\pi}{2}-Y_{11}$ | $\frac{\pi}{2}$ |
| $-Y_{11}$ | 0 | ⋱ | ⋮ | ⋮ | ⋰ | $\frac{\pi}{2}$ | $\frac{\pi}{2}+Y_{11}$ |
| ⋮ | ⋱ | 0 | $Y_{nn}$ | $\frac{\pi}{2}-Y_{nn}$ | $\frac{\pi}{2}$ | ⋰ | ⋮ |
| $-Y_{1n}$ | ..... | $-Y_{nn}$ | 0 | $\frac{\pi}{2}$ | $\frac{\pi}{2}+Y_{nn}$ | ..... | $\frac{\pi}{2}+Y_{1n}$ |
| $\frac{3\pi}{2}+Y_{1n}$ | ..... | $\frac{3\pi}{2}+Y_{nn}$ | $\frac{3\pi}{2}$ | $\pi$ | $\pi-Y_{nn}$ | ..... | $\pi-Y_{1n}$ |
| ⋮ | ⋰ | $\frac{3\pi}{2}$ | $\frac{3\pi}{2}-Y_{nn}$ | $\pi+Y_{nn}$ | $\pi$ | ⋱ | ⋮ |
| $\frac{3\pi}{2}+Y_{11}$ | $\frac{3\pi}{2}$ | ⋰ | ⋮ | ⋮ | ⋱ | $\pi$ | $\pi-Y_{11}$ |
| $\frac{3\pi}{2}$ | $\frac{3\pi}{2}-Y_{11}$ | ..... | $\frac{3\pi}{2}-Y_{1n}$ | $\pi+Y_{1n}$ | ..... | $\pi+Y_{11}$ | $\pi$ |

Fig. 1. Pattern of phases in a completely symmetric system.

by Sakaguchi and others [12]. It is these latter results that we wish to explore with particular attention on the loss of rigidly rotating waves as a certain parameter varies.

Let $H_\xi(\phi)$ be an $2\pi$-periodic function depending on a parameter, $\xi$, with $H_\xi(0)=0$ and such that $H_0(\phi)$ is odd. Let $X_{ij}$ denote the phase of an oscillator at position $(i,j)$ in a two-dimensional lattice. Consider the following set of equations:

$$\frac{dX_{ij}}{dt} = \omega_{i,j} + \sum_{i',j'} H_\xi(X_{i'j'} - X_{ij}), \tag{1.1}$$

where $(i'j')$ are the (up to) 4 nearest neighbors of the oscillator at $i,j$. We will assume that the lattice is $N \times N$ and that $N$ is an even number. This type of model arises naturally from more complicated kinetics if the coupling between cells is weak and diffusion-like. ([6,7]) The local frequency of each oscillator is $\omega_{i,j}$ and in absence of coupling each cell oscillates with a period of $2\pi/\omega_{i,j}$. If all of the frequencies are identical, then it is clear that one solution is $X_{i,j} = \omega t + C$ where $C$ is arbirary. This is the synchronous solution and it is asymptotically stable provided that $H'_\xi(0) > 0$ (see [5]). For $\xi$ sufficiently small, there are other solutions which are the analogues to rigidly rotating waves. The following theorem has recently been proven [11]:

*Theorem 1.* Suppose the lattice is $2m \times 2m$. Then, it can be viewed as a series of $m$ concentric rings of lengths 4, 12, ..., $8m-4$. If $H'_0(\phi) > 0$ for $\phi < \pi/2$ then for $\xi = 0$ there is a solution to (1.1) of the form:

$$X_{i,j} = \omega t + Y_{i,j} + C$$

such that the winding number around each ring is 1. (That is starting at one point in the "ring" and traversing it, the relative phase goes from 0 to $2\pi$.) Furthermore, this solution is asymptotically stable. (See Fig. 1)

Since the solutions are asymptotically stable, they persist under sufficiently small perturbations e.g., $H_\xi(\phi)$ for sufficiently small $\xi$. The examples that we will look at in detail are of the form $H_\xi(\phi) = H_0(\phi+\xi) - H_0(\xi)$. There are several effects of this parameter $\xi$ most notably, it induces a "twist" in the rotating wave so that it looks much more like a rotating spiral wave observed in active media. In Fig. 2a, a solution for a $30 \times 30$

