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OSCILLATOR DEATH IN SYSTEMS OF COUPLED NEURAL OSCILLATORS*

G. B. ERMENTROUT† AND N. KOPELL‡

Abstract. Phase-locking in a system of oscillators that are weakly coupled can be predicted by examining a related system in which the coupling is averaged over the oscillator cycle. This fails if the coupling is large. It is shown that in the presence of large interactions, a pair or a chain of oscillators may develop a new stable equilibrium state that corresponds to the cessation of oscillation. This phenomenon is robust for neural type interactions and does not happen in systems that are weakly coupled.

Key words. neural oscillators, coupled oscillators, neural networks

AMS(MOS) subject classifications. 34, 92

1. Introduction. Much of the analytical work on coupled oscillators deals with interactions that depend only on the differences of the phases [1]-[7]. (Using averaging theory [8], this includes more general work on weakly coupled oscillators.) A simple example of two oscillators coupled in such a way is

$$\frac{d\theta_1}{dt}=\omega_1+H_1(\theta_2-\theta_1), \qquad \frac{d\theta_2}{dt}=\omega_2+H_2(\theta_1-\theta_2) \tag{1.1}$$

where θ_i is the phase of the ith oscillator, ω_i is its uncoupled frequency, and H_i is a 2π-periodic, scalar-value function. For generic H_1, H_2 the behavior of (1.1) is very simple. If ω_1 and ω_2 are sufficiently close, the system "phase-locks," i.e., it has a stable periodic solution for which $\phi \equiv \theta_2-\theta_1$ is independent of time. For $|\omega_2-\omega_1|$ sufficiently large, the system drifts, with the faster oscillator processing the slower one [4] (see § 2.2 for further discussion of this).

This paper is concerned with several related observations:

(1) Models of coupled neural oscillators, even when they are reduced to phase models, have interactions that do not in general depend only on the differences of the phases. (We use the word "neural" to refer to *chemical* synaptic coupling, as opposed to electrotonic.)

(2) For more general coupling of oscillators (modeled by phase or by both phase and amplitude), there is a much larger repertoire of possible behavior. In particular, for sufficiently large interactions, the coupling can act to suppress the oscillation and lead to a stable steady state for the coupled system. Since none of the component oscillations need have a stable steady state, this phenomenon is inverse to the observation investigated by Turing [9], Smale [10], and Segel and Jackson [11] that nonoscillating systems, when coupled, can develop a stable oscillation.

(3) The disappearance of the stable limit cycle and its replacement by a stable critical point is not restricted to *pairs* of oscillators. It also happens to chains of oscillators coupled to nearest neighbors. Indeed, the phenomenon was discovered

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during an investigation of chains of oscillators modeling neural networks governing undulating locomotion [6], [12].

This investigation is motivated by the study of neural oscillators [6], [12]-[15], and the mechanism that we will discuss depends heavily on the assumption that the coupling is not equivalent to coupling by diffusion. In this mechanism, oscillator death turns out to come from a lack of uniformity of the local frequency along the limit cycle of the coupled system. That is, as oscillator death is approached (when some parameter is changed), the coupling causes the system to slow down near some point, and the frequency goes to zero. The mechanism contrasts with others that produce a death of the oscillator, including one that, unlike ours, requires very different frequencies among the coupled oscillators [7], [16], and one in which two coupled oscillators lock to equilibrium points in antiphase [17], [18].

The paper is organized as follows. Section 2 is concerned with oscillator death for phase equations of the form

$$\frac{d\theta_k}{dt} = \omega_k + h_k(\theta_k, \theta_j), \qquad j, k = 1, 2, \quad j \neq k. \tag{1.2}$$

We give emphasis to coupling functions $h_k(\theta_k, \theta_j)$ of the form $P(\theta_j)R(\theta_k)$ where P is a pulse-like function and R plays a role analogous to that of a "phase response curve," and we show how such equations are derived from assumptions about neural oscillators. In the Appendix, a numerical method is given for computing approximations to the functions $P(\theta)$ and $R(\theta)$ and applied to a pair of well-known models for neural oscillators that are not reduced to phase equations.

In § 3, we consider the latter nonphase models. We show, using analytical and numerical methods, that oscillator death also occurs in nonphase models. We also compare our mechanism of oscillator death to those of Aronson et al. [7], [16] and Crowley and Epstein [17].

Section 4 deals with chains of oscillators, and we show that the length of the chain has little effect on how easily the rhythmicity can be killed; i.e., there is no scaling effect. This result is obtained using procedures analogous to monotone methods for parabolic partial differential equations (PDEs). Section 5 contains a short discussion, including comments on the relevance of these models to the behavior of real chemical synapses which may have fatigue or delays. We also discuss the relation of our hypotheses to the notion of "synaptic coupling" used in previous papers [6], [12]-[14].

In a succeeding paper [15], we will discuss how neural networks can be designed so that the interactions of oscillating neurons, even if strong, do not cause a stoppage of the oscillations.

2. Phase equations for pairs of oscillators. We now show that equations of the form (1.2) in general undergo oscillator death if the coupling is sufficiently strong. We start by discussing some special cases that have particular significance for the coupling of neural oscillators.

2.1. Pulse-response coupling. Assume that the oscillators are given by

$$\frac{du_k}{dt} = F_k(u_k), \qquad k = 1, 2 \tag{2.1}$$

where $u_k \in R^n$ has an asymptotically stable limit cycle $U_k(t)$. We consider oscillators for which the attraction to the limit cycle is very strong. We can choose variables $\theta_k \in S^1$, and $y_k \in R^{n-1}$ such that θ_k parameterizes the limit cycle of (2.1) and y_k are

normal coordinates in a neighborhood (so $y_k = 0$ along the limit cycle). If $J_k(\theta, y)$ is the Jacobian matrix of this transformation, then (2.1) may be written as

$$\frac{d\theta_k}{dt} = \omega_k + O(\|y_k\|), \qquad \frac{dy_k}{dt} = a_k(\theta_k) \cdot y_k + o(\|y_k\|) \tag{2.2}$$

where the right-hand sides of (2.2) are the θ_k and y_k coordinates, respectively, of $J_k^{-1}(\theta_k, y_k)F_k(\theta_k, y_k)$.

Suppose now that we consider a pair of coupled oscillators

$$\frac{du_k}{dt} = F_k(u_k) + G_k(u_k, u_j), \qquad j \neq k. \tag{2.3}$$

In the coordinates described above, the set of coupled equations has the form

$$\begin{aligned} \frac{d\theta_k}{dt} &= \omega_k + h_k(\theta_k, \theta_j) + O(\|y_k, y_j\|), \\ \frac{dy_k}{dt} &= a_k(\theta_k) \cdot y_k + d_k(\theta_k, \theta_j) + O(\|y_j\|) + o(\|y_k\|), \end{aligned} \tag{2.4}$$

$j, k = 1, 2$ and $j \neq k$. Here a_k depends only on F_k, and h_k and d_k depend on the interaction terms G_j, G_k as well. For the product system (no coupling), there is a stable invariant torus that is the product of the limit cycles. By [19], this torus persists after perturbation. The amount of perturbation that can be permitted without loss of the invariant torus grows with the degree of attractivity of the limit cycles of (2.1) [19]. If there is no small parameter in the problem, the existence of the invariant manifold is not guaranteed, but [19] shows that an order one amount of coupling can be tolerated. If the rate of attraction to the limit cycle is very strong compared with the amount of coupling, the normal variables y_k stay small, and the θ component of solutions to (2.4) can be approximated by solutions to (1.2). (Even if the rate of attraction is not so strong, as long as there is an invariant torus the equations on it have the form (1.2), where the functions h in (2.4) and (1.2) need not be identical.)

Suppose that only one component of the coupling vector G_k is nonzero, and that this component is a product of a function of u_k times a function of u_j. It follows from the previous derivation that $h_k(\theta_k, \theta_j)$ has the form $P(\theta_j)R(\theta_k)$. This is true of Hodgkin–Huxley-like neural models in which the coupling is only via the voltage (see § 3). Thus we see that there is a natural derivation of the equations

$$\frac{d\theta_1}{dt} = \omega_1 + \alpha P(\theta_2)R(\theta_1), \qquad \frac{d\theta_2}{dt} = \omega_2 + \alpha P(\theta_1)R(\theta_2), \tag{2.5}$$

first proposed by Winfree [20]. We use (2.5) as a convenient subclass to illustrate more general results. In neural models, the function $P(\theta)$ comes from a conductance, and hence is nonnegative, while $R(\theta)$ can take both signs (see the Appendix).

It is instructive to understand the relationship of (2.5) to "phase response curves" (PRCs). For a given oscillator and a given brief stimulus, such a curve is experimentally determined by stimulating the oscillator, and waiting until the system relaxes back to its oscillation, except for a shift in phase. The PRC is the phaseshift $\bar{R}(\theta)$ that depends on the phase θ at which the stimulus was administered. If the relaxation time is short relative to the time between stimuli, the PRC may be used to find out for which frequencies of perturbation by the given stimulus the oscillator can be entrained. It

also gives information about the phase differences between the forced and forcing oscillators at different frequencies of forcing [21].

Let θ_m be the phase just before the mth stimulus, and assume that the relaxation time is short, as above. Then the $\{\theta_m\}$ are related by

$$\theta_{m+1} = \omega T + \theta_m + \bar{R}(\theta_m) \tag{2.6}$$

where ω is the natural frequency of the forced oscillator and T is the time between stimuli. (See [22] for work on (2.6), which is capable of very complex behavior, especially when $\bar{R}$ is large.) The difference equation (2.6) may be rewritten as a differential equation

$$\frac{d\theta}{dt} = \omega + \delta(t \bmod T)\bar{R}(\theta) \tag{2.7}$$

where δ is the Dirac delta function. If the δ-function is replaced by a smooth function $P(t)$, as in Fig. 2.1, then (2.7) becomes the forced analogue of (2.5), with $R(\theta)$ playing the role of the PRC. Indeed, since real stimuli are not instantaneous, it is probably more reasonable to use the distributed stimulus function P. By making P sufficiently sharp and narrow (2.5) mimics the effects of coupling two oscillators via their PRCs.

