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MONOTONICITY OF PHASELOCKED SOLUTIONS IN CHAINS AND ARRAYS OF NEAREST-NEIGHBOR COUPLED OSCILLATORS*

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Abstract. The existence of phaselocked solutions in chains of weakly coupled oscillators is proven rigorously. The solutions show interesting monotonicity which plays an important role for the existence proofs. Under some conditions, we show that two-dimensional arrays can be decomposed into two one-dimensional problems. With this theory of decomposition, target patterns can be explained. Numerical results are provided to illustrate the theorems on the chain problem and to show traveling waves in the chains and arrays.

Key words. coupled oscillators, target patterns, phaselocking, neurons

AMS subject classifications. 34C29, 34C15, 58F22, 92C20

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1. Introduction.

Coupled oscillators play an increasingly important role in our understanding of various types of repetitive activity in the nervous system. There have been numerous analytic and numerical studies of the behavior of systems of coupled oscillators. These range from models of cognitive processing and binding [1] to attempts to model locomotor patterns [2]. Several connection topologies have been explored primarily due to their mathematical tractability. The simplest topology is a one-dimensional chain of oscillators. Mathematically, the case in which the two ends are connected is the easiest to analyze, but in realistic applications, this rarely arises. However, the chain topology is quite natural for models of systems such as the lamprey swim central pattern generator [2] or the central pattern generator of the leech [3]. The behavior of weakly coupled oscillators in a chain has been the object of extensive work by several authors [4, 5, 6, 7, 8].

Two-dimensional arrays of oscillators have been subject to far less mathematical analysis; most work deals exclusively with numerical simulations. They arise more naturally than chains in attempts to understand oscillatory neural behavior in neural tissue which is typically arranged in distinct two-dimensional sheets. Furthermore, there are many phenomena that can occur in two- and three-dimensional systems of oscillators that are not possible in one dimension.

It was shown in [5] that the phaselocked behavior of a sufficiently long chain of weakly coupled oscillators can be described by the solutions of a singularly perturbed two-point boundary value problem. The point of this reduction is that the analysis of phaselocking and the behavior of the chain in the presence of inhomogeneities and anisotropic coupling is much easier for the continuum model than for its discrete analogue. In this paper, we will use another approach to investigate the phaselocked behavior with *any number of oscillators*. That is, we do not require the length of the chain to tend to infinity.

Coupled oscillators present an almost impossible problem to analyze in any generality. Thus, we will restrict our attention to a class of so-called phase models that

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arise when oscillatory elements are weakly coupled. As was the case in [5], we will restrict our attention in this paper to nearest-neighbor coupling. In a later paper, we investigate coupling with greater spread. We first consider one-dimensional chains of oscillators. Then, we turn our attention to two-dimensional arrays. Under the condition that the distribution of intrinsic frequencies is a sum of two stripe distributions: one with constant frequencies along each row and another with constant frequencies along each column, we are able to decompose the two-dimensional problem into a set of one-dimensional problems and from this gain insight into the global phaselocked behavior. The techniques for two-dimensional arrays can be generalized in an obvious fashion to three- and higher dimensional arrays.

The equations to be considered have the form

$$\theta_i' = \omega_i + H^+(\theta_{i+1} - \theta_i) + H^-(\theta_{i-1} - \theta i), \tag{1.1}$$

where $i = 1, \ldots, n+1$, both H^+ and H^- are smooth 2π-periodic functions of their arguments, and ω_i is the frequency for each oscillator. Note that (1.1) is a nearest-neighbor coupled system. The term H^- (respectively, H^+) will be ignored for $i = 1$ (respectively, $i = n+1$). Equation (1.1) arises naturally in systems of weakly coupled oscillators. We assume that without coupling, each component of the chain has an asymptotically stable limit cycle. Thus, without coupling, each oscillator is described by a single coordinate, the phase, θ_i. The phase space of the $n+1$ oscillators then lies in an $n+1$ torus. If the oscillators interact weakly, then this invariant torus persists, and it follows from averaging theory that the equations for the phases of the $n+1$ oscillators is exactly equation (1.1). (For details on the derivation of these equations, see, e.g., [4].) The interaction functions $H^\pm$ are easily computed once the uncoupled oscillation is known and a formula is given for the interaction between the oscillators.