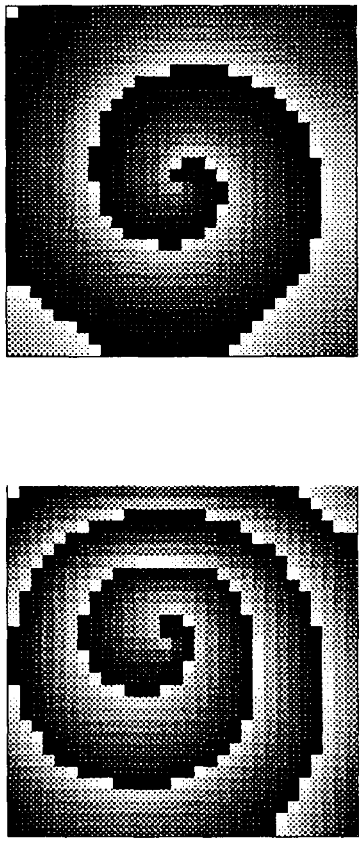

Fig. 2. The relative phase for a 30 × 30 array of oscillators for different values of $\xi$. (a) $\xi = \pi/6$ (b) $\xi = \pi/4$.

spiral is shown for $H_0(\phi) = \sin(\phi)$ and $\xi = \pi/6$. When $\xi = 0$ the equations are governed by a potential function, and the state of Theorem 1 is a local minimum. (The global minimum is the synchronous solutions, $X_{i,j} = C + \omega t$.) However, for $\xi \neq 0$ no potential function exists and the coupling is called "nonvariational" [12]. As $\xi$ increases, the spiral becomes more and more tightly wound until at a critical value, phase-locking is lost and the "core" of the spiral begins to wobble. Fig. 2b shows a snapshot of this for $\xi = \pi/4$. The "core" of the spiral is offset from the center and moves around the array. Note that the synchronous solution is still a stable solution since $H'(\xi) > 0$. This non-rotating appears to represent the bifurcation of a torus and thus leads to doubly periodic behavior.

Our goal is to understand the twist of the spiral and its dissolution as the parameter $\xi$ is increased. We also will try to shed insight onto the behavior once the rigidly rotating solution disappears. In the next section, we derive a simple one-dimensional chain model that quantitatively agrees with the behavior of the full two-dimensional system and gives a good prediction of the point of break-up as a function of the size of the domain. The reduced model also seems to work beyond the regime of phase-locking giving good quantitative agreement with quasi-periodic solutions of the full two dimensional array.

## 2. The model equations

Sakaguchi [12] suggests that the reason for the break-up of the spiral wave as the parameter $\xi$ increases is that the local frequency of the spiral wave in the center is sufficiently different from that of the edges so that it cannot phase-lock any longer. While, our model system has no intrinsic frequency differences, there are effective differences since the phase-differences in the "core" are much greater than the local phase-differences near the edges. Because of the asymmetry of the coupling (that is it is no longer odd) this induces a localized frequency difference between the core and outer areas of the spiral. Indeed, this is the discrete analogue of the localized curvature effects in continuous media that have been shown to be of importance by Keener and his collaborators [8].

We can make these analogies more precise by once again regarding the two-dimensional array as a collection of nested rings of oscillators. Referring to Fig. 1, we see that the $2m \times 2m$ array is comprised of $m$ rings: the inner or core has length 4, the next has length 12, and so on. Suppose for the moment that none of these rings is coupled to the others. Then, isolated, we have a system of $L_j$ differential equations for the $j$ different rings, where $L_j = 8j - 4$ is the length of the $j$th ring. (Here $j = 1$ is the "core" ring.) We note the following proposition:

*Proposition 1.* Consider a system of $L$ oscillators coupled in a ring with nearest neighbor coupling:

$$\frac{dY_k}{dt} = \omega + H_\xi(Y_{k+1} - Y_k) + H_\xi(Y_{k-1} - Y_k).$$

Here, we identify $L+1$ with 1 and 0 with $L$. Then the uniform travelling wave, $Y_k(t) = \Omega_L(\xi)t + \alpha(k-1)$ where $\alpha = 2\pi/L$ is a solution. Here $\Omega_L(\xi) = (\omega + H_\xi(\alpha) + H_\xi(-\alpha))$ is the frequency of the wave. A sufficient conditions for stability is that $H'_\xi(\pm\alpha)) > 0$.