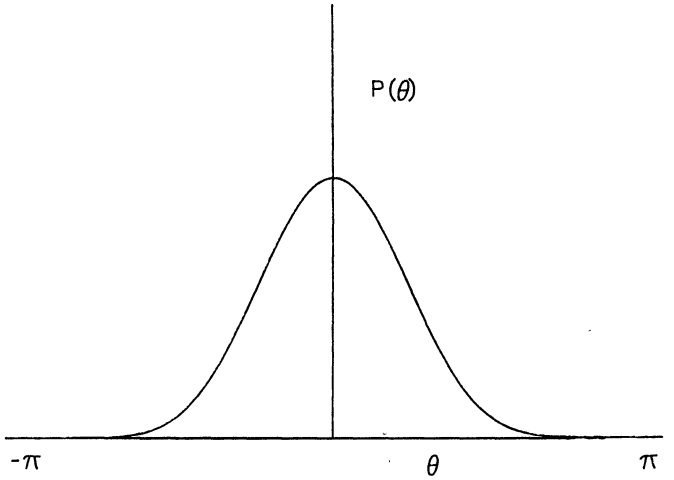

FIG. 2.1. *Periodic smooth pulse function.*

Alternatively, suppose that instead of instantaneously emitting a single pulse per cycle, the stimulus emits a sequence of pulses at times $t_j \in [0, T)$ with strength α_j. In the limit as more of these pulses are emitted, (2.7) becomes

$$\frac{d\theta}{dt} = \omega + P(t)\bar{R}(\theta)$$

where $P(t)$ is the envelope of the spikes emitted by the stimulus. If instead of a forced oscillator we consider two coupled oscillators with the same properties, then we obtain (2.5) where $R(\theta)$ is the PRC and $P(\theta)$ is the envelope of spikes emitted by the other oscillator. Thus, (2.5) may be considered a model for two coupled bursting cells.

2.2. Oscillator death in phase equations. Before demonstrating the existence of oscillator death for equations (1.2), we show that for (1.1) the phenomenon does not exist. Such equations can be reduced to a single equation for $\phi \equiv \theta_1 - \theta_2$:

$$\begin{aligned} \frac{d\phi}{dt} &= (\omega_1 - \omega_2) + [H_1(-\phi) - H_2(\phi)] \\ &\equiv \Delta + \hat{H}(\phi) \end{aligned} \tag{2.8}$$

where $\Delta = \omega_1 - \omega_2$. If Δ is in the range such that $0 = \Delta + \hat{H}(\phi)$ can be solved for ϕ_0 with $\hat{H}'(\phi_0) < 0$, then there is a stable phase-locked solution to (1.1) with $\theta_1 - \theta_2 = \phi_0$; otherwise, there is drift [2], [4]. Time-independent solutions to (1.1) exist only in the very nongeneric case that (2.8) can be solved *and* the solution ϕ_0 satisfies $\omega_1 + H_1(-\phi_0) = 0$. In that case, there is a circle of critical points on the two-torus (parameterized by θ_1 or θ_2), and so none of these points are asymptotically stable.

By contrast, for (1.2) the phenomenon is very robust. This is easiest to see in the special case $\omega_1 = \omega_2 \equiv \omega$ and $h_1(\theta_1, \theta_2) = h_2(\theta_2, \theta_1) \equiv \alpha h(\theta_1, \theta_2)$, where $\alpha > 0$ is a measure of the strength of the coupling. We may then look for solutions of the form $\theta_1(t) \equiv \theta_2(t) \equiv \theta(t)$. These solutions satisfy

$$\frac{d\theta}{dt} = \omega + \alpha h(\theta, \theta). \tag{2.9}$$

If $\omega + \alpha h(\theta, \theta) \neq 0$ for all θ, (2.9) has a periodic solution, and no time-independent (i.e., oscillator death) solution. The following proposition gives a sufficient condition for stability of this solution, a condition related to the stability of the time-independent solution when the latter exists (see Remark 2.2). Let $\partial_i h$ denote the partial derivative with respect to the ith argument of h.

PROPOSITION 2.1. *Suppose that* (2.9) *has a periodic solution* $\theta(t)$, *and along this solution* $\int_0^T \partial_1 h(\theta(t), \theta(t))\, dt < 0$. *Then* $\theta(t)$ *is stable as a solution to* (1.2).

Proof. Since (1.2) is an autonomous equation on T^2, we must show that one of the Floquet multipliers is less than one. (The equations are invariant under time translation, so the other Floquet multiplier is automatically equal to one.) The Jacobian matrix is

$$J = \left[\begin{matrix} \partial_1 h & \partial_2 h \\ \partial_2 h & \partial_1 h \end{matrix} \right] \Bigg|_{(\theta(t), \theta(t))} \tag{2.10}$$

Equation (1.2) is two-dimensional, so the nonunit multiplier is the exponential of the average of the trace of J over one period. Since the trace of J is $2\partial_1 h$, the conclusion follows immediately from the hypothesis. □

Remark 2.1. Suppose $h(\theta_1, \theta_2) = P(\theta_2) R(\theta_1)$ as in § 2.1. The hypothesis is then $\int P(\theta(t)) R'(\theta(t))\, dt < 0$. This holds when P is a pulse function centered around $\theta = 0$, as in Fig. 2.1, and R is approximately $-\sin\theta$. For more general pulse coupling, the stability hypothesis requires that the $\partial_1 h < 0$ where the pulse is concentrated.

It is easy to see that if $h(\theta, \theta)$ is negative for *any* θ, then for sufficiently large α, (2.9) has critical points. Furthermore, unlike (1.1), such a critical point may be stable as a solution to the full system (1.2). The following proposition gives sufficient conditions for stability.

PROPOSITION 2.2. (1) *Suppose that* $h(\theta, \theta)$ *takes both signs, has a negative minimum value* θ_m, *and* $(d^2/d\theta^2)[h(\theta, \theta)]|_{\theta = \theta_m} > 0$. *Then, for sufficiently large* α, (2.9) *has a pair of critical points, one a sink and one a saddle.*

(2) *Suppose that* $\hat{\theta}$ *solves*

$$0 = \omega + \alpha h(\hat{\theta}, \hat{\theta}), \tag{2.11}$$

and assume that $\hat{\theta}$ *is a stable critical point of* (2.9) (*i.e.*, $(\partial_1 h + \partial_2 h)(\hat{\theta}, \hat{\theta}) < 0$). *Suppose further that* $(\partial_2 h - \partial_1 h)(\hat{\theta}, \hat{\theta}) > 0$. *Then* $(\hat{\theta}, \hat{\theta})$ *is a stable solution to the full equations* (1.2).

Remark 2.2. The hypothesis for the stability of the limit cycle is that, on the average, $\partial_1 h < 0$. Suppose this is true uniformly for the family of limit cycles as α increases toward a bifurcation point $(\tilde{\alpha}, \tilde{\theta})$. Then $\partial_1 h < 0$ in a neighborhood of $(\tilde{\theta}, \tilde{\theta})$ in which a limit cycle near the bifurcation point spends most of its time. At the bifurcation point, this critical point is degenerate, i.e., $(\partial_1 h + \partial_2 h)(\hat{\theta}, \hat{\theta}) = 0$; thus, $\partial_2 h > 0$. It follows that, past the bifurcation point, but near it, the second hypothesis of the second half of Proposition 2.2 is then automatic.

Proof of Proposition 2.2. (1) It is clear that if θ_m is the minimum of $h(\theta, \theta)$, and $h(\theta_m, \theta_m) < 0$, then a bifurcation occurs at the value of α for which $0 = \omega + \alpha h(\theta_m, \theta_m)$. This bifurcation is supercritical and nondegenerate provided that $(d^2/d\theta^2)[h(\theta, \theta)]|_{\theta=\theta_m} \neq 0$, and that it has the opposite sign from $h(\theta_m, \theta_m) = (d/d\alpha)(\omega + \alpha h)(\theta, \theta)|_{\theta=\theta_m}$. Thus, by standard bifurcation theory, for α larger than the critical value $\tilde{\alpha}$, there are a pair of time-independent solutions to (2.9), a sink and a saddle. Since h takes both signs, (2.11) can be solved for all α larger than $\tilde{\alpha}$.

(2) The Jacobian for (1.2) around $\theta_1 = \theta_2 \equiv \hat{\theta}$ is (2.11), with $\theta(t) \equiv \hat{\theta}$. To ensure stability, we must have that $\operatorname{tr} J < 0$ and $\det J > 0$:

$$\begin{aligned} \det J &= [(\partial_1 h)^2 - (\partial_2 h)^2](\hat{\theta}, \hat{\theta}) \\ &= (\partial_1 h + \partial_2 h)(\partial_1 h - \partial_2 h)(\hat{\theta}, \hat{\theta}). \end{aligned} \tag{2.12}$$

By hypothesis, both factors of the right-hand side of (2.12) are negative. Together, the two inequalities imply $\partial_1 h < 0$, so $\operatorname{tr} J = 2\partial_1 h < 0$. Thus, we are done. □

Remark 2.3. Equations (1.2) do not have any critical points with $\theta_1 = \theta_2$ if $h(\theta, \theta)$ is positive for all θ (assuming, without loss of generality, that $\omega > 0$). However, if the coupling is to be effective in producing phase-locking, it is useful for the coupling term to be able to phase-advance or phase-delay the oscillator. If the coupling is by means of a pulse as in the previous example, this requires that $R(\theta)$ take both signs, implying that $h(\theta, \theta)$ does also.

Remark 2.4. In (1.1), $h(\theta_1, \theta_2)$ is dependent only on $\theta_2 - \theta_1$. Therefore, $\partial_1 h + \partial_2 h \equiv 0$, and so one of the hypotheses for the stability of the critical point always fails. The hypothesis $(\partial_1 h + \partial_2 h)(\hat{\theta}) \neq 0$ is a transversality condition that will reappear in the analysis of oscillator death in chains.

Remark 2.5. It is clear that it is not necessary that $h_1(\theta_1, \theta_2) = h_2(\theta_2, \theta_1)$ and $\omega_1 = \omega_2$ in order that (1.2) have a stable critical point. Indeed, the stability proved in the previous proposition implies that the phenomenon is robust under changes of parameter. In § 4 we will give a proof of the existence of death in chains of oscillators. The proof specializes to the case of $N = 2$ and generalizes this section to $h_1 \neq h_2$ and $\omega_1 \neq \omega_2$; however, it is not as transparent as the above proof.

