The Existence of Spiral Waves in an Oscillatory Reaction-Diffusion System

Joseph Paullet, Bard Ermentrout, William Troy

SIAM Journal on Applied Mathematics, Volume 54, Issue 5 (Oct., 1994), 1386-1401.

Your use of the JSTOR database indicates your acceptance of JSTOR's Terms and Conditions of Use. A copy of JSTOR's Terms and Conditions of Use is available at http://www.jstor.org/about/terms.html, by contacting JSTOR at jstor-info@umich.edu, or by calling JSTOR at (888)388-3574, (734)998-9101 or (FAX) (734)998-9113. No part of a JSTOR transmission may be copied, downloaded, stored, further transmitted, transferred, distributed, altered, or otherwise used, in any form or by any means, except: (1) one stored electronic and one paper copy of any article solely for your personal, non-commercial use, or (2) with prior written permission of JSTOR and the publisher of the article or other text.

Each copy of any part of a JSTOR transmission must contain the same copyright notice that appears on the screen or printed page of such transmission.

SIAM Journal on Applied Mathematics is published by Society for Industrial and Applied Mathematics. Please contact the publisher for further permissions regarding the use of this work. Publisher contact information may be obtained at http://www.jstor.org/journals/siam.html.

SIAM Journal on Applied Mathematics
©1994 Society for Industrial and Applied Mathematics

JSTOR and the JSTOR logo are trademarks of JSTOR, and are Registered in the U.S. Patent and Trademark Office. For more information on JSTOR contact jstor-info@umich.edu.

©2001 JSTOR

http://www.jstor.org/
Mon Aug 20 16:57:04 2001

SIAM J. APPL. MATH.
Vol. 54, No. 5, pp. 1386–1401, October 1994
© 1994 Society for Industrial and Applied Mathematics
011

THE EXISTENCE OF SPIRAL WAVES IN AN OSCILLATORY REACTION-DIFFUSION SYSTEM*

JOSEPH PAULLET†‡, BARD ERMENTROUT†‡§, AND WILLIAM TROY†§

Abstract. Rotating waves are proven to exist on the unit disk for an oscillatory reaction-diffusion equation with Neuman boundary conditions. The method of proof relies on a two-parameter shooting argument for the ensemble frequency and the radial derivative of the magnitude. Numerical solutions indicate that the waves are stable if the diffusion is sufficiently small. It is also shown that these solutions cease to exist for large diffusion. The origin of the rotating waves is discussed.

Key words. reaction-diffusion equations, spiral waves, oscillatory media

AMS subject classifications. 92C20, 92C05, 34B15

1. Introduction.

Spiral waves are a common feature in many active media such as chemical oscillators, excitable slime molds, epileptic waves, and cardiac fibrillation. (See [12] for a comprehensive review.) These phenomena are generally modelled as systems of reaction-diffusion equations where the reaction terms are either excitable or oscillatory. The mathematical problem is then to prove the existence of rotating Archimedean spirals in the domain of interest.

There have been many approaches to the solution of this problem. Keener and collaborators [7],[8] have used singular perturbation, asymptotics, and other formal arguments to construct spiral waves in the infinite plane as well as in three dimensions. They take advantage of the slow time scales of some of the variables as well as a small parameter multiplying diffusion. This approach has led to good qualitative and *quantitative* agreement with numerical results. Fife [4] views the existence of spirals in the plane as a free boundary problem. Recently, Xin [13] proves that a solution to Fife's free-boundary problem exists for a certain class of models that have symmetry and are in the oscillatory regime (in spite of the title of the paper). Cohen et al. [3] use fixed point methods to prove the existence of *logarithmic* spiral waves in a reaction-diffusion equation that has unusual local dynamics (in the sense that they *do not* arise as a normal form near a Hopf bifurcation). This result is rigorous and concerns the infinite plane. Greenberg [5] and Hagan [6] both consider oscillatory rather than excitable systems and formally construct spiral wave solutions in the plane. Their work is the main motivation for the results in the present paper.

Rather than work in the whole plane, we will restrict our attention to the question of existence of spiral waves in a finite circular domain. The restriction to a finite domain is not unreasonable as most experimental and numerical simulations are by necessity in a finite region. The restriction to a circular domain allows us to exploit the rotational symmetry and thus reduce the question of existence to a solution to a boundary value problem.

Here, we consider an oscillatory medium in a circular domain and rigorously prove the existence of rotating waves which in the limit of large domains approach the Archimedean spirals that have been formally constructed in infinite planar systems.

* Received by the editors May 6, 1993; accepted for publication (in revised form) November 29, 1993.

† Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15260.

‡ The work of this author was supported in part by National Institute of Mental Health grant 47150.

§ The work of this author was supported in part by National Science Foundation grant DMS-9002028.

We begin with the equation

$$u_t = u\lambda(u,v) - v\omega(u,v) + d\Delta u, \quad (1.1)$$
$$v_t = v\lambda(u,v) + u\omega(u,v) + d\Delta v,$$

$$n \cdot \nabla u = 0 \quad \text{on } \partial\Gamma, \quad (1.2)$$
$$n \cdot \nabla v = 0 \quad \text{on } \partial\Gamma,$$

where

$$\lambda(u,v) = 1 - u^2 - v^2, \quad (1.3)$$

$$\omega(u,v) = 1 + q(u^2 + v^2), \quad (1.4)$$

and Γ is the unit disk. This problem arises formally from the expansion of a general reaction diffusion system with scalar diffusion near a Hopf bifurcation:

$$c_t = d\Delta c + F(c,p),$$
$$n \cdot \nabla c = 0 \quad \text{on } \partial\Gamma,$$

where c is a vector of concentrations and p is some parameter. At $p = 0$ there is a supercritical Hopf bifurcation of the kinetics, F. After rescaling time and the amplitudes, (1.1) and (1.2) arise in a formal expansion near this critical parameter [10]. We note that if the diffusivity is not scalar, then the diffusion coefficient in (1.1) is a complex scalar. This makes the analysis much more difficult so we will restrict our attention to the scalar case. Thus, one can view our system better as a model of reacting chemical species which generally have similar diffusion coefficients than as a model for interacting pacemaker cells where only the potential diffuses. We only consider Neumann boundaries as these are the natural ones for a set of chemical species in a dish. Dirichlet conditions, while perhaps interesting from a mathematical point of view, introduce an unrealistic inhomogeneity in the medium. (Indeed, in real chemical systems, the homogeneous oscillation is always a solution; Dirichlet conditions prevent this.)

The problem (1.1) has been of interest to a number of researchers, particularly with respect to the existence of spiral waves. Hagan [6] analyzes it on the infinite domain and shows that when $q = 0$ there is a stable rotating wave solution of the form

$$u + iv = A(R)\exp(i(t + \phi)), \quad (1.5)$$

where (R, ϕ) are the polar coordinates on the (x, y) plane and $A(0) = 0, A(R) \to 1$ as $R \to \infty$. He uses the prior results of Greenberg [5] in order to obtain the existence of the function $A(R)$. Kopell and Howard [9] rigorously prove the existence of spiral waves in the whole plane by exploiting the smallness of the parameter q. Their approach is to treat the systems as a perturbed central force problem and then use dynamical systems and energy functions to obtain the proof. The proofs here have a similar flavor although we use standard methods from analysis rather than the geometric techniques that they exploit. Furthermore, by restricting our attention to the finite domain, we find some interesting behavior as a function of the domain size that is missed in the full plane analysis. The behavior in the finite plane then suggests a formal bifurcation analysis that sheds light on the stability of the spiral waves as well as their origin.