The proof of existence follows from direct substitution and stability follows from Ermentrout ([4,5]). The key point to note is that the frequency, $\Omega_L(\xi)$, of the oscillation depends on the ring size, $L$. If $\xi = 0$ and thus $H$ is an odd function, this dependence is trivial as the frequency, $\Omega_L(0) = \omega$ is then independent of $L$. For $H_0(\phi) = \sin(\phi)$ the frequency of a ring of length $L$ is:

$$\Omega_L(\xi) = \omega + 2\sin(\xi)(\cos(2\pi/L) - 1). \tag{2.1}$$

The effect of the parameter, $\xi$, is to add dispersion to the waves; frequency differences are large as $\xi$ increases. If $\xi > 0$, the frequency of the small rings is less than large rings with the "core" ring of length 4 having the smallest frequency, $\Omega_{\text{core}} = \omega - 2\sin(\xi)$. For large domains, the frequency of the outermost ring is close to $\omega$.

In order to derive a heuristic dynamics for (1.1) that explains the behavior in the presence of twist we appeal to numerical simulations of these equations. We find that even when the rigidly rotating spiral no longer exists, the phase relations within the "rings" comprising the two-dimensional array remain close to those of the isolated ring. That is, there is a fair approximation to a regular ring. In Fig. 3 we depict the outer ring for a $6 \times 6$ array and the corresponding wave for a ring of length 20. While there is some discrepancy, the relative phases are very close to those for the uncoupled ring. This suggests that the phases within each "ring" are essentially locked into the simple traveling wave and the value of the phase at one of the diagonals sets the phase at all the remaining cells in the ring. Thus, we will approximate the solutions to the full array by solving a much smaller system of equations for the phases of the upper left diagonal.

Consider, now, a $2m \times 2m$ array which thus consists of $m$ nested rings. We label the phases of the upper left corner of each of the $m$ rings, with the phase of the innermost ring labeled $\theta_1$ and the outermost $\theta_m$.

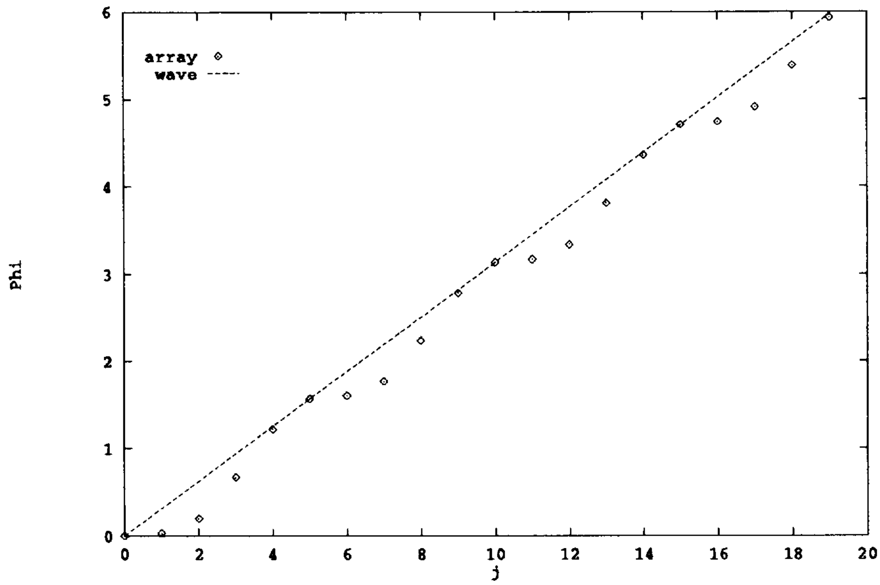


Fig. 3. Comparison of the relative phases around the edge of a 6 × 6 array with the travelling wave on a ring of the same length.