Remark 2.6. If $\omega_1 \neq \omega_2$, the system may go from oscillator death to phase-locking and finally to phase drift as the strength of the coupling decreases. We illustrate this with a simple example that has many of the qualitative properties of the more general system (2.5), although it does not satisfy $P(\theta) \geqq 0$. Consider the pair of product coupled oscillators:

$$\frac{d\theta_1}{dt} = \omega_1 - \alpha \cos(\theta_2) \sin(\theta_1), \qquad \frac{d\theta_2}{dt} = \omega_2 - \alpha \cos(\theta_1) \sin(\theta_2). \tag{2.13}$$

(We obtain these equations through the procedure described in § 2.1 if we let $F_k(u_k)$ be given by

$$F_k(u_k) = \begin{pmatrix} \lambda_k & -\omega_k \\ \omega_k & \lambda_k \end{pmatrix} u_k$$

where $\lambda_k = 1 - |u_k|^2$, and the coupling $G_k = (u_j^{(1)}, 0)^T, j \neq k$.) Letting $\phi = \theta_1 - \theta_2$, $\xi = \theta_1 + \theta_2$, we obtain

$$\dot{\phi} = \delta - \alpha \sin(\phi), \qquad \dot{\xi} = \sigma - \alpha \sin(\xi)$$

where $\sigma = \omega_1 + \omega_2$, $\delta = \omega_1 - \omega_2$. Fix $\omega_1 = 1$ and let ω_2 decrease below one. Note that if $\alpha < 1 - \omega_2 = |\delta|$ there is drift, and for $1 - \omega_2 < \alpha < 1 + \omega_2$ there is locking. For $\alpha > 1 + \omega_2$ there is death, so that the smaller ω_2 gets the easier it is to obtain oscillator death.

3. Oscillator death in nonphase equations for coupled oscillators. Recall that if two oscillators, each having a stable limit cycle, are coupled together weakly there is an invariant torus in the full equations near the product of the limit cycles. As the coupling is increased, the torus may cease to exist. In this section we discuss a pair of well-known models for neural oscillators, how they may be coupled, and describe some analytical and numerical results about their behavior. We show that for some ranges of the parameters and for some types of coupling, the oscillation disappears by the mechanism discussed in § 2, with the preservation of the invariant torus. In other ranges of parameters, the torus disappears and a stable critical point emerges via an inverse Hopf bifurcation.

3.1. Neural network models. The Wilson–Cowan equations [23] model the interactions between an excitatory and inhibitory population of neurons; we will treat one such network as a single oscillator even though, in fact, it represents a population of cells. We will then describe the consequences of coupling two such networks in a manner proposed in [23]. Let $E(t)$ and $I(t)$ represent the respective firing rates of the excitatory and inhibitory neural populations. The equations proposed in [23] take the form:

$$\frac{dE}{dt} = -E + S(\alpha_{ee}E - \alpha_{ie}I - \nu_e), \qquad \frac{dI}{dt} = -I + S(\alpha_{ei}E - \alpha_{ii}I - \nu_i). \tag{3.1}$$

The parameters α_{ee}, α_{ie}, α_{ei}, and α_{ii} are the synaptic strengths between the neurons, and the parameters ν_e and ν_i are the thresholds for firing of the populations. The function $S(u)$ is a monotone increasing saturating function with values between zero and one (see Fig. 3.1). Equations of the form (3.1), where S is a piecewise-linear function and the synaptic strengths are allowed to slowly change, are presently of great interest due to their connection with learning [29], [30]. In [24] it was shown that (3.1) admits an orbitally stable periodic solution under a wide range of parameters (indeed, a sufficient condition is that (3.1) have a unique unstable equilibrium point).

Suppose that we couple two such identical networks. The inputs are additive so that the equations take the form:

$$\begin{aligned} \frac{dE_j}{dt} &= -E_j + S(\alpha_{ee}E_j - \alpha_{ie}I_j + \beta_{ee}E_k - \beta_{ie}I_k - \nu_e), \\ \frac{dI_j}{dt} &= -I_j + S(\alpha_{ei}E_j - \alpha_{ii}I_j + \beta_{ei}E_k - \beta_{ii}I_k - \nu_i), \end{aligned} \tag{3.2}$$

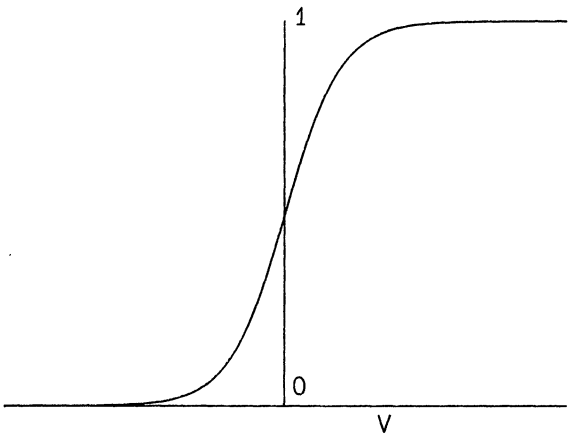

FIG. 3.1. *Nonlinear firing rate function* $S(V)$.

$j, k = 1, 2$, $j \neq k$. The parameters β_{ee}, β_{ie}, β_{ei}, and β_{ii} represent connections between the two populations; β_{ee} is the strength of connections between excitatory population j and excitatory population k, β_{ie} represents connections from inhibitory cells of j to excitatory cells of k, and so on. If the oscillators are identical and the coupling is symmetric, then one possible solution to (3.2) has $(E_1, I_1) = (E_2, I_2) = (E, I)$ where E and I satisfy:

$$\begin{aligned} \frac{dE}{dt} &= -E + S((\alpha_{ee} + \beta_{ee})E - (\alpha_{ie} + \beta_{ie})I - \nu_e), \\ \frac{dI}{dt} &= -I + S((\alpha_{ei} + \beta_{ei})E - (\alpha_{ii} + \beta_{ii})I - \nu_i). \end{aligned} \tag{3.3}$$

The effect of each form of coupling is to increase the appropriate synaptic weight of the symmetric system. We will restrict our attention to cases in which only one of the coupling parameters is nonzero. In particular, suppose that there is only excitatory-excitatory coupling. In Fig. 3.2(a) we show the phase-plane for (3.3) when $\beta_{ee} < \beta^* \approx 5.258$ (and $\alpha_{ee} = 12$, $\alpha_{ie} = 14$, $\alpha_{ei} = 18$, $\alpha_{ii} = 0$, $\nu_e = 1$, $\nu_i = 8$, $S(u) = .5(1 + \tanh(u))$). There is a stable periodic solution to (3.3) and, as β_{ee} increases, the amplitude and period of this oscillation increase. The shape of the E-nullcline changes and becomes larger in amplitude until at a critical value of $\beta_{ee} = \beta^*$, the E-nullcline is tangent to the I-nullcline. At this point (Fig. 3.2(b)) there is an infinite period solution and the invariant circle formed by the limit cycle has a saddle-node equilibrium point. Further increases in $\beta_{ee} > \beta^*$ cause the nullclines to cross at three points and lead to the appearance of two new equilibria, one a stable node and the other a saddle (Fig. 3.2(c)). The invariant circle persists and is formed by the pair of heteroclinic orbits joining the saddle-point and the node. Thus, as the coupling of the two oscillators is increased, the stable in-phase oscillation becomes larger in amplitude until, at some critical value, there is no longer periodic modulation; rather, the two networks are forced to lie in a state of tonic high frequency firing (recall that E and I are *firing rates*). This mechanism is very robust for these particular equations and occurs over a large range of reasonable parameters.

We now turn to the stability of the symmetric solutions with respect to the full four-dimensional system (3.2). For β_{ee} small, the symmetric periodic solution can be