In addition to the single-armed spirals, one could also look for multi-armed spirals as well. Indeed, Hagan formally finds a family of such waves in the infinite plane case. We have no doubt that such solutions could probably be found in a finite dish;

however, Hagan also shows that these modes are formally unstable. Thus, we have not attempted to prove existence for the multi-armed case.

It is clear from the form of the solution (1.5) that such a solution represents a rotating wave that is more of a "pinwheel" than a true spiral. Hagan then uses formal perturbation arguments to construct rotating waves of the form

$$u + iv = A(R) \exp i \left((1 - \Omega)t + \phi + \int_0^R k(s)ds \right). \tag{1.6}$$

Because $k(R) \sim k_0$ as $R \to \infty$ these solutions are similar to Archimedean spirals, for then

$$(u, v) \sim (\cos((1 - \Omega)t + \phi + k_0 R), \sin((1 - \Omega)t + \phi + k_0 R)), \tag{1.7}$$

that is, the contours of constant "concentration" satisfy $(\Omega + 1)t + \phi + k_0 R = C$. Notice that when $k_0 = 0$ the contours are radial lines and so are the aforementioned pinwheels. We note that when $q = 0$, it must be the case that $\Omega = 0$ and $k(R) \equiv 0$ (see §2.2 for details). Therefore, in order to get "geometric" twist in the spiral patterns, one must have nonzero values of q.

Without loss of generality, we may assume that the domain is the unit disk since the radius can be scaled into the diffusion parameter, d. We will rigorously prove the existence of solutions of the form (1.6) for q nonzero on the unit disk if d is sufficiently small. In fact, we show that if d is too large, then such solutions do not exist. Although we do not prove it, we conjecture that the critical diffusivity is given by $d^* = 1/\sqrt{z^*}$, where z^* is the first zero of the derivative of J_1 the first-order Bessel function. We also believe that these solutions will be stable only if d is sufficiently small. Thus, only sufficiently large domains can support stable spirals.

Substituting (1.6) into (1.1) and (1.2) we obtain the following boundary value problem:

$$A'' + \frac{A'}{R} - \frac{A}{R^2} = Ak^2 + \frac{A}{d}(A^2 - 1), \tag{1.8}$$

$$k' + k\left(\frac{1}{R} + \frac{2A'}{A}\right) = -\frac{\Omega + qA^2}{d}, \tag{1.9}$$

$$A(0) = k(0) = 0, \tag{1.10}$$

$$A'(1) = k(1) = 0. \tag{1.11}$$

The problem that Hagan formally analyzes is identical except that (1.11) is replaced by conditions at infinity:

$$A(\infty) = 1 \qquad k(R) \sim k_0 \quad \text{as } R \to \infty. \tag{1.12}$$

Remark 1.1. We note that if (A, k) solves (1.8)–(1.11) for some $q > 0$ and $\Omega < 0$ then there corresponds a second solution, $(A, -k)$ which solves (1.8)–(1.11) for the parameter values $\hat{q} = -q < 0$ and $\hat{\Omega} = -\Omega > 0$. Therefore, in Theorem 1 below we restrict our attention to positive values of q and negative values of Ω. Specifically, we prove the following theorems in the next sections.

THEOREM 1.2. *If $d > 0$ is sufficiently small there exists a value $q_1 > 0$ such that if $0 < q < q_1$ then the problem* (1.8)–(1.11) *has a solution for some $\Omega \in (-q, 0)$, and this solution satisfies $A' > 0$ over* [0,1). *If $q = 0$ then* (1.8)–(1.11) *has a solution for $\Omega = 0$, and this solution also satisfies $A' > 0$ over* [0,1).

THEOREM 1.3. *Let $d \geq 1$. Then no solution of* (1.8)–(1.11) *exists which satisfies $A > 0$ over* (0,1).

2. Outline of proofs and the $q = 0$ problem.

2.1. Outline. For Theorem 1 we employ a two-dimensional shooting argument. The parameters which we adjust are $d > 0$, $q > 0$, $\Omega < 0$ and $\alpha > 0$, where $A'(0) = \alpha$. In §2.2 we set $q = 0$ and conclude that $\Omega = 0$ and therefore $k = 0$ as long as $A > 0$. With $k = 0$ we then analyze (1.8) and show that if $d > 0$ and small there is an $\alpha_0 = \alpha_0(d) > 0$ for which $A' > 0$ on [0,1) and $A'(1) = 0$. For such small fixed d we then "perturb" off of the solution for which $\alpha = \alpha_0(d)$ and consider the full system (1.8)–(1.9) for small $q > 0$. We do this in §3. In particular, we find that Ω must lie in the interval $(-q, 0)$. Then, using a topological result due to McLeod and Serrin [11] we prove that there is a continuum Γ contained in the region $\alpha > 0, \Omega \in (-q, 0)$, which joins the lines $\{\Omega = 0, \alpha > 0\}$ and $\{\Omega = -q, \alpha > 0\}$ and such that $A' > 0$ on [0,1) and $A'(1) = 0$ for each $(\alpha, \Omega) \in \Gamma$. For $(\alpha, \Omega) \in \Gamma$ and close to $\Omega = 0$ we prove that $k(1) < 0$, while $k(1) > 0$ if Ω is close to $-q$. Since the interval $0 \leq R \leq 1$ is compact, then $k(1)$ is continuously dependent on (α, Ω). Thus, since Γ is a continuum, there must be an $(\bar{\alpha}, \overline{\Omega}) \varepsilon \Gamma$ for which $k(1) = 0$, and our theorem will then be proved. The proof of Theorem 2 relies on an analysis of the equation for $\rho = A'/A$. It is very short and straightforward and appears in §4.

2.2. The reduced problem for $q = 0$. With $q = 0$, (1.9) reduces to

$$k' + k\left(\frac{1}{R} + \frac{2A'}{A}\right) = -\frac{\Omega}{d}$$

and a simple integration gives

$$RA^2(R)k(R) = -\frac{\Omega}{d}\int_0^R tA^2(t)dt.$$

Since $A'(0) = \alpha > 0$ there is a small $\varepsilon > 0$ such that the solution exists for $0 \leq R \leq \varepsilon$, and therefore $L \equiv \int_0^\varepsilon tA^2(t)dt > 0$. Suppose that $\Omega \neq 0$ and that conditions (1.10)–(1.11) are satisfied. Then both $A(R)$ and $k(R)$ must be bounded. If $\Omega > 0$ the previous equation leads to $RA^2(R)k(R) \leq \frac{-\Omega L}{d} < 0$ for $\varepsilon \leq R \leq 1$. Thus, because $k(1) = 0$, we see that $A^2(R) \to \infty$ as $R \to 1$, contradicting the boundedness of A. Similarly Ω cannot be negative. Thus, if (1.10)–(1.11) are to be satisfied, it must be the case that $\Omega = 0$. This reduces the initial value problem for A to

$$A'' + \frac{A'}{R} - \frac{A}{R^2} = \frac{A}{d}(A^2 - 1), \tag{2.1}$$

$$A(0) = 0, \quad A'(0) = \alpha. \tag{2.2}$$

We let $A(R, \alpha)$ denote the solution of (2.1)–(2.2). Wherever it is convenient, however, we will omit the dependence of A on α. The rest of this section is devoted to the analysis of the behavior of solutions of (2.1)–(2.2). In order to construct our topological shooting argument for the full problem (i.e., $q \neq 0$) we shall need a few preliminary results. The first is the following.