We point out that if two oscillators are coupled by diffusion, then the interaction functions $H^\pm$ vanish at 0. Thus, if there are no local differences in the oscillators (ω_i is independent of i) then the synchronous state $\theta_i(t) = \omega t$ is one possible solution. However, if the coupling between oscillators is based on chemical transmission then one does not expect that $H^\pm(0)$ will vanish. Because oscillators on the boundary (at the ends in one dimension, on the edges in two dimensions, etc.) receive less synaptic input than oscillators in the interior, this sets up a natural frequency difference between the oscillators. This makes it possible to induce a pattern of relative phases such as a traveling wave in one dimension and target patterns in two dimensions. In [5] we analyzed chains of oscillators in which there is an intrinsic anisotropy in the coupling so that H^+ and H^- are not necessarily the same. This was exploited in order to suggest a mechanism for the uniform traveling wave of electrical activity in the lamprey spinal cord. In this paper, we are mainly concerned with couplings for which $H^\pm$ are identical. In [7] the behavior of the chain is understood by letting n get very large and converting to a continuum equation. Here we do not restrict the size of n; the results hold for both small and large n. The main reason that we first analyze the one-dimensional chain is that we can then use these results to analyze a class of solutions in two and higher dimensions.

In section 2, we shall take the technique used in [9] to prove the existence of phaselocked solutions for several general cases. The monotonicity of the phaselocked solutions is also obtained. The monotonicity does not have any specific implication for traveling wave, but it does play a critical role in the existence proof of the phaselocked solutions.

In section 3, we shall investigate the two-dimensional arrays of weakly coupled oscillators based on the existence results of section 2. As in the one-dimensional case, we restrict our attention to nearest-neighbor coupling, but the coupling in each of the four directions need not be the same. Under some conditions on the frequencies ω_{ij}, we can reduce this problem to two independent chain problems such that we can apply the results obtained from section 2 to describe the behavior of two-dimensional arrays of weakly coupled oscillators. One of the main results is that with isotropic "synaptic coupling," target patterns spontaneously form and synchrony cannot occur. This is due to the effects of boundaries in synaptically coupled cells.

Finally, we discuss some other two-dimensional solutions as well as how small chains can qualitatively differ from very long chains.

2. Chains of oscillators. For convenience, the equations (1.1) are written in the form

$$\begin{aligned} \theta_1' &= \omega_1 + H^+(\theta_2 - \theta_1), \\ \theta_i' &= \omega_i + H^-(\theta_{i-1} - \theta_i) + H^+(\theta_{i+1} - \theta_i), \\ \theta_{n+1}' &= \omega_{n+1} + H^-(\theta_n - \theta_{n+1}). \end{aligned} \tag{2.1}$$

We take $\phi_i = \theta_{i+1} - \theta_i, \beta_i = \omega_{i+1} - \omega_i, i = 1, \ldots, n$. Also, we define two functions f and g related to H^+ and H^- as $f(\phi) + g(\phi) = H^+(\phi)$ and $f(\phi) - g(\phi) = H^-(-\phi)$. In (2.1), if the ith equation is subtracted from the $(i+1)$th one, we have

$$\begin{aligned} \phi_1' &= \beta_1 + f(\phi_2) + g(\phi_2) - 2g(\phi_1), \\ \phi_i' &= \beta_i + f(\phi_{i+1}) - f(\phi_{i-1}) + g(\phi_{i+1}) - 2g(\phi_i) + g(\phi_{i-1}), \\ &\qquad i = 2, \ldots, n-1, \\ \phi_n' &= \beta_n - f(\phi_{n-1}) - 2g(\phi_n) + g(\phi_{n-1}). \end{aligned} \tag{2.2}$$

Two numbers ϕ_L and ϕ_R need to be considered. They are defined as $f(\phi_L) = g(\phi_L)$, i.e., $H^-(-\phi_L) = 0$, and $f(\phi_R) = -g(\phi_R)$, i.e., $H^+(\phi_R) = 0$.

We assume some hypotheses on f and g in a sufficiently large interval J around $\phi = 0$:

(H1) $g'(\phi) > |f'(\phi)|$ *for* $\phi \in J$;

(H2) *There exists a unique solution* ϕ_L *(respectively,* ϕ_R*) to* $f = g$ *(respectively,* $f = -g$*) for* $\phi \in J$.