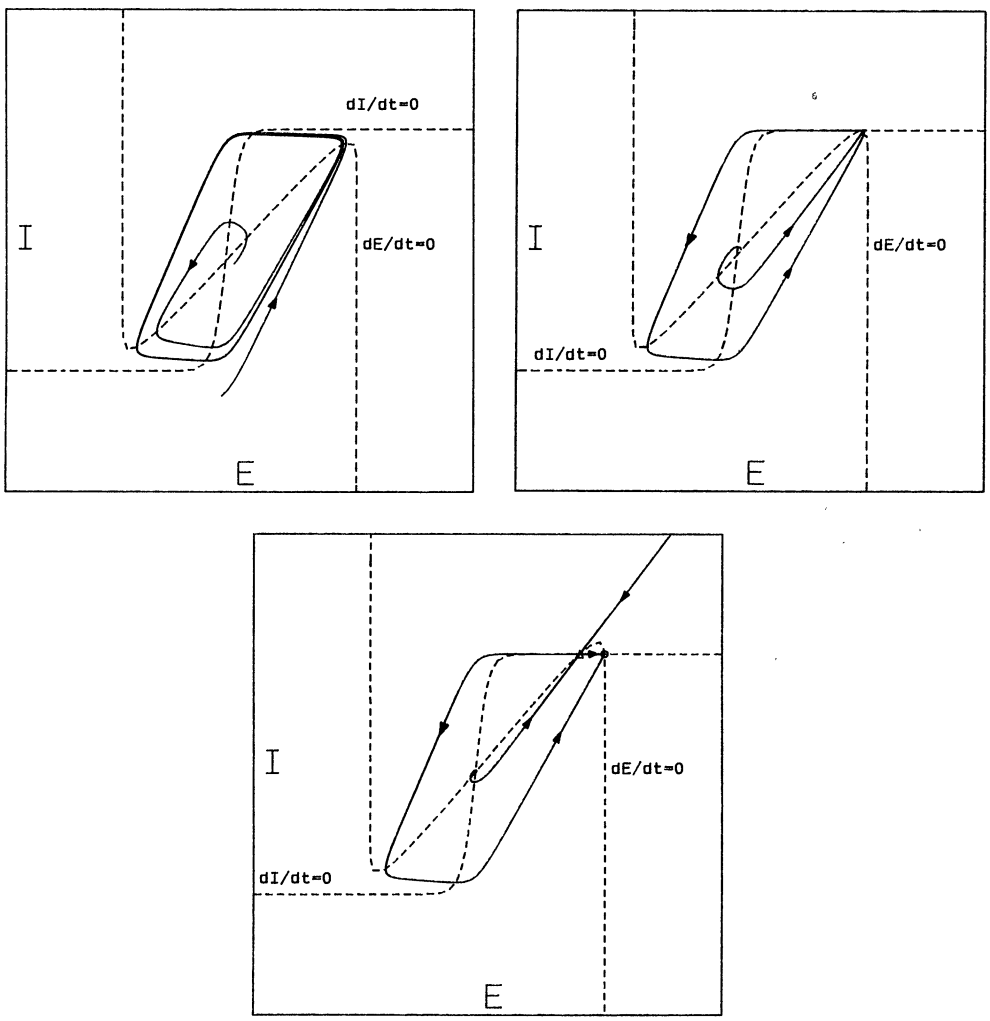

FIG. 3.2. *Phase plane for the Wilson–Cowan equations with various strengths of excitatory–excitatory coupling.* (a) $\beta_{ee} = 4.5$. *There is a symmetric limit cycle that is globally attracting.* (b) $\beta_{ee} = 5.25$. *At criticality, there is a saddle node and a trajectory with infinite period. The stable manifold is also shown.* (c) $\beta_{ee} = 6.0$. *Beyond criticality, there are two new equilibria and a heteroclinic cycle joining them. Also shown is the stable manifold for the saddle point.* (*Parameters*: $\nu_e = 1$, $\nu_i = 8$, $\alpha_{ee} = 12$, $\alpha_{ie} = 14$, $\alpha_{ii} = 0$, $\alpha_{ei} = 18$, $S(u) = .5(1 + \tanh(u))$.)

shown to be stable by using an averaging technique. For larger β_{ee}, numerical simulations of (3.2) indicate that the symmetric *oscillatory* solution is stable as a solution to the full equations (3.3). We will show that the symmetric *equilibrium* solution to (3.3) is always stable if the coupling is excitatory-excitatory. The next proposition can be used to analyze coupling in a large class of systems.

PROPOSITION 3.1. *Let A and B be $n \times n$ real matrices. The eigenvalues of the matrix*

$$M = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

are identical to the eigenvalues of the two matrices $A + B$ and $A - B$.

Proof. Let

$$P=\begin{bmatrix} I & -I \\ I & I \end{bmatrix}$$

so that

$$P^{-1}=\frac{1}{2}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}.$$

It is easily seen that $P^{-1}MP$ is the diagonal matrix, diag $[A+B, A-B]$, whose eigenvalues are those of $A+B$ and $A-B$ as required. □

Remark 3.1. The transformation P can also be applied to the linear differential equation $dz/dt = Mz$ when the arrays A and B are time-dependent. Thus, to establish the stability of the symmetric time-dependent solutions, we need only analyze two $n \times n$ systems rather than a $2n \times 2n$ system.

COROLLARY 3.1. *Consider the coupled neural network of* (3.2). *Let* $(\bar{E}, \bar{I}, \bar{E}, \bar{I})$ *be a symmetric equilibrium solution that is stable as a solution to the reduced system* (3.3). *Then for excitatory-excitatory coupling, this symmetric solution is an asymptotically stable solution to the full system* (3.2).

Proof. The matrix of linearization for (3.2) has the form $\left[\begin{smallmatrix} A & B \\ B & A \end{smallmatrix}\right]$, where

$$A=\begin{bmatrix} -1+S_e\alpha_{ee} & -S_e\alpha_{ie} \\ S_i\alpha_{ei} & -1-S_i\alpha_{ii} \end{bmatrix} \quad \text{and} \quad B=\begin{bmatrix} S_e\beta_{ee} & -S_e\beta_{ie} \\ S_i\beta_{ei} & -S_i\beta_{ii} \end{bmatrix}.$$

$S_e = S'((\alpha_{ee}+\beta_{ee})\bar{E}-(\alpha_{ie}+\beta_{ie})\bar{I}-\nu_e)$ and S_i is defined similarly. Suppose that the coupling is only excitatory-excitatory, so that all of the entries of B are zero except for the upper left one, which is positive since S is monotone increasing. We have assumed that $(\bar{E}, \bar{I})$ is a stable solution for (3.3), so that $A+B$, the matrix of linearization for (3.3), has all of its eigenvalues in the left half plane. This means that the trace of $A+B$ is negative and the determinant is positive. We must show that the eigenvalues of $A-B$ all have negative real parts:

$$\operatorname{tr}(A-B)=\operatorname{tr}(A+B)-2S_e\beta_{ee}<0,$$

$$\det(A-B)=\det(A+B)+2S_e\beta_{ee}(1+S_i\alpha_{ii})>0.$$

These two inequalities imply that the real parts of the eigenvalues of $A-B$ are negative. □

Remark 3.2. For cross-inhibitory coupling (either β_{ei} or β_{ie} nonzero), phase death cannot occur in the manner described here. The effect of increasing β_{ei} is to sharpen the I-nullcline, so that new intersections with the E-nullcline cannot occur. Increasing β_{ie} reduces the size of the height of the E-nullcline, so again these intersections cannot take place. Furthermore, increasing either of these two weights leads to very complicated behavior in the full system (3.2). There can be coexistent stable periodic solutions as well as aperiodic behavior for large values of either of these coupling strengths. Disinhibitory coupling (β_{ii} nonzero) can result in death of the symmetric oscillation, but numerical studies indicate that these solutions are unstable as solutions to the full model system (3.2).

3.2. Membrane models. A typical model for membrane oscillations based on the Hodgkin-Huxley formalism has the form:

$$C\frac{dV}{dt}=\sum_i g_i(V_i-V)+I \tag{3.4}$$

where $V(t)$ is the membrane potential, C is the membrane capacitance, g_i are conductances that are controlled by auxiliary variables having voltage and time-dependences, V_i are the reversal potentials of the corresponding ion channels, and I is the applied current. Many models, including the Hodgkin-Huxley model, can be reduced to a pair of differential equations for the voltage and some "recovery" variable (see, e.g., Rinzel [25] for such a reduction applied to the original Hodgkin-Huxley system). The fact that we consider planar systems is not important; it simplifies the discussion and makes the mechanism more transparent. We consider a membrane model proposed by Morris and Lecar [26] that simulates barnacle neuron oscillations. This model has the form:

$$\begin{aligned} C\frac{dV}{dt} &= g_L(-V_L - V) + g_{Ca}m_\infty(V)(V_{Ca} - V) + g_K w(-V_K - V) + I \\ &\equiv f(V, w) + I, \\ \frac{dw}{dt} &= \lambda(V)(w_\infty(V) - w). \end{aligned} \tag{3.5}$$

Here g_L is the leakage conductance, g_K, g_{Ca} are the potassium and calcium conductances, respectively, and V_L, V_K, V_{Ca} are the reversal potentials for each of the three channels. $\lambda(V)$ is a voltage dependent rate and $m_\infty(V)$ and $w_\infty(V)$ are positive sigmoid gating functions (as in Fig. 3.1). When parameters are chosen as in Fig. 3.3(a), the system admits a stable periodic oscillation. (These are the parameters that were implied in [26] and incorrectly given in that paper. We have made (3.5) dimensionless by introducing the variables V' and t' where $V \mapsto V_{Ca}V'$ and $t \mapsto (C/g_{Ca})t'$; the equations are then identical to (3.5), but with all voltages scaled by V_{Ca} and all conductances scaled by g_{Ca}. The current I and the time constant of the recovery variable are also rescaled. With no loss in generality, we can assume that C, V_{Ca}, and g_{Ca} are unity.)

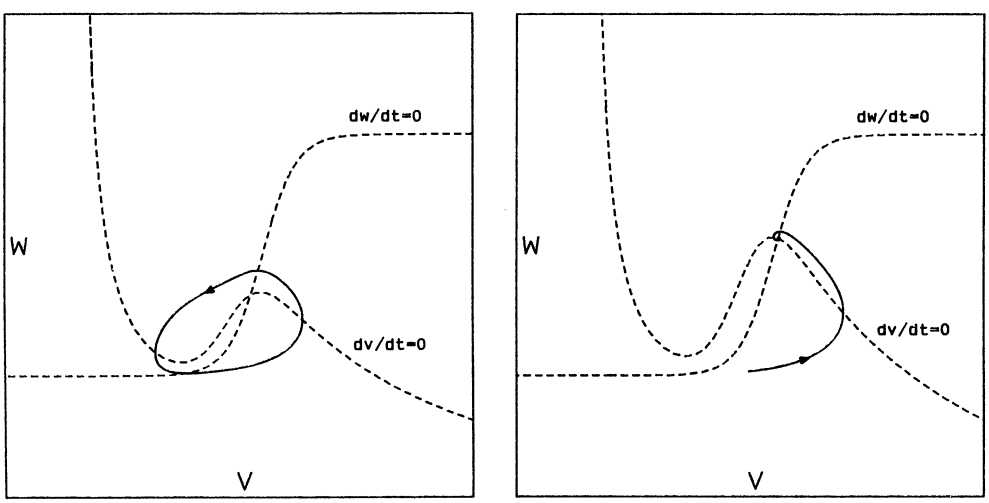

FIG. 3.3. *Phase plane for Lecar-Morris system showing nullclines and stable oscillations or equilibrium points.* (a) *No coupling. There is a stable oscillation.* (b) *Excitatory coupling,* $g_{ex}(V) = \beta_e m_\infty(V)$, $V_{ex} = 1$, $\beta_e = 0.5$. *There is a stable critical point.* (*Parameters*: $V_1 = -0.01$, $V_2 = 0.15$, $V_3 = 0.1$, $V_4 = 0.145$, $\lambda_0 = 0.33$, $g_L = 0.5$, $g_K = 2$, $V_L = 0.4$, $V_K = 0.7$, $V_{Na} = 1.0$, $I = 0.125$. *These are dimensionless versions of the values in* [26]. *Here* $m_\infty(V) = .5(1 + \tanh((V - V_1)/V_2))$, $n_\infty(V) = .5(1 + \tanh((V - V_3)/V_4))$, $\lambda(V) = \lambda_0 \cosh((V - V_3)/V_4)$.)