LEMMA 2.1. *Let $d > 0$ and $\alpha > 0$. If $A'(\overline{R}) = 0$ for some first $\overline{R} > 0$, then $A''(\overline{R}) < 0$.*

Proof. The definition of $\overline{R}$ implies that $A''(\overline{R}) \leq 0$. If $A''(\overline{R}) = 0$ then $A'''(\overline{R}) = -2A(\overline{R})/\overline{R}^3 < 0$. Thus $A'' > 0$ on an interval $(\overline{R}-\varepsilon, \overline{R})$. This implies that $A'(R) < 0$ on $(\overline{R} - \varepsilon, \overline{R})$, contradicting the definition of $\overline{R}$. Therefore we must conclude that $A''(\overline{R}) < 0$, and the proof is complete. □

Next, we consider small values of $d > 0$ and $\alpha > 0$ and prove the following.

LEMMA 2.2. *There are values $d_1 \in (0,1)$ and $\bar{\alpha} > 0$ such that if $0 < d < d_1$ and $0 < \alpha < \bar{\alpha}$, then $A'(\overline{R}) = 0$ at some first $\bar{R} \in (0,1)$.*

Proof. We assume, on the contrary, that the conclusion of the lemma is false. Then there are positive, decreasing sequences $\{\alpha_i\}_{i\in N}$ and $\{d_i\}_{i\in N}$ with $\lim_{i\to\infty}(\alpha_i, d_i) = (0,0)$ and such that for each $i, A' > 0$ for $R \in (0,1)$ as long as the solution exists. For ease of notation we delete the dependence of α_i and d_i on i. The variation of parameters formula for A' is given by

$$A' = \alpha + \frac{1}{2d}\int_0^R \left(1 + \frac{t^2}{R^2}\right) A(A^2-1)\, dt. \tag{2.3}$$

It is apparent from (2.3) that $A' \leq \alpha$ and so $A \leq \alpha R$ as long as $0 < A < 1$. Thus we restrict our attention to $0 < \alpha < 1$ so that $0 < A < 1$, and the solution exists with $A' > 0$ on the entire interval (0,1). Next, setting $\rho = A'/A$ we find that ρ satisfies

$$\rho' = -\rho^2 - \frac{\rho}{R} + \frac{1}{R^2} + \frac{A^2-1}{d}. \tag{2.4}$$

Note that $\rho > 0$ on (0,1) since we are assuming that $A' > 0$ over (0,1). We consider the interval $\frac{1}{2} \leq R \leq \frac{3}{4}$. Since $A \leq \alpha R$ we may choose $d > 0$ and $\alpha > 0$ sufficiently small so that (2.4) reduces to

$$\rho' \leq -\left(\rho^2 + \frac{1}{4d}\right) \quad \text{for } \frac{1}{2} \leq R \leq \frac{3}{4}. \tag{2.5}$$

Integrating this differential inequality from $\frac{1}{2}$ to R we conclude that

$$\tan^{-1}(2\sqrt{d}\rho(R)) \leq \tan^{-1}\left(2\sqrt{d}\rho\left(\frac{1}{2}\right)\right) - \frac{1}{2\sqrt{d}}\left(R - \frac{1}{2}\right). \tag{2.6}$$

Since $0 < \rho < \infty$, then $\tan^{-1}(2\sqrt{\lambda}\rho(\frac{1}{2})) \leq \frac{\pi}{2}$. Thus for d small it follows that, at $R = 3/4, \tan^{-1}(2\sqrt{d}\rho(3/4)) \leq \frac{\pi}{2} - \frac{1}{8\sqrt{d}} < 0$; hence $\rho < 0$ and $A' < 0$, a contradiction since A' is assumed to be positive over (0,1). This completes the proof. □

We now restrict d to $0 < d < d_1$ and consider large values of α.

LEMMA 2.3. *Let $0 < d < d_1$ and $\alpha \geq \frac{1}{d} + \frac{2}{9\sqrt{3}}$. Then $A(R_1) = 1$ at some first $R_1 \in (0, d]$, and $A' > 0$ for all $R \in [0, R_1]$.*

Proof. Recall that the variation of constants formula for A' is given by

$$A' = \alpha + \frac{1}{2d}\int_0^R (1 + t^2/R^2)A(A^2-1)dt. \tag{2.7}$$

Thus, since $A(A^2-1) \geq \frac{-2}{3\sqrt{3}}$, (2.7) reduces to

$$A' \geq \alpha - \frac{1}{3d\sqrt{3}} \int_0^R \left(1 + \frac{t^2}{R^2}\right) dt = \alpha - \frac{4}{9d\sqrt{3}} R. \tag{2.8}$$

If $0 \leq R \leq d$, then $A' > 0$ for $\alpha > \frac{4}{9\sqrt{3}}$. A further integration gives $A(R) > \alpha R - \frac{2R^2}{9d\sqrt{3}}$. From this it follows that $A(R_1) = 1$ at some first $R_1 \in (0, d]$ if $\alpha \geq \frac{1}{d} + \frac{2}{9\sqrt{3}}$. □

Hereafter we assume that $d \in (0, d_1)$ is fixed. It is possible that solutions of (1.8)–(1.9) exist which satisfy $A \to \infty$ at finite $R \in (0,1)$. Thus, in order to proceed with our shooting argument, we first need to prove that there is a value of α for which the solution exists over the entire interval [0,1], and that $A' > 0$ over this interval. This is accomplished within the next lemma.

LEMMA 2.4. *There exists a value $\alpha^* > 0$ such that the solution $A(R, \alpha^*)$ satisfies $A' > 0$ for all $R > 0$ and $\lim_{R\to\infty} A = 1$.*

Proof. Define the set $E = \{\hat{\alpha} > 0 |$ if $\alpha > \hat{\alpha}$ then there exists a first $R_1 = R_1(\alpha) > 0$ such that the solution of (2.1)–(2.2) satisfies $A' > 0$ for $0 \leq R \leq R_1$ and $A(R_1) = 1\}$. □

It follows from Lemma 3 and continuity that E is nonempty and open. Let $\alpha^* = \inf E$. Then, by Lemma 2, $\alpha^* \geq \bar{\alpha}$. Let $\alpha = \alpha^*$. If it were the case that $A' = 0$ at some first $\bar{R}$ then Lemma 1 shows that $A''(\overline{R}) < 0$. Since $A(\overline{R}) > 0, A'(\overline{R}) = 0$, and $A''(\overline{R}) < 0$, equation (2.1) shows that $A(\overline{R})/\overline{R}^2 + A(\overline{R})/d(A^2(\overline{R}) - 1) < 0$. Thus, $A(\overline{R}) < 1$. Furthermore, we conclude from these observations that there is an interval $(\overline{R}, \overline{R} + \delta)$ in which $A(R) > 0$ and $A'(R) < 0$. This and continuity imply that $\alpha \notin E$ if $\alpha - \alpha^* > 0$ and sufficiently small, contradicting the definition of α^*. Thus $A' > 0$ for all $R > 0$. Again, continuity and the definition of α^* imply that $A < 1$ on $[0, \infty)$. It then follows from (2.1) that $\lim_{R\to\infty} A = 1$ and $\lim_{R\to\infty} A' = 0$.