These conditions are proposed in [5] with other conditions. Note that $\phi_R < 0 < \phi_L$ if $f(0) > |g(0)|$ and $\phi_L < 0 < \phi_R$ if $f(0) < -|g(0)|$.

2.1. Isotropic case with $\boldsymbol{\beta_i = 0, i = 1, \ldots, n}$. We investigate the case with $H^+ = H^-$ and $\beta_i = 0, i = 0, \ldots, n$. In this case, f is an even function and g an odd one. And we have $\phi_L = -\phi_R$. Then (2.2) can be rewritten as

$$\begin{aligned} \phi_1' &= f(\phi_2) + g(\phi_2) - 2g(\phi_1), \\ \phi_i' &= f(\phi_{i+1}) - f(\phi_{i-1}) + g(\phi_{i+1}) - 2g(\phi_i) + g(\phi_{i-1}), \\ &\qquad i = 2, \ldots, n-1, \\ \phi_n' &= -f(\phi_{n-1}) - 2g(\phi_n) + g(\phi_{n-1}). \end{aligned} \tag{2.3}$$

First of all, let's look at the initial value problem (IVP) (2.1) with $\theta_i(0) = c$ where c is any real number. Then by the facts that $H^+ = H^-$ and $\omega_i \equiv \omega$ (since $\beta_i = 0$, $i = 1, \ldots, n$), we have $\theta_i(t) = \theta_{n+2-i}(t)$ for $t \geq 0, i = 1, \ldots, n+1$. Then the IVP (2.3) with $\phi_i(0) = 0$ shall yield $\phi_i(t) \equiv -\phi_{n+1-i}(t)$. That inspires us to study the system including only half the number of equations of (2.3).

Lemma 2.1. *Let $n = 2m - 1$. Assume that f and g satisfy the conditions* (H1), (H2), *and $f(0) > 0$; then the IVP* (2.3) *with $\phi_i(0) = 0, i = 1, \ldots, n$, has the following monotonicity along the trajectory:*

$$\phi_L > \phi_1(t) > \phi_2(t) > \cdots > \phi_{m-1}(t) > \phi_m(t) \equiv 0 \tag{2.4}$$

and

$$\phi_i'(t) > 0, i = 1, \ldots, m-1 \tag{2.5}$$

for $0 < t < \hat{t}$, where $\hat{t}$ is such that $\phi_i'(\hat{t}) = 0$, $i = 1, \ldots, m$, or $\hat{t} = +\infty$.

Remark. The fixed point always happens at $t = +\infty$ for an autonomous system. So we should have $\hat{t} = +\infty$ here. But a finite positive $\hat{t}$ does not affect our results. Hence we define $\hat{t}$ in the above way for the convenience of proof.

Proof. As we mentioned, $\phi_m(t) = -\phi_{n+1-m}(t) = -\phi_m(t)$ for $t \geq 0$. Then $\phi_m(t) \equiv 0$ is obvious. Since we only use half the number of equations (2.3), we restate them as

$$\begin{aligned} \phi_1' &= f(\phi_2) + g(\phi_2) - 2g(\phi_1), \\ \phi_i' &= f(\phi_{i+1}) - f(\phi_{i-1}) + g(\phi_{i+1}) - 2g(\phi_i) + g(\phi_{i-1}), \\ &\quad i = 2, \ldots, m-2, \\ \phi_{m-1}' &= f(0) - f(\phi_{m-2}) - 2g(\phi_{m-1}) + g(\phi_{m-2}). \end{aligned} \tag{2.6}$$

Therefore $\phi_1'(0) = f(0) > 0, \phi_i'(0) = 0, i = 2, \ldots, m-1$ (where we use the fact that $g(0) = 0$ since g is odd).

Furthermore, one can show by induction that

$$\begin{aligned} \phi_1'(0) &> 0, \\ \phi_i'(0) = \cdots = \phi_i^{(i-1)}(0) &= 0, \\ \phi_i^{(i)}(0) = g'(0)\phi_{i-1}^{(i-1)}(0) &> 0, \quad i = 2, \ldots, m-1. \end{aligned} \tag{2.7}$$

Remark. By (H1), $g'(0) > 0$ such that $\phi_i^{(i)}(0) = [g'(0)]^{i-1}\phi_1'(0) > 0$.