Suppose that we wish to couple two such membrane oscillators. The simplest coupling is via a gap junction or electrotonic synapse. However, as we noted in § 2, that type of coupling, which depends on the *difference* between the membrane potentials, cannot lead to phase death. A number of recent papers have modeled chemical synapses between two cells by a voltage dependence conductance with or without delays (see [27]). This results in the following model:

$$\begin{aligned} C\frac{dV_j}{dt} &= f(V_j, w_j) + I + g_{ex}(V_k)(V_{ex} - V_j) + g_{in}(V_k)(-V_{in} - V_j) \\ &\equiv F(V_j, w_j, V_k) + I, \\ \frac{dw_j}{dt} &= g(V_j, w_j), \end{aligned} \tag{3.6}$$

$j, k = 1, 2$, $j \neq k$. The functions $g_{ex}(V)$ and $g_{in}(V)$ are similar in form to $m_\infty(V)$ and $w_\infty(V)$, respectively, and V_{ex} (respectively, V_{in}) can be assumed to equal V_{Ca} (respectively, V_K). The first type of synapse depolarizes the membrane while the second type hyperpolarizes it. Thus, these can be thought of as excitatory and inhibitory synapses. In this simple model we include no habituation, no delay, nor any synaptic fatigue. Our point here is to illustrate some of the consequences of these simple models. See § 5 for further discussion.

As in § 3.1, we can look for symmetric solutions to (3.6). The resulting equation is identical to (3.6) with all subscripts removed. Suppose that the coupling is excitatory so that $g_{in}(V) = 0$ and let $g_{ex}(V) = \alpha m_\infty(V)$. For $\alpha = 0$, we have the oscillation depicted in Fig. 3.3(a). As α increases, the V-nullcline moves upward in such a way as to cross the w-nullcline, leading to a stable critical point of the symmetric system (see Fig. 3.3(b)). This is a form of oscillator death that is different from that described in § 3.1, since the notion of phase is not preserved. The invariant torus that exists for weak coupling collapses as the symmetric oscillation undergoes an inverse Hopf bifurcation to result in a stable nonoscillatory solution. We can use Proposition 3.1 to show that this equilibrium is a stable solution to the full-coupled system (3.6). Numerical methods and averaging can be used to show that, before the symmetric oscillation disappears, it is a stable solution to the full problem. Similar methods can be used to show that for inhibitory coupling ($g_{ex}(V) = 0$, $g_{in}(V) = \alpha g_K(V)$) the symmetric solutions are not stable as solutions to (3.6).

For a different parameter set, the Morris–Lecar system can undergo phase death in the manner described in § 3.1. Furthermore, the symmetric solutions appear to be stable solutions to the full system (3.6). (For the interested reader, the parameters are as in Fig. 3.3(a), but $V_1 = 0.1$, $V_2 = 0.05$, $V_4 = 0.02$, $I = 0.5$, and α ranging between zero and 1.6, with a saddle-node occurring on the cycle for $\alpha = 1.34$.)

3.3. Other mechanisms for the cessation of rhythmicity. As remarked earlier, the essential reason for the death of the oscillations in the equations discussed in § 2 is the lack of uniformity in the angular speed. For the equations in § 3, depending on the parameters, the rhythmicity could disappear either by that mechanism, or by the disappearance of the invariant torus containing the limit cycle. In both cases, the necessary feature of the coupled system is that the coupling does not vanish identically when the oscillators are in the same phase. In particular, for general identical diffusively coupled systems, it is clear that the in-phase solution is always oscillatory so that "symmetric" oscillator death will not occur. The stability of the in-phase solution depends on the details of the diffusion matrix; if, for example, it is a positive scalar multiple of the identity, the in-phase solution is stable [28].

There are other examples of disappearance of rhythmicity that depend on very different features of the coupled system. For example, Bar-Eli has shown numerically that certain oscillators modeling chemical reactions, coupled by diffusion, can cease to oscillate if the frequencies of the two oscillators are sufficiently different [16]. In [7] it has been shown that, for a large class of equations, this is essentially a linear effect; if each (uncoupled) oscillator has an unstable equilibrium point, and if the frequencies are sufficiently different, then there is a range of intermediate coupling strengths for which the equilibrium point of the coupled system is stabilized by the coupling. For simple oscillator equations having some special symmetries, this can be understood in the context of nonlinear equations, and it is seen that, if the frequencies of the oscillators are sufficiently different, the invariant torus containing the periodic solution of the coupled system can shrink to zero as the coupling is increased [7]. Beyond another threshold in the coupling strength, a limit cycle reemerges, but without the invariant torus. If each of the oscillators has its local frequency dependent on the local amplitude of the trajectory, the bifurcation structure is very complicated [7].

Still another mechanism for the cessation of oscillation is given in [17]. Crowley and Epstein have simulated a pair of oscillators, each a model of the oscillatory Belousov–Zhabotinskii reaction, coupled by diffusion. As in the mechanism discussed in § 2, the oscillators are permitted to be close to identical, and death is characterized by the existence of a new stable steady state that appears to be on or near the old limit cycle. In [17], however, the oscillators are near antiphase at the new steady state, rather than near synchrony, as in this paper. The relaxation nature of the oscillators is essential for the phenomenon described in [17], but a similar behavior is found for "sinusoidal" oscillators in [18].

4. Oscillator death in chains of oscillators. The phenomena discussed in § 2 persists for chains of $N \geqq 2$ oscillators; however, this is considerably less apparent than for $N = 2$. As in [6], we assume that each oscillator has a stable limit cycle solution and is coupled only to its nearest neighbors. Without assuming that the coupling is "weak," we assume that the coupling is in a range in which there still exists an invariant N-torus, parameterized by the phases. (The size of the coupling for which this is correct does not decrease to zero as $N \to \infty$ [6].) If the oscillators have the same frequency, the resulting equations are

$$\begin{aligned} &\frac{d\theta_k}{dt} = \omega + h^-(\theta_k, \theta_{k-1}) + h^+(\theta_k, \theta_{k+1}), \qquad k = 2, \cdots, N-1, \\ &\frac{d\theta_1}{dt} = \omega + h^+(\theta_1, \theta_2), \qquad \frac{d\theta_N}{dt} = \omega + h^-(\theta_N, \theta_{N-1}). \end{aligned} \tag{4.1}$$

Here h^+ and h^- represent the ascending and the descending coupling. The derivation of (4.1) follows that for the $N = 2$ case in § 2. Without loss of generality, we may assume that $\omega > 0$. We are looking for solutions in which $\theta_k' = 0$ for all k, i.e., the θ_k satisfy

$$\begin{aligned} 0 &= \omega + h^-(\theta_k, \theta_{k-1}) + h^+(\theta_k, \theta_{k+1}), \qquad k = 2, \cdots, N-1, \\ 0 &= \omega + h^+(\theta_1, \theta_2), \qquad 0 = \omega + h^-(\theta_N, \theta_{N-1}). \end{aligned} \tag{4.2}$$

As in § 2, the above hypothesis on the frequencies is far from necessary to encounter oscillator death, but we will leave the extensions to the interested reader. (For $N = 2$, the extension to $\omega_1 \neq \omega_2$ is done in Remark 4.4 after the proof.) One point that we wish to make is that the phenomenon does not scale with N; that is, death does not occur more easily or less easily in a long chain than in a short one.

The main result of this section is Theorem 4.1.

THEOREM 4.1. *Assume there are* θ_+, θ_-, *and* θ_C *such that* $\theta_C < \theta_\pm$, $h^\pm(\theta_\pm, \theta_\pm) + \omega = 0$ *and* $h^+(\theta_c, \theta_C) + h^-(\theta_C, \theta_C) + \omega = 0$. *Assume further that for all* θ, $\bar{\theta}$ *in an interval* J *containing* $\theta_\pm$, θ_C *we have* $(\partial_1 h^\pm + \partial_2 h^\pm)(\theta, \theta) < 0$, $\partial_1 h^\pm(\theta, \bar{\theta}) < 0$, $\partial_2 h^\pm(\theta, \bar{\theta}) > 0$ *and* $h^\pm(\theta_C, \theta_C) < 0$. *Then there is a unique stable, time-independent solution to* (4.1).

Remark 4.1. To better understand the hypotheses and motivation for Theorem 4.1, it is instructive to consider the isotropic product coupling model, $h^\pm(\theta_1, \theta_2) = R(\theta_1)P(\theta_2)$, where $P(\theta)$ is a symmetric positive peaked function and $R(\theta)$ is qualitatively like $-\sin(\theta)$. Figure 4.1(a) illustrates these two functions and Fig. 4.1(b) shows the function $h^\pm(\theta, \theta)$. $P(\theta)R(\theta)$ is positive for $-\pi < \theta < 0$ and negative on the interval $0 < \theta < \pi$. Since $h^+ = h^-$ in this example, θ_C satisfies $P(\theta_C)R(\theta_C) = -\omega/2$ and $P(\theta_\pm)R(\theta_\pm) = -\omega$. Let $J = [\theta_C, \theta_\pm]$ where θ_C, $\theta_\pm > 0$. It is clear from Figs. 4.1(a) and 4.1(b) that the hypotheses are satisfied; in particular, $(\partial_1 h^\pm + \partial_2 h^\pm)(\theta, \theta) = P'(\theta)R(\theta) + P(\theta)R'(\theta) = [P(\theta)R(\theta)]' < 0$, $\partial_2 h^\pm(\theta, \hat{\theta}) = P'(\hat{\theta})R(\theta) > 0$, and $\partial_1 h^\pm(\theta, \hat{\theta}) = P(\hat{\theta})R'(\theta) < 0$. The hypotheses continue to be satisfied if $P(\theta)$ and $R(\theta)$ are perturbed so as to lose their symmetry properties.