Next, we further restrict our attention to the range $0 < \alpha < \alpha^*$. At $\alpha = \alpha^*$ we have shown in Lemma 4 that the solution $A(R, \alpha^*)$ satisfies $A'(R, \alpha^*) > 0$ and $A < 1$ on $[0, \infty)$. For $\alpha \in (0, \alpha^*)$ we will prove in Lemma 5 that the solution $A(R, \alpha)$ of (2.1)–(2.2) remains below $A(R, \alpha^*)$ for $R \in (0,1)$ as long as $A'(R, \alpha) > 0$. This crucial property prevents $A(R, \alpha)$ from becoming unbounded at some finite $\tilde{R}$ before $A'(R, \alpha)$ has a zero. In turn this observation will then allow us to proceed with the construction of our shooting argument for the full system (1)–(2). Thus, for each $\alpha \in (0, \alpha^*)$ we define $\overline{R} = \overline{R}(\alpha)$ by

$$\overline{R} = \sup\{\hat{R} \in (0,1) | A'(R, \alpha) > 0 \text{ for } 0 \leq R < \hat{R}\}.$$

LEMMA 2.5. *Let $0 < d < d_1$ and $0 < \alpha < \alpha^*$. Then*

$$0 < A(R, \alpha) < A(R, \alpha^*) \quad \textit{for } 0 < R < \bar{R}(\alpha).$$

Proof. We set $v = \frac{\partial A}{\partial \alpha}$ and analyze the behavior of solutions of the problem

$$v'' + \frac{v'}{R} - \frac{v}{R^2} = \frac{v}{d}(3A^2 - 1), \tag{2.9}$$

$$v(0) = 0, \quad v'(0) = 1. \tag{2.10}$$

We show that $v' > 0$ and therefore $v > 0$ over $(0, \overline{R})$. This implies that $\frac{\partial A}{\partial \alpha}(R, \alpha) > 0$; hence $A(R, \alpha) < A(R, \alpha^*)$ over $(0, \bar{R})$. To do this we use the comparison function

$\psi = \frac{v'}{v} - \frac{A'}{A}$, which was introduced by Greenberg [5]. The use of (2.1) and (2.9), and two applications of L'Hopital's rule, give

$$\lim_{R\to 0^+} \psi(R) = 0. \tag{2.11}$$

Furthermore, ψ satisfies

$$\psi' + \psi\left(\frac{1}{R} + \psi + \frac{2A'}{A}\right) = \frac{2A^2}{d}. \tag{2.12}$$

From (2.11), (2.12) and the definition of $\overline{R}$ it immediately follows that $\psi > 0$ over $(0, \overline{R})$. Therefore $\frac{v'}{v} > \frac{A'}{A} > 0$ over $(0, \overline{R})$ and the lemma follows. □

We need one last technical result for (2.1)–(2.2). We assume that $0 < d < d_1$ is fixed and that $\alpha^* = \alpha^*(d)$ satisfies Lemma 4. We then completely determine the behavior of the reduced system (2.1)–(2.2) over an appropriate range of α values in the interval $(0, \alpha^*)$. This analysis will then allow us to proceed with our topological shooting argument for the full system (1.8)–(1.9).

LEMMA 2.6. *Let $0 < d < d_1$. There exist positive values α_* and α_0 with $0 < \alpha_* < \alpha_0 < \alpha^*$ such that (see Fig. 1)*

(i) *if $\alpha_0 < \alpha < \alpha^*$ then $A'(R, \alpha) > 0$ and $A(R, \alpha) < A(R, \alpha^*)$ for $0 < R \leq 1$;*

(ii) *$A'(R, \alpha_0) > 0$ for $R \in [0, 1)$ and $A'(1, \alpha_0) = 0$;*

(iii) *if $\alpha_* < \alpha < \alpha_0$ there is a first $\overline{R} = \bar{R}(\alpha) \in (0, 1)$ such that $A'(\overline{R}(\alpha), \alpha) = 0$, and $A'(R, \alpha) < 0, A(R, \alpha) > 0$ for all $R \in (\bar{R}, 1]$.*

Proof. The first step is to define α_0. For this we consider the set $G = \{\hat{\alpha} \in (0, \alpha^*) |$ if $\hat{\alpha} < \alpha < \alpha^*$ then $A'(R, \alpha) > 0$ for all $R \in [0, 1]\}$. It follows from continuity of solutions with respect to initial values, and Lemma 4 that G is open and nonempty. We define $\alpha_0 \equiv \inf G$. Then Lemma 2 implies that $\alpha_0 > 0$. Next, consider the solution $A(R, \alpha_0)$. If $A'(\overline{R}, \alpha_0) = 0$ at some first $\overline{R} > 0$, then Lemma 1, continuity, and the definition of α_0 force $\overline{R} = 1$. If $A'(R, \alpha_0) > 0$ for all $R \in [0, 1]$, then $\alpha_0 \in G$, and continuity implies that $\alpha \in G$ if $\alpha_0 - \alpha > 0$ is sufficiently small, contradicting

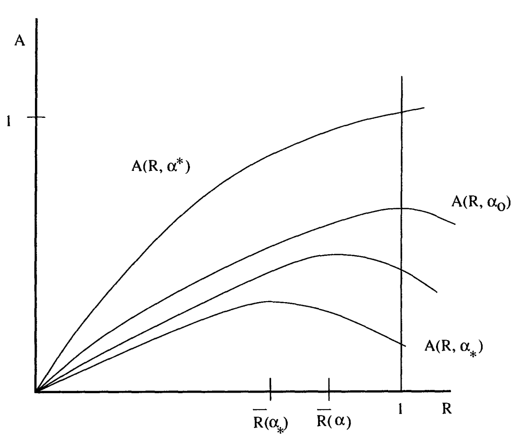

FIG. 1. *$A(R)$ for various values of α.*

the definition of α_0. Thus, it must be the case that $A'(1,\alpha_0) = 0$ and part (ii) holds. Lemma 5 and the definitions of G and α_0 lead immediately to the proof of (i). For part (iii) we first conclude from (ii) and Lemma 1 that $A'(1,\alpha_0) = 0$ and $A''(1,\alpha_0) < 0$. Thus the implicit function theorem guarantees the existence of a continuously differentiable function $\bar{R}(\alpha)$ defined on a small interval $(\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$ over which $A'(\bar{R}(\alpha),\alpha) = 0$. Differentiating both sides of the equation with respect to α, and appealing to Lemmas 1 and 5, we conclude that $d\overline{R}/d\alpha = -v'(\overline{R})/A''(\overline{R},\alpha) > 0$ for $\alpha_0 - \varepsilon < \alpha < \alpha_0 + \varepsilon$ and small ε. Furthermore, it follows from (1) and the definition of $\overline{R}$ that $\overline{R} > d^{1/2}$. Thus, for small $\varepsilon > 0$ we see that $(d)^{1/2} < \overline{R}(\alpha) < 1$ if $\alpha_0 - \varepsilon < \alpha < \alpha_0$. We need to show that if $\varepsilon > 0$ is small enough, then $A'(R,\alpha) < 0$ and $A(R,\alpha) > 0$ over $(\overline{R}, 1]$. Recall that $A(1,\alpha_0) > 0$ and $A'(1,\alpha_0) < 0$. Then $A(R,\alpha_0) > 0$ over an interval $(0, 1+\mu)$ independent of ε. Thus, by continuity, $A(R,\alpha) > 0$ over $(0, 1+\frac{\mu}{2})$ for $\alpha_0 - \varepsilon < \alpha \leq \alpha_0$ and ε small. This and (1) imply that $A'(R,\alpha)$ cannot have a zero over $(\bar{R}, 1+\mu/2)$ if $\alpha_0 - \varepsilon < \alpha < \alpha_0$. Therefore we set $\alpha_* = \alpha_0 - \varepsilon$ and the proof is complete. □