Once these are given, our simplified dynamics requires the each of the remaining phases within a given ring be slaved to this diagonal phase. Referring to the set of three consecutive rings illustrated in Fig. 4, we want to derive equations for $\theta_j$. In absence of coupling the $j$th ring has a frequency of $\Omega_{L_j}$. To derive the equations for phases, we average the effects over all interactions around the $j$th ring with its two neighbors. For notational convenience, we drop the subscript on $\theta_j$ and let $\theta^{\pm}$ denote $\theta_{j\pm 1}$. The three wave-numbers $\alpha, \gamma$, and $\beta$ denote the local phase differences within the uncoupled ring and are respectively, $2\pi/L_j, 2\pi/L_{j-1}$, and $2\pi/L_{j+1}$. The inputs from top row of the outer ring are:

$$H_\xi(\theta^+ + \beta - \theta) + H_\xi(\theta^+ + 2\beta - (\theta + \alpha)) + \ldots + H_\xi(\theta^+ + 2j\beta - (\theta + \pi/2)).$$

By symmetry, the total inputs are just 4 times this quantity as the sides and bottom of the rings are shifted by the same multiples of $\pi/2$. The inputs from top row of the inner ring are

$$H_\xi(\theta^- - (\theta + \alpha)) + H_\xi(\theta^- + \gamma - (\theta + 2\alpha)) + \ldots + H_\xi(\theta^- + \pi/2 - (\theta + (2j-2)\alpha)).$$

The sum of all these inputs must be divided by $L_j$, the length of the ring, to get the average.

Putting these together with the frequency information of each ring, we obtain a system of $m$ coupled nearest neighbor coupled oscillators with a frequency gradient:

$$\frac{d\theta_j}{dt} = \omega + H_\xi\left(\frac{\pi}{4j-2}\right) + H_\xi\left(-\frac{\pi}{4j-2}\right) + \frac{4}{L_j}\sum_{k=0}^{2j-1} H_\xi\left(\theta_{j+1} - \theta_j + \frac{\pi}{2}\frac{2(j-k)-1}{4j^2-1}\right) + \frac{4}{L_j}\sum_{k=0}^{2j-3} H_\xi\left(\theta_{j-1} - \theta_j + \frac{\pi}{2}\frac{2(k-j)+3}{(2j-3)(2j-1)}\right). \tag{2.2}$$

For $j = 1$ the inner contribution (second sum) is deleted and for $j = m$ the outer contribution (first sum) is deleted.

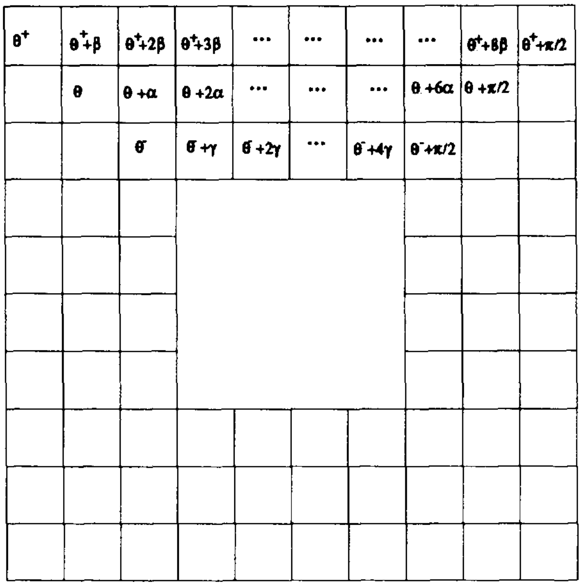


Fig. 4. Interactions between the rings comprising a two dimensional array.

This simplified dynamics is a substantial reduction in computation over the full two-dimensional array. Furthermore, it allows us to understand how the existence of a rigidly rotating wave is lost by studying a simpler well-known problem: the linear chain with a frequency gradient. Before continuing, we make several remarks. First, if $\xi = 0$ and the coupling is odd, then the local "frequency" of each ring is the same. Furthermore, $\theta_j = \omega t$ solves (2.2) since the terms $2(j-k)-1$ are antisymmetric about $k = j, j-1$ and so, since $H_0$ is odd, they will cancel. Thus, the "straight-armed" spiral is the solution for odd coupling functions. As $\xi$ increases, the frequencies begin to change and the phases will disperse. For small enough values of $\xi$ we expect there to remain a phaselocked solution; that is one for which $\theta_j = At + z_j$ for $A$ a constant (the ensemble frequency) and $z_j$ independent of time. A phaselocked solution to (2.2) thus corresponds to a rigidly rotating spiral wave for the two-dimensional medium. As $\xi$ increases even more, it is possible that no locked solutions exist for this chain.