So there exists small $\delta > 0$ such that (2.4) and (2.5) hold for $0 < t < \delta$ if one applies the Taylor's expansion for $\phi_i(t)$ and $\phi_i'(t)$ around $t = 0$. Starting with this result, we need to show that (2.4) and (2.5) are always true for $t > 0$.

By contradiction, suppose that there is a first place t_0 where (2.4) and (2.5) break down. Then we need to study the following cases.

Case 1. $\phi_L = \phi_1(t_0) \geq \phi_2(t_0) \geq \cdots \geq \phi_{m-1}(t_0) > \phi_m(t_0) \equiv 0$ *and* $\phi_i'(t_0) \geq 0$, $i = 1, \ldots, m-1$.

Then

$$\begin{aligned} 0 \leq \phi_1'(t_0) &= f(\phi_2(t_0)) + g(\phi_2(t_0)) - 2g(\phi_1(t_0)) \\ &= f(\phi_2(t_0)) + g(\phi_2(t_0)) - 2g(\phi_L) \\ &= f(\phi_2(t_0)) + g(\phi_2(t_0)) - f(\phi_L) - g(\phi_L) \\ &= [f'(\xi) + g'(\xi)](\phi_2(t_0) - \phi_L) \\ &\leq 0, \end{aligned}$$

where $\xi \in (\phi_2(t_0), \phi_L)$ by the mean value theorem and $(f' + g')(\xi) > 0$ by (H1). This leads to $\phi_2(t_0) = \phi_L$.

By induction on i, we shall gain $\phi_i(t_0) = \phi_L$, $i = 2, \dots, m-1$.

Then we have

$$\begin{aligned} 0 &\le f(0) - f(\phi_L) - g(\phi_L) \\ &= f(0) + g(0) - f(\phi_L) - g(\phi_L) \\ &= (f' + g')(\xi)(0 - \phi_L), \end{aligned}$$

which implies $\phi_L \le 0$. This leads to contradiction since $\phi_L > 0$. Therefore *Case* 1 *is impossible.*

CASE 2. $\phi_L > \phi_1(t_0) > \cdots > \phi_j(t_0) = \phi_{j+1} \ge \cdots \ge \phi_{m-1}(t_0) > \phi_m(t_0) = 0$ *for some* $j \in \{1, 2, \dots, m-2\}$ *and* $\phi_i'(t_0) \ge 0$ $\forall i \in \{1, \dots, m-1\}$.

Then

$$\begin{aligned} 0 \le \phi_{j+1}'(t_0) &= f(\phi_{j+2}(t_0)) - f(\phi_j(t_0)) + g(\phi_{j+2}(t_0)) - 2g(\phi_{j+1}(t_0)) + g(\phi_j(t_0)) \\ &= f(\phi_{j+2}(t_0)) - f(\phi_{j+1}(t_0)) + g(\phi_{j+2}(t_0)) - g(\phi_{j+1}(t_0)) \\ &= [f' + g'](\xi)(\phi_{j+2}(t_0) - \phi_{j+1}(t_0)) \\ &\le 0, \end{aligned}$$

which implies $\phi_{j+2}(t_0) = \phi_{j+1}(t_0)$ (since $f' + g' > 0$ in J and $\phi_{j+1}(t_0) \ge \phi_{j+2}(t_0)$).

By induction, we have $\phi_L > \phi_1(t_0) > \cdots > \phi_j(t_0) = \phi_{j+1}(t_0) = \cdots = \phi_{m-1}(t_0) > \phi_m(t_0) = 0$.

Then

$$\begin{aligned} 0 \le \phi_{m-1}'(t_0) &= f(0) - f(\phi_{m-1}(t_0)) - g(\phi_{m-1}(t_0)) \\ &= f(0) + g(0) - f(\phi_{m-1}(t_0)) - g(\phi_{m-1}(t_0)) \\ &= [f' + g'](\xi)(0 - \phi_{m-1}(t_0)), \end{aligned}$$

which implies $\phi_{m-1}(t_0) \le 0$: a contradiction!

Therefore we *eliminate the possibility of Case* 2.