3. Completion of Proof of Theorem 1. In order to complete the proof of Theorem 1 we employ a two-dimensional topological shooting arguement. Throughout we assume that $d \in (0, d_1)$ is held fixed, where d_1 is the value found in §2 for the reduced problem (2.1)–(2.2). It follows from continuity and Lemma 6 that there exists a small value $q_1 = q_1(d) > 0$ such that if $0 < q < q_1, \Omega \in (-q_1, 0)$ and $\alpha_* \leq \alpha \leq \alpha^*$, then the solution of the initial value problem (1.8)–(1.10), with $A'(0) = \alpha$, must satisfy $0 < A < 1$ over $(0,1]$. Also, $A'(1) < 0$ if $\alpha = \alpha_*$, and $A'(1) > 0$ if $\alpha = \alpha^*$. (See Fig. 2.)

Assume now that $q \in (0, q_1)$ is held fixed. This leaves us with the two parameters α and Ω to be further adjusted. We restrict our attention to the parameter set $D = \{(\alpha,\Omega) \big| \alpha_* \leq \alpha \leq \alpha^* \text{ and } -q < \Omega < 0\}$. The choice for the range of Ω is justified in the next lemma.

LEMMA 3.1. *If there is a solution of* (1.8)–(1.10) *which satisfies* (1.11), *and* $0 < A < 1$ *over* (0,1], *then* $\Omega \in (-q, 0)$.

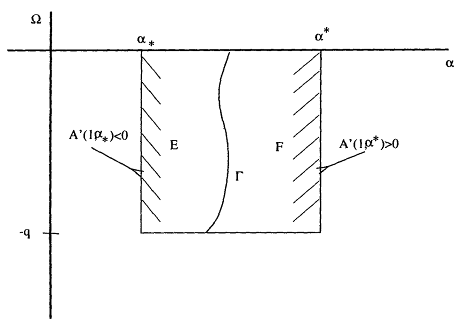

FIG. 2. *The shooting set for the topological argument.*

Proof. An integration of (1.9) gives

$$RA^2(R)k(R) = -\int_0^R \frac{tA^2}{d}(\Omega + qA^2)dt, \quad 0 < R \leq 1. \tag{3.1}$$

At $R = 1$, condition (1.11) implies that

$$\int_0^1 tA^2(\Omega + qA^2)dt = 0. \tag{3.2}$$

Since $0 < A < 1$ it then follows immediately from (3.2) that $-q < \Omega < 0$, proving the lemma. □

We are now prepared to define our topological shooting sets. These are

$$E = \{(\alpha, \Omega) \in D \big| A'(1) < 0\}$$

and

$$F = \{(\alpha, \Omega) \in D \big| A'(1) > 0\}.$$

It follows from our earlier discussion, and continuity, that E and F are relatively open, nonempty subsets of D. Furthermore, the sets E and F contain, respectively, the line segments $\{\alpha = \alpha_*, -q \leq \Omega \leq 0\}$ and $\{\alpha = \alpha^*, -q \leq \Omega \leq 0\}$.

In order to complete the first part of our proof of Theorem 1 we need the following topological result proved by McLeod and Serrin [11].

THEOREM 3.2. *Let I be the closed unit square $\{0 \leq x \leq 1, 0 \leq y \leq 1\}$ in the (x, y) plane, and let S^- and S^+ be disjoint, relatively, open subsets of I, respectively, containing the lines $y = 0$ and $y = 1$. Then the complement U of S^+ and S^- in I contains a continuum joining the lines $x = 0$ and $x = 1$.*

The McLeod–Serrin result applies to any closed rectangle in the plane. We need to apply this result to the closure of D. According to Theorem 3, the properties of E and F which we have now established allow us to conclude that there exists a continuum Γ in the set $D \setminus (E \cup F)$ which joins the line segments $\{\Omega = 0, \alpha_* \leq \alpha \leq \alpha^*\}$ and $\{\Omega = -q, \alpha_* \leq \alpha \leq \alpha^*\}$. It is clear from the definitions of E and F and the fact that $A'(1)$ exists that

$$(\alpha, \Omega) \in \Gamma \Rightarrow A'(1) = 0.$$

We now let (α, Ω) pass along Γ from $\Omega = 0$ to $\Omega = -q$. Near $\Omega = 0$ we have the following.

LEMMA 3.3. *There is a value $\Omega_1 \in (-q, 0)$ such that if $\Omega_1 < \Omega < 0$ and $(\alpha, \Omega) \in \Gamma$ then $k(1) < 0$.*

Proof. At $\Omega = 0$ we see that (3.1) reduces to

$$RA^2(R)k(R) = -\int_0^R \frac{tA^2}{d}(qA^2)dt < 0 \quad \text{for } 0 < R \leq 1.$$

Thus $k(1) < 0$. This and continuity of solutions with respect to α and Ω lead immediately to the conclusion of the lemma. □

Next, we consider the behavior of solutions near $\Omega = -q$.

LEMMA 3.4. *There exists $\Omega_2 \in (-q, \Omega_1)$ such that if $-q < \Omega < \Omega_2$ and $(\alpha, \Omega) \in \Gamma$ then $k(1) > 0$.*

Proof. Setting $\Omega = -q$, we see that (3.1) becomes

$$RA^2(R)k(R) = \frac{q}{d}\int_0^R tA^2(1-A^2)dt \quad \text{for } 0 < R \le 1. \tag{3.3}$$

Since $A' > 0$ over (0,1] and $A'(1) = 0$, then $A''(1) \le 0$. It then follows from (1.8) that $A(1) + A(1)(k^2(1) + \frac{1}{d}(A^2(1) - 1)) \le 0$. Since $A(1) > 0$ this inequality implies that $A(1) < 0$. Thus $A(R) < 0$ for all $R \in (0,1)$ and we conclude from (3.3) that $k(1) > 0$. Again, continuity leads us to conclude that $k(1) > 0$ if $(\alpha, \Omega) \in \Gamma$ and $|\Omega + q|$ is sufficiently small. This completes the proof. □

We now finish the proof of Theorem 1. First, since [0,1] is compact, and Γ is contained in the bounded set D, it must be the case that $k(1)$ depends continuously on $(\alpha, \Omega) \in \Gamma$. Thus we conclude from Lemmas 8 and 9 that as (α, Ω) passes along Γ from $\Omega = 0$ to $\Omega = -q$, then $k(1) = 0$ at some $(\bar{\alpha}, \bar{\Omega}) \in \Gamma$. Therefore at $(\alpha, \Omega) = (\bar{\alpha}, \bar{\Omega})$ condition (1.11) is satisfied and our proof is complete.