Generally, the sums in (2.2) cannot be evaluated in closed form, but if $H_\xi$ consists of only a few trigonometric terms, then we can sum these. In particular, we will set $H_\xi(\phi) = \sin(\phi + \xi) - \sin(\xi)$. Then, (2.2) becomes:

$$\frac{d\theta_j}{dt} = \omega + \sin\xi\, C_j$$
$$+A_j\left(\sin(\theta_{j+1} - \theta_j + \xi) - \frac{2j}{2j-1}\sin\xi\right) + B_j\left(\sin(\theta_{j-1} - \theta_j + \xi) - \frac{2j-2}{2j-1}\sin\xi\right), \qquad (2.3)$$

where

$$C_j = 2(\cos[\pi/(4j-2)] - 1),$$
$$A_j = \frac{\cos[\pi/2(2j+1)] - \cos[\pi/2(2j-1)]}{(2j-1)(1-\cos[\pi/(4j^2-1)])},$$
$$B_j = \frac{\cos[\pi/2(2j-1)] - \cos[\pi/2(2j-3)]}{(2j-1)(1-\cos[\pi/(2j-1)(2j-3)])}.$$

The parameter $C_j$ defines the frequency effect due to the ring-diameter. It is strictly negative and monotonically increases from $C_1 = -2$ to zero as $j$ increases. Thus, for $\xi > 0$, the "natural" frequency of the inner ring is less than that of the outer rings. The coefficients $A_j$ and $B_j$ respectively describe the strength of coupling of the outer $(j+1)$ and inner rings $(j-1)$ to the reference ring $(j)$. $A_j > 1$ and monotonically decreases from $A_1 \approx 1.297$ to 1 as $j$ increases. $B_j$ increases monotonically from a value $B_2 = 1/\sqrt{3}$ to 1 as $j$ increases. Thus, the inner ring has a lesser effect on the phase lags than does the outer ring.

The diminished influence of the inner rings on the outer ones as well as the large intrinsic frequency gradient near the "core" ring suggest that as $\xi$ increases, if phaselocking is lost, then it will be the center ring of oscillators that slips from the remaining ones. Since the chain of oscillators with a gradient is not explicitly solvable if the coupling is not odd, we must resort to numerical simulations of the reduced model.

In Figs. 5a,b, we compare the behavior of a $10 \times 10$ array of oscillators and the $m = 5$ version of (2.2) as $\xi$ increases. for simplicity, we have plotted only the phase difference between the core and outer oscillators in (a) and the difference between the outer and the third ring in (b). The agreement is very good and the predicted value of the loss of phaselocking in the chain (near $\xi = 0.55$) is very close to the computed value in the array (about $\xi = 0.58$.) The behavior of the array once locking is lost consists of a slow drift of the core phases and periodic modulation around their "rest states" of the remaining oscillators in the array. This is precisely what happens in the chain. Fig. 6 shows the projections of the relative phases of the third and fourth diagonal entries for the reduced model and for the full array when $\xi = 0.6$. The magnitude and general shape of the oscillations are quite similar even though a path around one of the rings is far from the regular travelling wave. The transition from a stable rigidly rotating spiral to the quasi-periodic solutions appears to be via the following mechanism. The first oscillator (representing the core) breaks away from the remaining ones thus giving an invariant two-dimensional torus on the full (in this case) 5 dimensional torus. This is in qualitative agreement with the behavior of spiral waves in an excitable continuous medium where it has been suggested that the meander of the core is due to a Hopf bifurcation. Since the Hopf bifurcation will not generally have a frequency commensurate with the natural rotation of the spiral, the result of these two periodic motions is a two dimensional invariant torus. This bifurcation can be more readily understood if one considers the four dimensional torus formed by the phase-differences between successive oscillators. The phaselocked solution is then a stable fixed point for this 4 dimensional system. As $\xi$ increases, this fixed point merges with another unstable fixed point and disappears leaving in its wake a one dimensional invariant circle (see [6]). The frequency of rotation on this circle is proportional to $\sqrt{(\xi - \xi^*)}$. Thus, in the full five-dimenssional system, we have the appearance of a two-torus. This bifurcation is described in detail in [6] for a chain of oscillators with a gradient in frequency. The simple chain model we have described has the property that the largest frequency change occurs in the first or "core" ring so that we expect in general that this will break away from the others (via the torus bifurcation described above) once locking is lost. The consequence of this is that only the core oscillators drift and the remaining ones stay relatively close to their phase-locked values.