Remark 4.2. As in [2], the discrete system (4.2) behaves as a singularly perturbed two-point boundary value system. This continuum plays no role, other than heuristic,

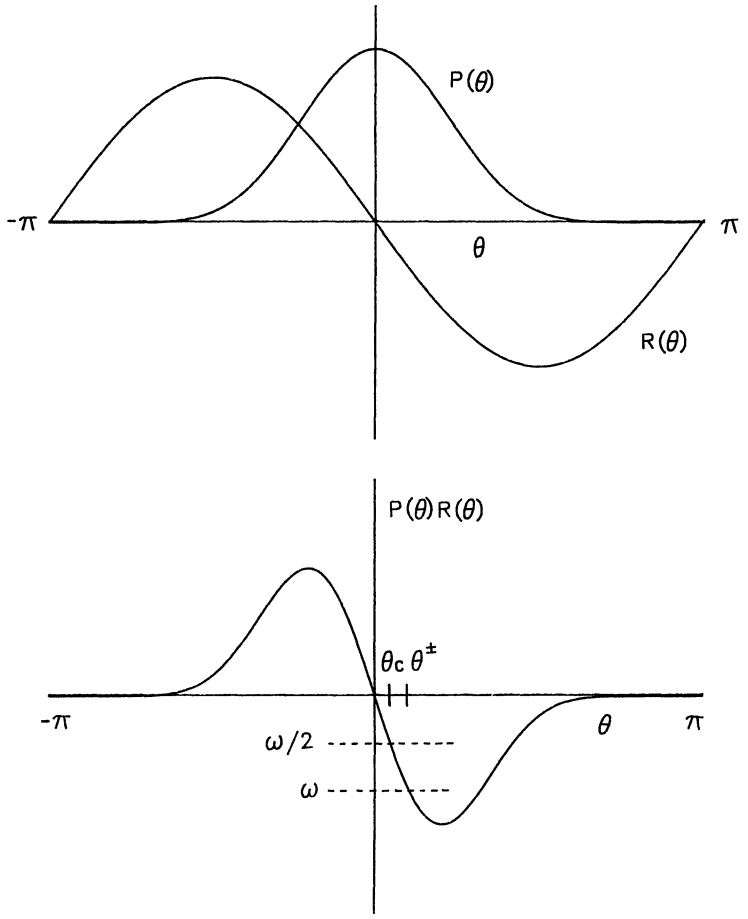

FIG. 4.1. (a) *Pulse function,* $P(\theta)$ *and response function* $R(\theta)$ *for a simple "product" coupled model.* (b) *The product* $h(\theta, \theta) = P(\theta)R(\theta)$.

in the proof to be given below. Hence, we derive it only for the case $h^+ = h^- \equiv h$, for which the derivation is most transparent. We rewrite (4.2) as

$$0 = \omega + 2h(\theta_k, \theta_k) + [h(\theta_k, \theta_{k+1}) - 2h(\theta_k, \theta_k) + h(\theta_k, \theta_{k-1})]. \tag{4.3}$$

By multiplying and dividing the term in the square brackets by $1/N^2$, and considering $\theta_{k\pm1}$ as close to θ_k, we see that (4.3) is formally similar to

$$0 = \omega + 2h(\theta, \theta) + (1/N^2)\partial_2 h(\theta, \theta)\theta_{xx} \tag{4.4}$$

where $0 \leqq x \leqq 1$ and $\theta_k \approx \theta(k/N)$, so the $\{\theta_k\}$ form a discretization of the function $\theta(x)$. The boundary conditions for (4.4) come from the second and third equations of (4.2):

$$0 = \omega + h(\theta, \theta) \quad \text{at } x = 0 \text{ and } x = 1. \tag{4.5}$$

(This is equivalent to $\theta = \theta_\pm$ at $x = 0$ and $x = 1$.) It can be shown that the solution to (4.4), (4.5) is essentially constant over most of $[0, 1]$, with boundary layers at both ends. The constant value satisfies the "outer equation" $0 = \omega + 2h(\theta, \theta)$, i.e., $\theta = \theta_C$.

Remark 4.3. The hypothesis $h^\pm(\theta_C, \theta_C) < 0$ turns out to be automatic, providing that $|h^+ - h^-|$ is not too large; thus this hypothesis is to be interpreted as giving bounds on the allowable amount of anisotropy. The other hypotheses are necessary for stability.

A proof of Theorem 4.1 can be made by mimicking the proof of the solution to (4.4), (4.5); such a proof was made for the analogous discrete system in [6]. Because we are assuming that $\omega_k \equiv \omega$, it is possible to construct a simpler proof using ideas similar to monotone methods for parabolic PDEs. We will define upper and lower solutions for (4.2), and show that the upper (respectively, lower) solutions decrease (respectively, increase) to a time-independent solution. The proof is similar to the proof of stability of the solution in [6].

DEFINITION 4.1. An upper (respectively, lower) solution to (4.1) is a solution $\{\theta_k(t)\}$ such that, for all $1 \leqq k \leqq N$, $d\theta_k/dt < 0$ (respectively, >0) at $t = 0$.

LEMMA 4.1. *Suppose that θ_k^1 and θ_k^2 are two solutions to* (4.1) *such that $\theta_k^1(0) < \theta_k^2(0)$. Then $\theta_k^1(t) \leqq \theta_k^2(t)$ for all $t > 0$ for which θ_k^i lie in J.*

Proof. From (4.1)

$$\frac{d}{dt}(\theta_k^2 - \theta_k^1) = [h^+(\theta_k^2, \theta_{k-1}^2) - h^-(\theta_k^1, \theta_{k+1}^1)] + [h^-(\theta_k^2, \theta_{k-1}^2) - h^-(\theta_k^1, \theta_{k-1}^1)]. \tag{4.6}$$

Suppose that $\theta_k^2(\tau) = \theta_k^1(\tau)$ for some k and some τ; without loss of generality, we may assume that this is the smallest such τ, and the k is the smallest such integer at that τ. Using the hypothesis that $\partial_2 h^\pm > 0$, we see that the right-hand side of (4.6) > 0 at that k and that τ. Thus, θ_k^2 cannot cross θ_k^1 for any value of k or τ. □

COROLLARY 4.1. *If θ_k^u (respectively, θ_k^l) is an upper (respectively, lower) solution to* (4.1), *then $(d/dt)\theta_k^u \leqq 0$ for all t (respectively, $(d/dt)\theta_k^l \geqq 0$ for all t) such that θ_k^u (respectively, θ_k^l) lie in J.*

Proof. Consider an upper solution θ_k^u. Since, at $t = 0$, $(d/dt)\theta_k^u < 0$, we have that $\theta_k^u(t+\delta) < \theta_k^u(t)$ for $t = 0$ and all δ sufficiently small. By Lemma 4.1, it follows that $\theta_k^u(t+\delta) \leqq \theta_k^u(t)$ for all $t > 0$, implying that $(d/dt)\theta_k^u \leqq 0$ for all $k \leqq N$ and all $t > 0$. A similar argument holds for the lower solution. □

Before giving the proof of the theorem, we need another lemma, which will be used to show the uniqueness of a solution in the interval J. In this lemma, we treat (4.2) as a difference equation, and use a shooting method. That is, we rewrite the first

equation of (4.2) as follows:

$$h^+(\theta_k, \theta_{k+1}) - h^+(\theta_k, \theta_k) = h^-(\theta_k, \theta_k) - h^-(\theta_k, \theta_{k-1}) - [h^+(\theta_k, \theta_k) + h^-(\theta_k, \theta_k) + \omega]. \tag{4.7}$$

If $\partial_2 h^+ \neq 0$, then given θ_1, (4.7) can be used to define a sequence of values $\theta_2, \theta_3, \cdots$. Note that for $k=1$, the term involving θ_{k-1} is missing, so θ_2 is defined using only θ_1; for higher k, it is necessary to know θ_k and θ_{k-1} to obtain θ_{k+1}. We shall restrict ourselves to trajectories $\{\theta_i\}$ that lie in J, and hence $\partial_2 h^+ > 0$.

LEMMA 4.2. *Let $\{\theta_k\}$ be defined from θ_1 using* (4.7). *Then $d\theta_{k+1}/d\theta_1 > d\theta_k/d\theta_1 > 1$, for all k such that the trajectory stays in the region in which the hypotheses of Theorem* 4.1 *hold. Similarly, if $\{\theta_k\}$ is defined from θ_N using* (4.7), *then $d\theta_k/d\theta_N > d\theta_{k+1}/d\theta_N > 1$.*

Proof. For $k=1$, (4.7) is $0 = \omega + h^+(\theta_1, \theta_2)$. Differentiating this with respect to θ_1, we get $\partial_1 h^+ + \partial_2 h^+ \cdot (d\theta_2/d\theta_1) = 0$. By hypothesis, $|\partial_1 h^+| > |\partial_2 h^+|$; hence, using the fact that $\partial_1 h^+ < 0$ and $\partial_2 h^+ > 0$, we have that $d\theta_2/d\theta_1 > 1$.

Now assume by induction that $d\theta_j/d\theta_1 > d\theta_{j-1}/d\theta_1$, $j = 2, \cdots, k$. Differentiating (4.2) with index k with respect to θ_1, we obtain

$$0 = \partial_1 h^-(\theta_k, \theta_{k-1}) \cdot \left(\frac{d\theta_k}{d\theta_1}\right) + \partial_2 h^-(\theta_k, \theta_{k-1}) \cdot \left(\frac{d\theta_{k-1}}{d\theta_1}\right) + \partial_1 h^+(\theta_k, \theta_{k+1}) \cdot \left(\frac{d\theta_k}{d\theta_1}\right) + \partial_2 h^+(\theta_k, \theta_{k+1}) \cdot \left(\frac{d\theta_{k+1}}{d\theta_1}\right).$$

Again, since $|\partial_1 h^\pm| > |\partial_2 h^\pm|$, we can conclude that

$$0 < \partial_1 h^-(\theta_k, \theta_{k-1}) \left[\frac{d\theta_k}{d\theta_1} - \frac{d\theta_{k-1}}{d\theta_1}\right] + \partial_1 h^+(\theta_k, \theta_{k+1}) \left[\frac{d\theta_k}{d\theta_1} - \frac{d\theta_{k+1}}{d\theta_1}\right]. \tag{4.8}$$

By induction, $d\theta_k/d\theta_1 > d\theta_{k-1}/d\theta_1$. Thus, (4.7) implies that $d\theta_{k+1}/d\theta_1 > d\theta_k/d\theta_1$, and we are done. □

Proof of Theorem 4.1. We first show that $\theta_C < \theta_\pm$. From the definition of $\theta_\pm$, we have that $h^\pm(\theta_\pm, \theta_\pm) < 0$. From the definition of θ_C, we have that

$$\omega + h^+(\theta_C, \theta_C) = -h^-(\theta_C, \theta_C). \tag{4.9}$$

Since $h^-(\theta_C, \theta_C)$ is <0, the left-hand side of (4.9) is >0; this, plus the definition of θ_+ implies that $h^+(\theta_C, \theta_C) > h^+(\theta_+, \theta_+)$. Since $\partial_1 h^+ + \partial_2 h^+ < 0$, that implies $\theta_C < \theta_+$. A similar argument shows that $\theta_C < \theta_-$.