4. Proof of Theorem 2. The proof of our second main result relies on an analysis of the differential equation for $\rho = A'/A$, namely,

$$\rho' = -\rho^2 - \frac{\rho}{R} + \frac{1}{R^2} + \frac{(A^2 - 1)}{d} + k^2. \tag{4.1}$$

LEMMA 4.1. *Let $d \ge 1$ and let $\alpha > 0$. Then $\rho > 0$ for $0 < R \le 1$ as long as the solution exists.*

Proof. Since $\alpha > 0$ and we see that $A' > 0$ and $A > 0$ at least on a small interval $(0, \varepsilon)$. Thus, $\rho > 0$ on $(0, \varepsilon)$. If $\rho(\hat{R}) = 0$ at some first $\hat{R} \in (0, 1]$, then

$$\rho'(\hat{R}) \le 0. \tag{4.2}$$

Furthermore, $A(\hat{R}) > 0$ since $\rho = A'/A > 0$ over $(0, \hat{R})$. This and (4.1) lead to

$$\rho'(\hat{R}) = \frac{1}{\hat{R}^2} + \frac{A^2(\hat{R}) - 1}{d} + k^2(\hat{R}) > 0$$

since $d \ge 1$. This contradicts (4.2). Thus $\rho > 0$ over $(0, 1]$ as long as the solution exists and the lemma is proved. □

We now complete the proof of Theorem 2. Let $d \ge 1, \alpha > 0$, and Ω, q be real. If the corresponding solution of (1.8)–(1.10) exists for all $R \in [0, 1]$, then Lemma 10 implies that $A' > 0$ for all $R \in [0, 1]$ so that condition (1.11) cannot hold. Thus no solution of the boundary value problem (1.8)–(1.11) exists and the proof is complete.

5. Discussion and numerical results. In Figs. 3a, b we show contours of the solutions on a disk with $d = .08$ and $q = 0, 8$ respectively. The figure with zero twist ($q = 0$) shows that the rotating waves themselves have no curvature and the arms are straight. In contrast is Fig. 3b which shows a distinct curvature. To see how the solutions to the boundary value problem change with the twist, q, we plot $|k| \equiv \int_0^1 |k(t)|dt$ at several values of q as well as the ensemble frequency and the magnitude of the spiral at $R = 1$ in Fig. 4. The frequency is initially very close to a linear function of q but seems to saturate as q increases. The amplitude decreases as q increases. The magnitude of k increases as is expected since this is what imparts the radial dependence on the phase and gives the rotating waves their characteristic

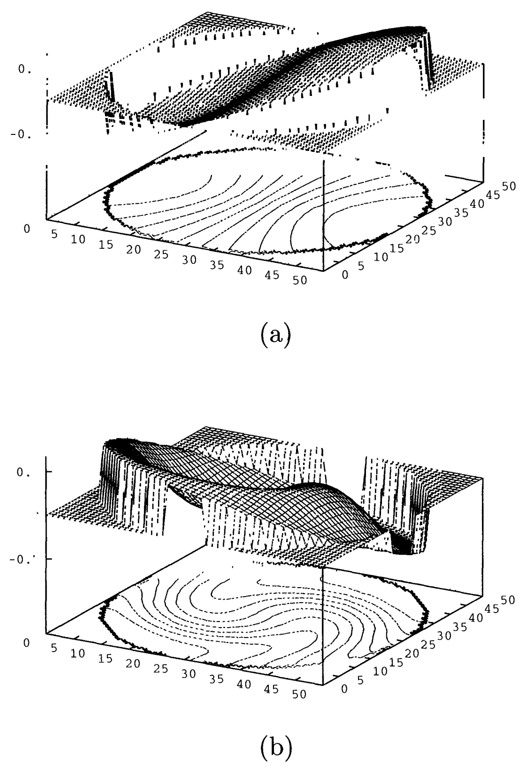

FIG. 3. *The magnitude and contours of* $u(x, y, t)$ *at a fixed value of* t *for* $d = .08$ *and* (a) $q = 0$; (b) $q = 8$.

geometric twist. Hagan obtains essentially the same results in the infinite domain, but no amplitude effects are seen as all waves approach 1 in magnitude as $R \to \infty$.

While the behavior of the wave curvature and the frequency of the spiral waves is similar in the finite- and infinite-dimensional cases, the magnitude of the waves is very dependent on the domain size (a feature obviously irrelevant in the infinite case.) As the domain size is proportional to $1/\sqrt{d}$, we will fix q and vary the diffusion size. As $d \to 0$ the spiral will approach that computed by Hagan and Greenberg. Theorem 2 shows that for d sufficiently large ($d \geq 1$), the spirals do not exist. We obtain fairly good estimates of the critical value of d. To obtain a better quantitative value of the critical diffusivity, we numerically solve the boundary value problem. In Fig. 5, we depict the solution to (1.8)–(1.11) as a function of d by showing the magnitude of the spiral at the edge of the medium. Clearly, for $d \to 0$ the magnitude tends to 1 and as d increases, the wave shrinks to 0. This indicates that the wave magnitude is small and since q appears only in the nonlinear terms, the critical value of d is independent of q. The numerical value at which the wave vanishes is close to .3. The way in which the spiral disappears is suggestive of a Hopf bifurcation. This leads us to consider this possibility by looking at the linearization of (1.1) on

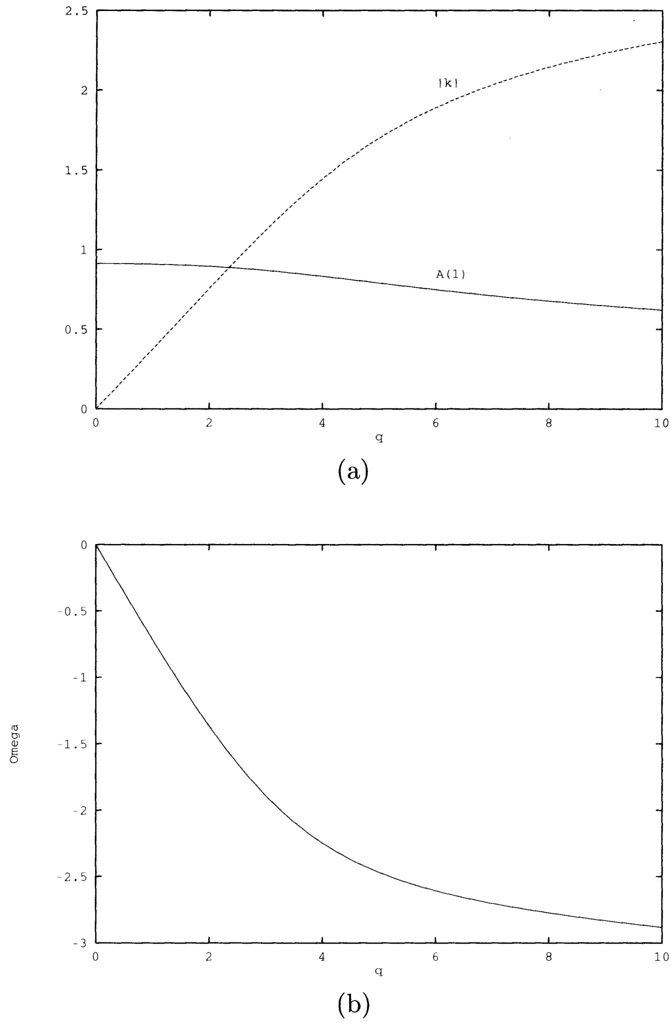

FIG. 4. *The behavior of solutions to* (1.8)–(1.11) *for* $d = .08$ *as a function of* q. (a) $A(1)$ *and* $|k|$; (b) Ω.