One can simplify (2.2) even further by ignoring the $\alpha_j$ terms thus eliminating the sums:

$$\frac{d\theta_j}{dt} = \omega + H(\alpha_j) + H(-\alpha_j) + \frac{L_{j+1} - 4}{L_j} H(\theta_{j+1} - \theta_j) + \frac{L_j - 4}{L_j} H(\theta_{j-1} - \theta_j). \tag{2.4}$$

The dotted line in Fig. 5a, shows the behavior of this compared to (2.2) And the array at the innermost ring. This simplification is not very good and consistently underestimates the relative phase (i.e., the simple chain remains more tightly coupled than the array and (2.2). The reason for this is that the sums in (2.2) and the actual coupling within the array result in a smaller coupling magnitude than the simple scaled coupling in (2.4). Thus, the frequency gradient results in a much larger phase gradient in the more "weakly" coupled models.

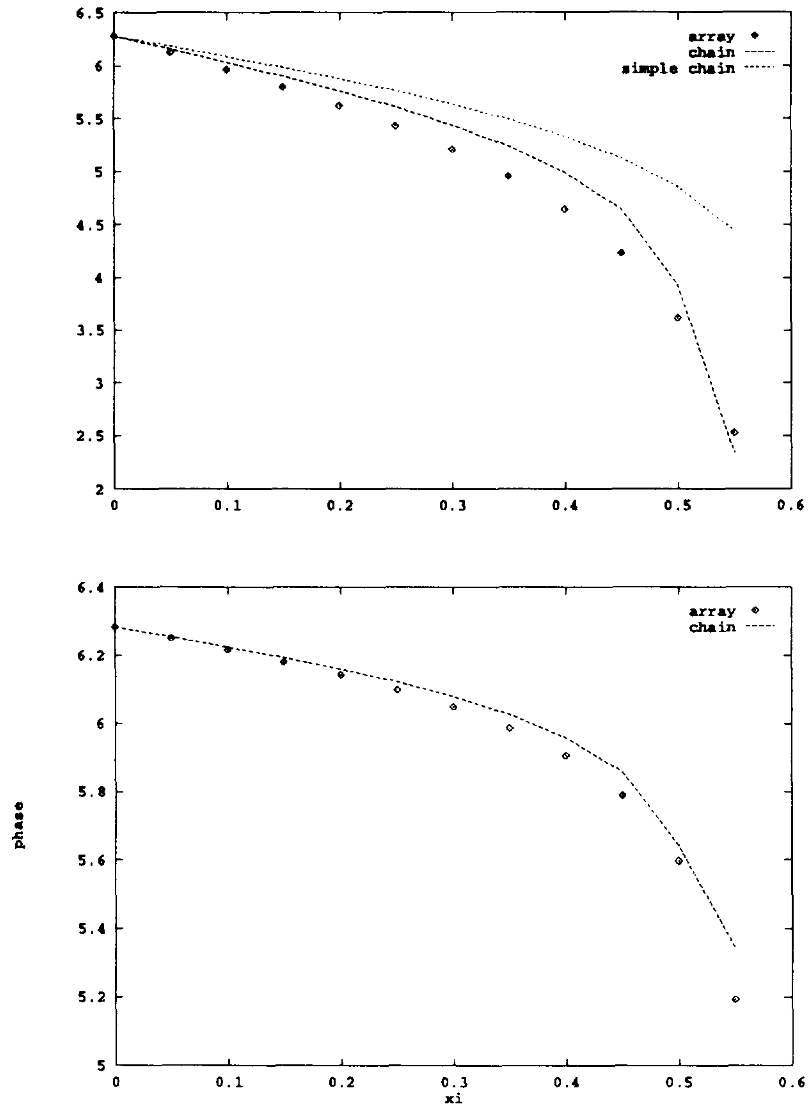


Fig. 5. Comparison of the chain model (2.2) with the behavior of the full 10 × 10 array as a function of the twist $\xi$. (a) Comparison of the core ring, the reduced chain model, and a simpler chain model (2.4). (b) Comparison of the third ring with the reduced chain model.