We may now construct upper and lower solutions to (4.1). These are easy to construct, since we may use constant functions (independent of k). Let $\theta_B = \max\{\theta_+, \theta_-\}$ and let $\theta_k^u \equiv \theta_B + \varepsilon$ for ε small enough so that $\theta_B + \varepsilon$ lies in J. As shown above, $h^\pm(\theta_B, \theta_B) < 0$. If $\varepsilon = 0$, (4.1) implies that $(\theta_k^u)'(0) < 0$ for $k \neq 1$, N and $=0$ for $k = 1$, N. For ε small but positive, the hypothesis $\partial_1 h^\pm + \partial_2 h^\pm < 0$ then implies that $(\theta_k^u)' < 0$ for all k. Thus, θ_k^u is an upper solution. Let $\theta_k^l \equiv \theta_C - \varepsilon$. For $\varepsilon = 0$, $(\theta_k^l)'(0) = 0$ for $k = 2, \cdots, N-1$. For $k=1$, $(\theta_1^l)' = \omega + h^+(\theta_C, \theta_C)$. Since $\theta_C < \theta_+$, $(\theta_1^l)'(0) > 0$, and similarly for $(\theta_N^l)'(0)$. As above, for ε not too large θ_k^l is a lower solution. By Lemma 4.1 and Corollary 4.1, $\theta_k^u > \theta_k^l$ for all k, t, so both upper and lower solutions lie in J for all t.

From the existence of an upper and lower solution we can construct an approximate time-independent solution to (4.1). To construct an actual time-independent solution, we need a further transversality condition which also shows the solution is locally unique; this condition is that the Jacobian $\bar{M}$ of (4.2) is invertible. The jth row of $\bar{M}$

is tridiagonal, with nonzero elements $\partial_2 h^-(\theta_j, \theta_{j-1})$, $\partial_1 h^+(\theta_j, \theta_{j+1}) + \partial_1 h^-(\theta_j, \theta_{j-1})$, $\partial_2 h^+(\theta_j, \theta_{j+1})$.

We use the Gerschgorin theorem, which states if λ is an eigenvalue of $\bar{M} = m_{ij}$, then for each i, $|\lambda - m_{ii}| \leqq \sum_j |m_{ij}|$. By the hypotheses, $\partial_1 h^{\pm} < -\partial_2 h^{\pm} < 0$. Hence, the Gerschgorin theorem implies that each eigenvalue λ of $\bar{M}$ is strictly contained in the left half complex plane. Thus, $\lambda = 0$ can never be an eigenvalue and so $\bar{M}$ is invertible at each point in the interval of J of the theorem. This proves local uniqueness and completes the proof. □

Remark 4.4. Although the above proof is motivated by large N (since the proof is modeled on ideas from continuum equations) it is valid for any $N > 1$, and in particular for $N = 2$. Thus it provides an alternative proof, but one that is much less transparent than that in § 2. This proof has the advantage that it works for $h^+ \neq h^-$; furthermore, as we now show, for $N = 2$ the proof generalizes to different frequencies $\omega_1 \neq \omega_2$.

The structure of the proof is the same as above, so we merely sketch the differences. We define θ_{B1}, θ_{B2}, and θ_C by

$$\omega_1 + h_1(\theta_{B1}, \theta_{B1}) = 0, \qquad \omega_2 + h_2(\theta_{B2}, \theta_{B2}) = 0, \tag{4.10}$$

$$\frac{\omega_1 + \omega_2}{2} + h_1(\theta_C, \theta_C) + h_2(\theta_C, \theta_C) = 0. \tag{4.11}$$

Instead of assuming that $h_i(\theta_C, \theta_C) < 0$, we make the assumption that

$$h_i(\theta_C, \theta_C) + \frac{\omega_i - \omega_j}{2} < 0, \qquad i, j = 1, 2, \quad i \neq j. \tag{4.12}$$

This inequality should be interpreted as providing restrictions on the anisotropy and the frequency differences; (4.12) is automatically satisfied if $h_1 = h_2$ and $\omega_1 = \omega_2$. From (4.11) and (4.12), we can conclude that $\omega_i + h_i(\theta_C, \theta_C) > 0$ and hence, as before, $\theta_C < \theta_{Bi}$, $i = 1, 2$. The constructions of the lower and upper solutions are as before.

5. Discussion. (A) The oscillator death of this paper occurs for differential equations using models of synapses that are much simplified. In particular, the model synapses do not exhibit fatigue. Thus, if a pair of oscillators modeling bursting neurons go into tonic firing when they are coupled, the system may still be capable of oscillation if fatigue and recovery of the synapses are introduced. However, the mechanism of this network oscillation is quite different from that of the original oscillations; it is still correct to say that the coupling turned off the original oscillators. Since this kind of mathematical description of a synapse is common in the modeling literature, it is important to understand the behavior of such models. For example, when the oscillator death occurs by the creation of a saddle-node on the limit cycle, as in the phase models and some of the full models, the frequency first almost slows down to zero. Hence, we see that excitatory coupling can lead to a slower oscillation.

In addition to fatigue, another possible problem with the present model is that it does not take synaptic delay into effect. That is, if two identical neural oscillators are coupled synaptically, it is likely that there will be a delay between the firing of one and the effect on the other. There are many sources for such delays, including axon conductance time, transmitter release, etc. It is a priori conceivable that the delay could destabilize the in-phase phase-death solution and thus prevent phase death. We have numerically studied the Morris–Lecar model and cannot find any destabilization. Indeed, we can prove an analogue of Proposition 2.2 for a pair of phase models with a delay.

PROPOSITION 5.1. *Consider the two oscillator model:*

$$\frac{d\theta_1}{dt}=\omega+h(\theta_2(t-\tau),\theta_1(t)),\qquad \frac{d\theta_2}{dt}=\omega+h(\theta_1(t-\tau),\theta_2(t)). \tag{5.1}$$

Assume that there is an equilibrium solution (ϕ,ϕ) and that it is asymptotically stable when $\tau=0$. Then if $\tau>0$, it is also asymptotically stable.

Proof. Let $b=\partial_1 h(\phi,\phi)$ and $a=\partial_2 h(\phi,\phi)$, where the subscripts denote derivatives with respect to the first or second argument. In absence of delays, the eigenvalues of the linear problem are $b\pm a$. The assumption of asymptotic stability implies that both are negative; hence, $b<0$, $b<\pm a$ as in Proposition 2.2. The linearized equations for (5.1) about the equilibrium solution have the form

$$y_1'=by_1+ay_2(t-\tau),\qquad y_2'=by_2+ay_1(t-\tau). \tag{5.2}$$

The eigenvalue equation for this is

$$(\lambda-b-ae^{-\lambda\tau})(\lambda-b+ae^{-\lambda\tau})=0. \tag{5.3}$$

We must show that the roots of the transcendental equations $\lambda+b\pm ae^{-\lambda\tau}=0$ are all in the left half complex plane. If we multiply $\lambda-b\pm ae^{-\lambda t}$ by $\tau e^{\lambda\tau}$ and let $z=\lambda\tau$, $p=b\tau$, and $q=\pm a\tau$, the equation becomes $pe^z+q-ze^z=0$. By a theorem in [31], the roots of this equation are in the left half plane if and only if

$$\text{(i)}\quad p<1,\qquad \text{(ii)}\quad p<-q<\sqrt{\beta^2+p^2}$$

where $0<\beta<\pi$ and $\beta=p\tan(\beta)$. The first condition follows since $p=b\tau<0$. The second follows from the fact that $|b\tau|>|a\tau|$. □

(B) As remarked in § 3.3, the essential feature of the coupling $h(\theta_k,\theta_j)$ needed to obtain oscillator death is that h does not vanish identically if $\theta_k=\theta_j$. Thus, the coupling $G_k(u_k,u_j)$ of (2.3) may not be mathematically equivalent to diffusion, i.e., given by $D(u_j-u_k)$, where D is a positive definite matrix. In a previous paper, we have defined and used a related notion of "synaptic coupling" that is also inequivalent to diffusion.The coupling function $h_k(\theta_k,\theta_j)$ is said to be "synaptic" if *on the average* h does not vanish if $\theta_k=\theta_j$, i.e.,

$$\int_0^{2\pi} h_k(\theta,\theta)\,d\theta\neq 0.$$

We can now see that synaptic coupling is not necessary to obtain oscillator death. Indeed, the example in Remark 2.6 has coupling that vanishes on the average, but not identically, if $\theta_1=\theta_2$, and there is oscillator death. However, standard models of neural coupling *are* also synaptic.

Appendix. In this Appendix, we derive an explicit approximation to $h_k(\theta_k,\theta_j)$ in terms of the coupling and the limit cycles; this formula has greater accuracy the greater the strength of attraction of the limit cycle. We then apply this formula to the Lecar-Morris equations and Wilson-Cowan equations.