the disk. The eigenfunctions of the Laplacian on the disk are $J_n(\nu R)\cos(n\theta)$ where $J_n'(\nu) = 0$ with eigenvalues, $-\nu^2$. Thus, the eigenvalues of the linearized problem are $1 - d\nu^2 \pm i$. As d increases, there will be Hopf bifurcations at $d^* = 1/\nu^2$. The smallest value of d at which this occurs is when $n = 1$ and ν is the first zero of the derivative of J_1. A simple calculation reveals that $d^* \approx 0.2949889302$, which is very close to the numerically observed value and close to the bound obtained in Theorem 2. The numerical picture in Fig. 5 shows that the bifurcation is supercritical. However, the solution bifurcates from the trivial branch which is already unstable; thus, this initial branch is unstable and has two eigenvalues with positive real parts. The results of Hagan show that in the infinite domain, if q is small enough, the spiral wave is stable. Numerical integration of (1.1) for d small reveals that the spiral we have computed is also stable. Thus, as d decreases, the initially unstable branch of spiral waves stabilizes at some value d_s and remains stable for all smaller values. Put another way,

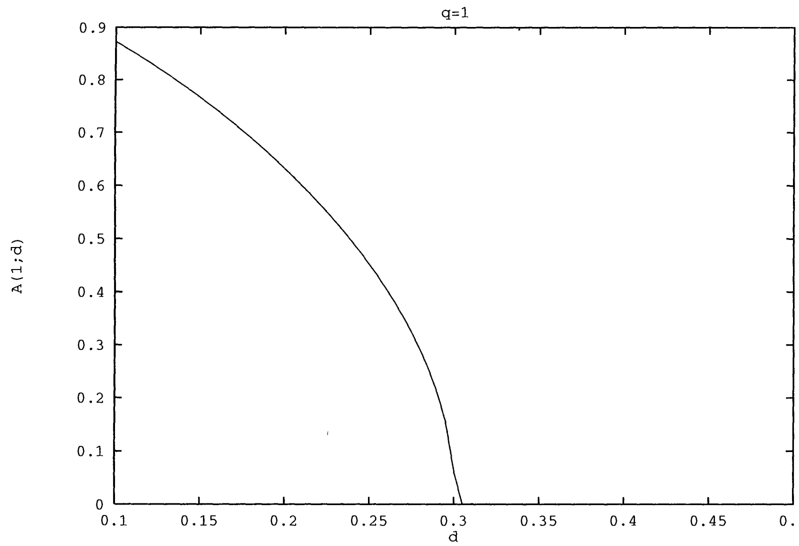

FIG. 5. *The magnitude of the wave, $A(1)$ for $q = 1$ as a function of d.*

as the domain size increases from very small, the following sequence occurs: (i) a small amplitude spiral wave bifurcates from the unstable origin and is initially unstable; (ii) as the domain increases, the spiral grows in amplitude and at a critical domain size it stabilizes. Below, we confirm this heuristic argument with a formal bifurcation analysis.

A complete bifurcation analysis of this solution is difficult, however, if we add a free parameter governing the "amplitude" and fix the diffusion coefficient d to be small, it is possible to perform a formal analysis on the interaction of the rotating wave with the spatially homogeneous periodic solution. This analysis enables us to understand the nature of the bifurcation that ultimately stabilizes the rotating waves. Thus, we consider (1.1), (1.2) with (1.3) replaced by

$$\lambda(u, v) = \lambda_0 - u^2 - v^2. \tag{5.1}$$

Our formal approach will be to consider λ_0 the bifurcation parameter and to let d be small so that near $\lambda_0 = 0$ there will be another complex conjugate pair of eigenvalues with a real part close to (but not identically) zero. Using formal secondary bifurcation methods, we can then derive a set of bifurcation equations for the amplitudes of the different modes. It is easiest to work in the complex coordinate $z = u + iv$ when we analyze

$$z_t = z(\lambda_0 + i - (1 - iq)z\bar{z}) + d\Delta z \tag{5.2}$$

with the Neumann boundary conditions on the unit disk. The normalized eigenfunctions of the Laplacian on the unit disk with these boundary conditions are

$$E_{nm}(r, \theta) = c_{nm} \exp(in\theta) J_n(\alpha_{nm} r)$$

with eigenvalues, $-\alpha_{nm}^2$. Here n is any integer and α_{nm} is the mth zero of the derivative of J_n. The normalization constant c_{nm} is chosen so that

$$\frac{1}{\pi}\int_0^{2\pi} d\theta \int_0^1 r dr E_{nm}(r,\theta)\bar{E}_{nm}(r,\theta) = 1.$$

Note, for example, that when $n = 0$, $\alpha_{01} = 0$, the normalized eigenfunction is 1, and is thus spatially homogeneous. The first nonzero eigenvalue is for $n = 1$ and corresponds to the branch of the spiral wave described above. With real d the eigenvalues of the linearization of (5.2) are

$$\nu_{nm} = \lambda_0 - d\alpha_{nm}^2 + i.$$

For $\lambda_0 = 0$ there is (as is obvious) a Hopf bifurcation. From construction of the model, we see that it is supercritical and stable. Note that for $\lambda_0 = d\alpha_{11}^2$ there is another Hopf bifurcation from the trivial state. Similarly, there is a Hopf bifurcation for each value of n, m at successively higher values of λ_0. One can perform a standard bifurcation analysis on each of the branches bifurcating from rest and see that they are all supercritical. However, due to the fact that the rest state from which they bifurcate is unstable, they too will be unstable. Nevertheless, for d small (that is, for a large domain) the bifurcations occur close to each other and there may be some interaction. Since the first branch is homogeneous and leads to *stable* periodic solutions and the next one is the branch of our spiral waves, we will formally look at the interaction of these two modes. This approach is a standard one in secondary bifurcation theory (see, e.g., Cohen [2]). Thus, we seek solutions bifurcating from the origin of the form

$$z(t) = \varepsilon(z_0(\tau)e^{it} + z_1(\tau)J_1(\alpha_{11}r)e^{i(t+\theta)} + z_2(\tau)J_1(\alpha_{11}r)e^{i(t-\theta)}) + O(\varepsilon^2),$$

where $\tau = \varepsilon^2 t$ is slow time. We assume that λ_0 and d are small so that after routine calculations, we obtain the *formal* equations

$$\begin{aligned}
\partial_\tau z_0 &= z_0(\lambda_0 - (1-iq)(z_0\bar{z}_0 + 2z_1\bar{z}_1 + 2z_2\bar{z}_2)),\\
\partial_\tau z_1 &= z_1(\lambda_0 - d\alpha_{11}^2 - (1-iq)(az_1\bar{z}_1 + 2az_2\bar{z}_2 + 2z_0\bar{z}_0)),\\
\partial_\tau z_2 &= z_1(\lambda_0 - d\alpha_{11}^2 - (1-iq)(az_2\bar{z}_2 + 2az_1\bar{z}_1 + 2z_0\bar{z}_0)),
\end{aligned}$$

where $a = 2\int_0^1 (c_{11}J_1(\alpha_{11}r))^4 r dr \approx 1.168$. As with all systems such as this, one need only look at magnitudes of the solutions in order to study the stability and bifurcation from rest. Letting $|z_j| = s_j$ and s' denote the derivative with respect to τ, the magnitudes satisfy