## 3. Discussion

We have proposed a simple reduction theory in order to explain the behavior of a two-dimensional array of coupled oscillators. The appearance of a meandering core in the evolution of spiral waves in this simplified model appears to occur through the analogue of the Hopf bifurcation of the invariant circle corresponding to a rigidly rotating spiral. Barkley suggests that a normal form based on the interaction of a Hopf bifurcation with translation of the spiral tip suffices to explain the meandering of the core as well as the oscillatory modulation of the spiral wave. Since the present model is a system of phase equations, the analogue of a Hopf bifurcation is the appearance of a stable invariant circle on the torus of relative phases. One apparent difference between our

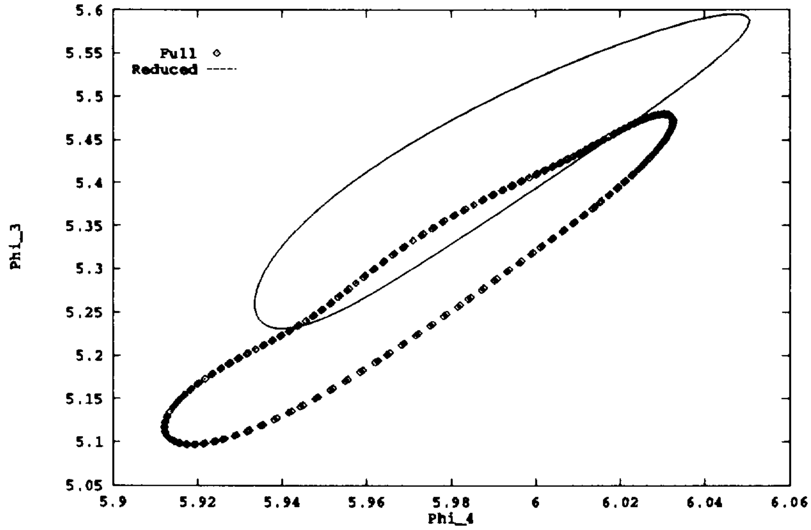


Fig. 6. Projection of the relative phases for $\xi = 0.6$ beyond the regime of rigidly rotating waves. The reduced chain model and the full $10 \times 10$ array are shown.

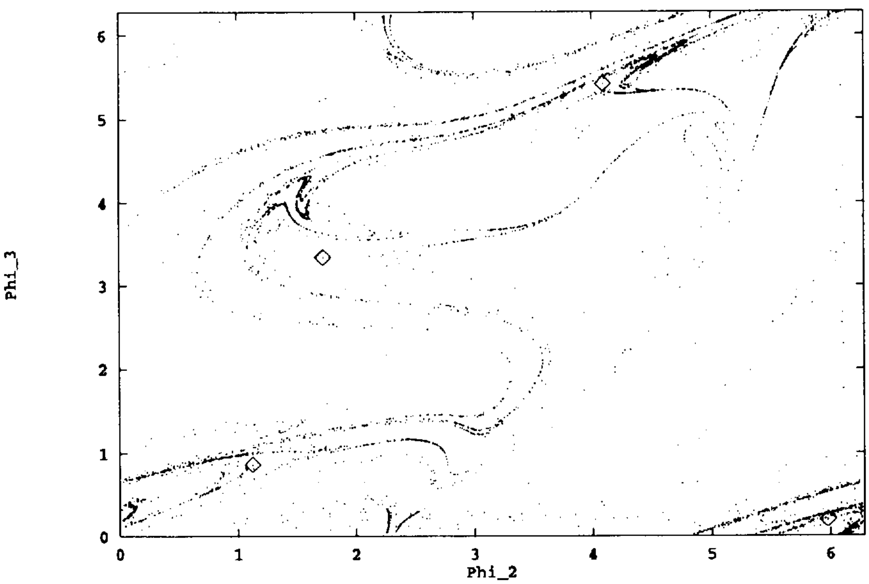


Fig. 7. Poincaré section of the reduced chain for an $8 \times 8$ array of oscillators for $\xi = 1.4$. The period 4 soluiton is shown by the diamonds.

model and that of Barkley's is that the onset of core modulation occurs with infinite period. This phenomena is true for both the full array and for the chain reduction. Ermentrout and Kopell [6] prove that such a bifurcation occurs in a chain of oscillators with a frequency gradient. In the present case, the frequency gradient is imposed through the dispersive properties of the oscillatory waves.