Consider the pair of coupled oscillators (2.3). We assume as before that $du_k/dt=F_k(u_k)$ has an asymptotically stable limit cycle solution that we will call $U_k(t/\omega_k)$, where ω_k is the frequency of the limit cycle and $U_k(\theta)$ is 2π-periodic in its argument. Let $u_k=T_k(\theta_k,y_k)$, $y\in\mathbf{R}^{n-1}$, be coordinate transformations chosen so that

$$u_k(t)=U_k(\theta_k(t))+M_k(\theta_k(t))y_k(t)+O(|y_k^2|) \tag{A1}$$

where $M_k(\theta)$ is an $n\times(n-1)$ matrix and

$$M_k(\theta)^TM_k(\theta)=1_{(n-1)\times(n-1)},\qquad U_k'(\theta)^TM_k(\theta)=0_{1\times(n-1)}. \tag{A2}$$

We use $'$ to denote derivatives with respect to θ. The Jacobian of the transformation is

$$J_k(\theta_k, y_k) = (U_k'(\theta_k) + M_k'(\theta_k)y_k,\ M_k(\theta_k)) + O(|y_k^2|).$$

For $y_k = 0$, J_k is an orthogonal matrix, so its inverse is easily computed. Furthermore, if B is small, to lowest order in B the inverse of $(A+B)$ is $(I - A^{-1}B)A^{-1}$. Hence, if y_k is small, the inverse of J_k is

$$\begin{aligned}&(I - J_{k0}^{-1}[M_k' y_k, 0])J_{k0}^{-1} + O(|y_k|^2)\\ &\quad = \begin{bmatrix} (1-(U_k')^T M_k' y_k/\rho_k)(U_k')^T/\rho_k \\ -(M_k)^T M_k' y_k (U_k')^T/\rho_k + (M_k)^T \end{bmatrix} + O(|y_k|^2)\end{aligned} \tag{A3}$$

where we have used

$$J_{k0}^{-1} = \begin{bmatrix} (U_k')^T/\rho_k \\ (M_k)^T \end{bmatrix}, \qquad \rho_k = |U_k'|^2. \tag{A4}$$

Next, we note that

$$\begin{aligned}F_k(U_k + M_k y_k) &= F_k(U_k) + D_k M_k y_k + O(|y_k|^2)\\ &= U_k' + D_k M_k y_k + O(|y_k|^2)\end{aligned} \tag{A5}$$

where D_k is the Jacobian of F_k evaluated at U_k. U_k', D_k, and M_k are all periodic in θ and independent of y_k to lowest order. Finally, we observe that by differentiating (A2) with respect to θ and using the fact that $U_k'' = D_k U_k'$, we see that

$$-(U_k')^T M_k' = (U_k')^T (D_k)^T M_k. \tag{A6}$$

Additionally, if we differentiate the first equation in (A2) we find that

$$-(M_k)^T M_k' = (M_k')^T M_k.$$

Substituting (A3), (A5), (A6) into (2.3) gives us the equations in (θ_k, y_k):

$$\begin{aligned}\omega_k^{-1}\frac{d\theta_k}{dt} &= 1 + \rho_k^{-1}(U_k')^T[(D_k + (D_k)^T)M_k y_k + G_k],\\ \omega_k^{-1}\frac{dy_k}{dt} &= ((M_k)^T D_k M_k + (M_k')^T M_k)y_k + (M_k)^T G_k.\end{aligned} \tag{A7}$$

(Note that ω_k^{-1} arises from differentiation of $U_k(\theta_k/\omega_k)$ with respect to time.) We will use (A7) as the basis for the rest of the calculations in this Appendix.

In the limit of "infinite attraction" to the limit cycle, $y_k \to 0$, and (A7) becomes (2.5) with

$$h_k(\theta_k, \theta_j) = \omega_k \rho_k^{-1}(\theta_k)(U_k')^T(\theta_k) G_k(U_k(\theta_k), U_j(\theta_j)). \tag{A8}$$

The significance of (A8a) is that it allows us to compute an approximate phase model, which is more accurate the greater the strength of the attraction of the limit cycle.

As discussed in § 2, under some important circumstances involving coupling through only one variable, equations (1.2) have product coupling of the form (2.5). To use (A8) on the Wilson-Cowan equations, we need to recast the latter in "voltage" variables so that the coupling is additive. The membrane potential of the excitatory (respectively, inhibitory) cells is $U = \alpha_{ee}E - \alpha_{ie}I$ (respectively, $V = \alpha_{ei}E - \alpha_{ii}I$). As long

as this change of variables is nonsingular, then we can rewrite (3.2) in terms of the new variables (U_j, V_j):

$$\text{(A9)}\qquad \begin{aligned}\frac{dU_j}{dt} &= -U_j + \alpha_{ee}S(U_j-\nu_e) - \alpha_{ie}S(V_j-\nu_i) + \beta_{ee}S(U_k-\nu_e) - \beta_{ie}S(V_k-\nu_i),\\ \frac{dV_j}{dt} &= -V_j + \alpha_{ei}S(U_j-\nu_e) - \alpha_{ii}S(V_j-\nu_i) + \beta_{ei}S(U_k-\nu_e) - \beta_{ii}S(V_k-\nu_i),\end{aligned}$$

$j, k = 1, 2, k \neq j$. (Note that, since $\alpha_{ii} = 0$, the transformation is never singular if an oscillation exists, since $\alpha_{ei}\alpha_{ie} \neq 0$.)

We are now in a position to use (A8). There are four combinations of phase models possible corresponding to the four types of coupling. Table A1 summarizes them, and in Figs. A1(a) and A1(b), we sketch two pulse functions and two response functions. If the interactions are composite, to obtain the function h multiply by the appropriate synaptic weight (e.g., $-\beta_{ie}$ or β_{ee}) and add all synaptic interactions.

We now give the pulse function $P(\theta)$ and the response function $R(\theta)$ for the Lecar-Morris model (whose Floquet exponent is very small) with excitatory or inhibitory coupling. Let $(V_0(t), w_0(t))$ be the stable limit cycle for (3.5), and let τ be its period. Let $P_e(\theta)$ (respectively, $P_i(\theta)$) and $R_e(\theta)$ (respectively, $R_i(\theta)$) denote the pulse and response functions for excitatory (respectively, inhibitory) coupling. Then from (3.7) and (A8)

$$\text{(A10)}\qquad P_e(\theta) = g_{ex}(V_0(\theta)), \qquad R_e(\theta) = \omega V_0'(\theta)(V_{ex} - V_0(\theta))/\rho(\theta)$$

TABLE A1

This gives the formula for the pulse and response functions for the Wilson-Cowan equations for the parameters in Fig. 3.2. Here, $(U_0(t), V_0(t))$ is the stable limit cycle and $\rho(\theta) = U_0'^2(\theta) + V_0'^2(\theta)$.

	Pulse	Response
EE	$U_0'(\theta)/\rho(\theta)$	$S(U_0(\theta)-\nu_e)$
EI	$V_0'(\theta)/\rho(\theta)$	$S(U_0(\theta)-\nu_e)$
IE	$U_0'(\theta)/\rho(\theta)$	$S(V_0(\theta)-\nu_i)$
II	$V_0'(\theta)/\rho(\theta)$	$S(V_0(\theta)-\nu_i)$

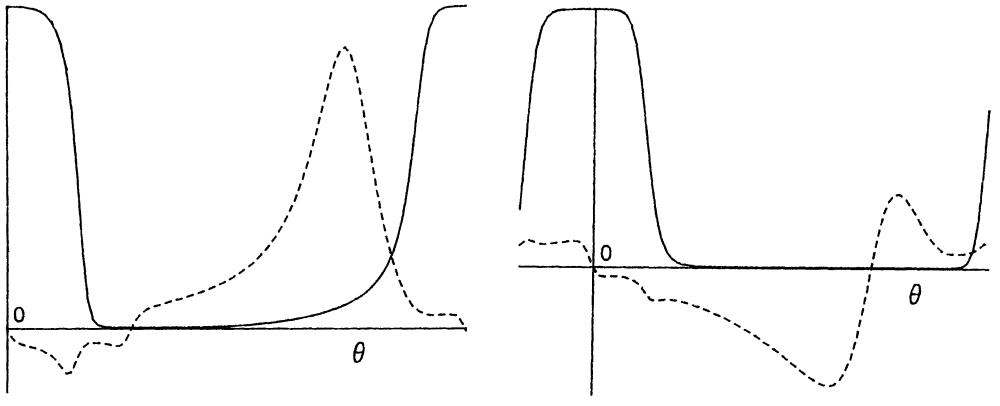

FIG. A1(a) *Pulse (solid) and response (dotted) functions for excitatory-excitatory coupling of the Wilson-Cowan equations, parameters as in Fig.* 3.2. (b) *The same as* (a) for inhibitory-inhibitory coupling.

where $\rho(\theta) = V_0'^2(\theta) + w_0'^2(\theta)$. Similarly,

$$\text{(A11)} \qquad P_i(\theta) = g_{in}(V_0(\theta)), \qquad R_i(\theta) = \omega V_0'(\theta)(-V_{in} - V_0(\theta))/\rho(\theta).$$

If the nonlinear conductances g_{ex} and g_{in} are monotone, then $R_e = 0$ (respectively, $R_i = 0$) at the point at which P_e (respectively, P_i) is maximal. In Figs. A2(a) and A2(b) we sketch these two curves for a typical set of parameters (corresponding to the values in Fig. 3.3). The response functions, especially R_e, are qualitatively sinusoidal; a good approximation is $-\alpha[\sin(\theta+\xi) - \sin(\xi)]$.

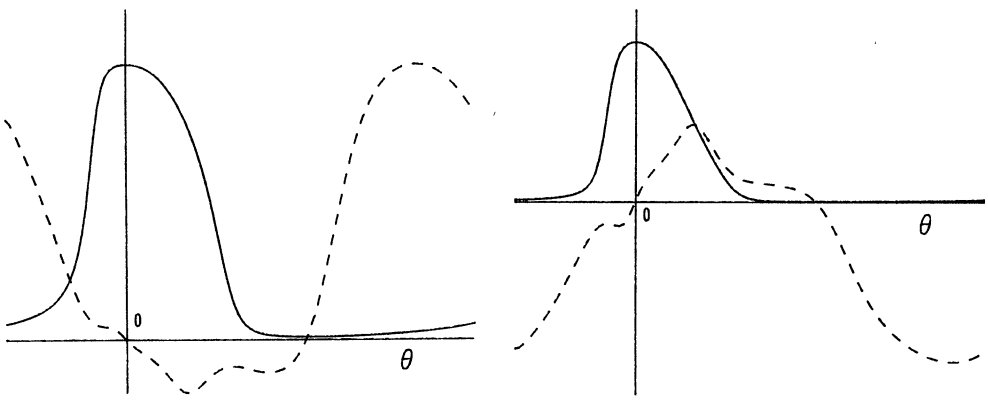

FIG. A2(a) *Pulse (solid) and response (dotted) functions for excitatory coupling of the Lecar-Morris equations, parameters as in* (3.3). (b) *Same as* (a) *for inhibitory coupling*; $V_{in} = V_k$, $g_{in}(V) = w_\infty(V)$.

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