(5.3) $$s_0' = s_0(\lambda_0 - s_0^2 - 2s_1^2 - 2s_2^2),$$

(5.4) $$s_1' = s_1(\lambda_0 - d\alpha_{11}^2 - as_1^2 - 2as_2^2 - 2s_0^2),$$

(5.5) $$s_2' = s_2(\lambda_0 - d\alpha_{11}^2 - as_2^2 - 2as_1^2 - 2s_0^2).$$

The homogenous oscillation corresponds to the solution $s_0^2 = \lambda_0$ and $s_1, s_2 = 0$. The eigenvalues about this solution are $-2\lambda_0, -\lambda_0 - d\alpha_{11}^2, -\lambda_0 - d\alpha_{11}^2$, so that the interaction with the inhomogeneous state never destabilizes the homogeneous oscillations. (If the formal analysis is to be taken seriously, this had better be the case as the homogeneous oscillation is always a stable solution to (1.1).) The spiral wave solution corresponds to $s_0 = 0$ and either of s_1, s_2 nonzero. Without loss of generality, we take $s_2 = 0$ and $s_1^2 = (\lambda_0 - d\alpha_{11}^2)/a$. Thus, this solution bifurcates for a higher value of λ_0 than does the homogeneous solution. The eigenvalues about this solution are

$$\nu_1 = \lambda_0 - \frac{2}{a}(\lambda_0 - d\alpha_{11}^2),$$
$$\nu_2 = -2(\lambda_0 - d\alpha_{11}^2),$$
$$\nu_3 = -(\lambda_0 - d\alpha_{11}^2).$$

(Note that these correspond to complex conjugate pairs of eigenvalues for the full complex system.) It is clear that just past the bifurcation point, this solution is unstable since $\nu_1 > 0$. However, since $a < 2$ it is possible for ν_1 to become negative if either λ_0 is large enough or d is sufficiently small. For example, fixing λ_0 positive and viewing d as the "bifurcation" parameter, one sees that as d decreases to $d^{**} = \lambda_0(1-a/2)/\alpha_{11}^2$ the "spiral" wave becomes stable via a Hopf bifurcation. The stabilization is a result of the nonlinear interaction with the branch of homogeneous oscillations. One can readily show that there are no stable "mixed mode" solutions. The argument is purely formal but seems to agree with our numerical results. We have neglected other modes although Hagan has shown (at least in the infinite domain) that these are likely to be unstable. This analysis also points out some differences between the infinite and finite domains. The spiral wave in the infinite domain is always stable, whereas in the finite domain it is initially unstable and then stabilizes for large enough domains or large enough λ_0.

One should note that the conjectured Hopf bifurcation that leads to the stabilization of the spiral wave is not the same as that discussed by Barkley [1]. Barkley [1] and others have run extensive simulations of spirals in the circular domain in order to understand the loss of stability of the regular rotating spiral and the appearance of the so-called wobble solutions. His instability is related to the destabilization of the spiral as the equivalent of the parameter q in our model is increased. Again, appealing to numerical results, we have found that as q increases, the spiral winds more tightly and at a critical value loses stability to a Hopf bifurcation. The resultant solution is no longer a solution to an ordinary differential equation and instead becomes a spiral that meanders about the core. Barkley (personal communication) has recently developed a formal analysis of this bifurcation based on normal forms of the interaction of a Hopf bifurcation with the almost translational invariance of the spiral core. Since we are able to explicity solve for the spiral wave we hope to apply some of these techniques to the behavior of the present model as q increases.

The bifurcation argument above does not depend on any special properties of (1.1). Thus, any system of reaction-diffusion equations on a disk that has scalar diffusion and an unstable equilibrium with the instability arising from a pair of complex conjugate eigenvalues with positive real parts will have a Hopf bifurcation at a critical domain size or diffusivity. This branch of solutions will be unstable but can possibly stabilize as the domain increases in size. The symmetry of this solution is similar to that of the rotating waves in excitable media and is closely related to the spiral we have computed in the present paper. Recent numerical results on spatially discrete arrays as well as some preliminary analysis reveal that once the discrete analogue of the spiral wave is well established it is possible to change a parameter that controls the transition from oscillatory to excitable in such a way that the spiral wave persists. Thus, we are led to conjecture that the well-known spiral waves in excitable media are *continuations* of branches that arise in oscillatory media and that these latter branches arise themselves from Hopf bifurcations from an unstable steady state. This conjecture has been numerically verified in (spatially discrete) square domains where there is a continuous change in the shape of the spiral as one goes from an oscillatory to an excitable system.

REFERENCES

[1] D. BARKLEY, *Linear stability analysis of rotating spiral waves in excitable media*, Phys. Rev. Letters, 68(1992), pp. 2090–2093.

[2] D. S. COHEN, *Bifurcation from multiple complex eigenvalues,* J. Math. Anal. Appl., 57(1977), pp. 505–521.

[3] D. S. COHEN, J. C. NEU, AND R. ROSALES, *Rotating spiral wave solutions of reaction-diffusion equations*, SIAM J. Appl. Math., 35(1978), pp. 536–547.

[4] P. C. FIFE, *Understanding the patterns in the B-Z reagent*, J. Stat. Phys., 39(1985), pp. 687–701.

[5] J. M. GREENBERG, *Spiral waves for* $\lambda-\omega$ *systems*, SIAM J. Appl. Math., 39(1982), pp. 301–309.

[6] P. S. HAGAN, *Spiral waves in reaction-diffusion systems*, SIAM J. Appl. Math., 42(1982), pp. 762–786.

[7] J. P. KEENER, *A geometrical theory for spiral waves in excitable media*, SIAM J. Appl. Math., 46(1986), pp. 1039–1059.

[8] J. P. KEENER AND J. J. TYSON, *Spiral waves in the B-Z reaction*, Phys. D., 21(1986), pp. 307–324.

[9] N. KOPELL AND L. N. HOWARD, *Target pattern and spiral solutions to reaction-diffusion equations with more than one space dimension,* Adv. Appl. Math., 2(1981), pp. 417–449.

[10] Y. KURAMOTO, *Chemical Oscillations, Waves, and Turbulence*, Springer-Verlag, New York, 1984.

[11] J. B. MCLEOD AND J. SERRIN, *The existence of similarity solutions for some laminar boundary layer problems*, Arch. Rational Mech. Anal., 36(1968), pp. 288–303.

[12] H. SWINNEY AND V. KRINSKY, *Waves and patterns in chemical and biological media*, Physica D, 49(1991).

[13] JACK XIN, *Existence of a class of symmetric rotating spiral waves on finite disc domains in excitable media*, Indiana U. Math. J., 42 (1993), pp. 1305–1338.