The reduced model may additionally shed some light on the complex behavior seen in two-dimensional arrays for large values of $\xi$. The entire domain of oscillators breaks up and no spatially coherent structures

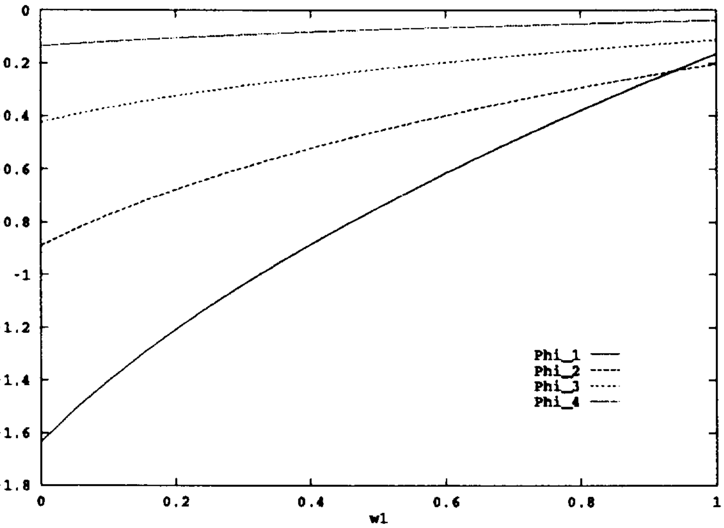


Fig. 8. The relative phases as a function of the core frequency.

seem to exist. For example, in an $8 \times 8$ array of oscillators, the chain of relative phases is a dynamical system on a three-dimensional torus. This is the minimal size of a chain which can exhibit aperiodic behavior. We find that for this chain, at $\xi \approx 0.55$ the locked solution disappears and there is a periodic solution satisfying $\phi_1(t+T) = \phi_1(t) + 2\pi$ and $\phi_j(t+T) = \phi_j(t)$ for $j = 2, 3$. Thus, only the "core" traverses all phases. Continued increases in $\xi$ cause the periodic solution to evolve. Letting, $\phi_1 = \pi$ be a Poincaré section, the periodic solutions $\phi_j(t)$ become fixed points. At $\xi \approx 1.18$ the fixed points become period 4 points. This period 4 behavior persists until $\xi \approx 1.4$ at which point, the behavior suddenly becomes aperioidic. In Fig. 7, we show the behavior of the Poincaré map in the $\phi_2$–$\phi_3$ plane along with the period 4 solution. The behavior is reminiscent of the toral chaos described by Baesens, et al. [1] in a system of three coupled oscillators. Whether the transition to chaos for the reduced chain model is relevant to the breakdown of the two-dimensional array remains to be seen. Numerical computations for the full array are delicate; there are large regimes for the parameter $\xi$ for which the medium, after a long complicated transient, synchronizes. Synchrony in the chain implies that there is a rigidly rotating spiral and cannot occur unless $\xi = 0$.

We finally mention the numerical work of Sakaguchi, et al. They study (1.1) with $H(\phi)$ as above. They obtain similar numerical results and suggest that the mechanism for the twist of the rotating wave is due to the differences in frequency of the "core" oscillators from the edge oscillators. They then increase the local frequency of the core oscillators and "straighten" out the rotating wave. In fact, the frequency change is only part of the picture as the coupling is between rings is also inhomogeneous. But the main effect of *xi* is on the dispersion. We can mimic the Sakaguchi results easily with the chain reduction. Consider the reduction of the $10 \times 10$ chain with $\xi = 0.45$. This is close to the critical value at which the spiral breaks up. As we noted above, changing $\xi$ results in a steep decrease in the local frequency of the core ring of oscillators. Thus we can increase the local frequency of the core oscillator and examone the relative phases. In Fig. 8, we show the four relative phases as a function of the applied core frequency. Note how the phases tend toward zero which corresponds to the "straight-armed" spiral. For large values of $\xi$, e.g., $\xi = 0.65$ the spiral and the chain no longer admit phase-locked solutions. However, adding a small frequency term, $\omega_1$ to the core oscillator (for $\xi = 0.65$, $\omega_1 \approx 0.5$ ) enables the system to phase-lock once again and leads to rigidly rotating waves.

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