CASE 3. $\phi_L > \phi_1(t_0) > \cdots > \phi_{m-1}(t_0) > \phi_m(t_0) = 0$ *and* $\phi_i'(t_0) \ge 0$ $\forall i$ *and* $\phi_j'(t_0) = 0$ *for some* $j \in \{1, \dots, m-1\}$.

First of all, if $j = 1$, i.e., $\phi_1'(t_0) = 0$, then we must have $\phi_2'(t_0) = 0$. Otherwise $\phi_2'(t_0) > 0$; then for $\varepsilon > 0$ small enough, we have

$$\begin{aligned} \phi_1'(t_0 - \varepsilon) &= \phi_1'(t_0) - \phi_1''(t_0)\varepsilon + o(\varepsilon^2) \\ &= \phi_1'(t_0) - [f'(\phi_2(t_0)) + g'(\phi_2(t_0))]\phi_2'(t_0)\varepsilon + 2g'(\phi_1(t_0))\phi_1'(t_0)\varepsilon + o(\varepsilon^2) \\ &= -[f'(\phi_2(t_0)) + g'(\phi_2(t_0))]\phi_2'(t_0)\varepsilon + o(\varepsilon^2) \\ &< 0. \end{aligned}$$

This is a contradiction since t_0 is the first place where (2.4) and (2.5) break down.

Furthermore, we can get $\phi_i'(t_0) = 0, i = 2, \dots, m-1$ by using the techniques of induction and contradiction. Taking $\hat{t} = t_0$, we are done with the proof.

Secondly, assume that $\phi_i'(t_0) > 0, i = 1, \dots, j-1$, and $\phi_j'(t_0) = 0$ for some $j \in \{2, \dots, m-1\}$. Then by applying the same technique above and noting that $g' - f' > 0$ in J, we will obtain $\phi_{j-1}'(t_0) = 0$, which is a contradiction. Hence *we eliminate Case* 3.

Now by getting rid of Cases 1–3, we can conclude that either there exists a $\hat{t} > 0$ such that (2.4) and (2.5) hold for $0 < t < \hat{t}$ and $\phi_i'(\hat{t}) = 0, i = 1, \dots, m$, or the first place t_0 where (2.4) and (2.5) break down does not exist. The proof is completed. ∎

Remark 1. In the proof of Lemma 2.1, the monotonicity of solution along the trajectory plays an important role. In order to get monotonicity at the start of the trajectory, we need the initial vector $\phi_i(0) = 0, i = 1, \dots, n$. For other initial vectors, monotonicity fails.

Remark 2. Throughout this paper, we always start from $\phi_i(0) = 0$. As we can see in the following sections, if the monotonicity fails on the trajectory, we cannot continue the proof theoretically. But numerical experiments show that the solution trajectory of (2.2) always converges to the same equilibrium for any initial vector. It seems that the basin of attraction is infinitely large.

LEMMA 2.2. *Let $n = 2m$. Assume that f and g satisfy the conditions* (H1), (H2), *and $f(0) > 0$; then the IVP* (2.3) *with $\phi_i(0) = 0, i = 1, \dots, n$, has the following monotonicity along the trajectory:*

$$\phi_L > \phi_1(t) > \phi_2(t) > \cdots > \phi_{m-1}(t) > \phi_m(t) > 0 \tag{2.8}$$

and

$$\phi_i'(t) > 0, \quad 0 < t < \hat{t}, \quad i = 1, \dots, m, \tag{2.9}$$

where $\hat{t}$ is such that $\phi_i'(\hat{t}) = 0$, $i = 1, \dots, m$, or $\hat{t} = +\infty$.

Proof. The proof is similar to the proof of Lemma 2.1. The difference is that we should restate the equations of (2.3) as

$$\begin{aligned} \phi_1' &= f(\phi_2) + g(\phi_2) - 2g(\phi_1), \\ \phi_i' &= f(\phi_{i+1}) - f(\phi_{i-1}) + g(\phi_{i+1}) - 2g(\phi_i) + g(\phi_{i-1}), \\ &\qquad i = 2, \dots, m-1, \\ \phi_m' &= f(\phi_m) - f(\phi_{m-1}) - 3g(\phi_m) + g(\phi_{m-1}). \end{aligned} \tag{2.10}$$

All the techniques from Lemma 2.1 can be applied here so we ignore the details. □

THEOREM 2.3. *Assume f and g satisfy the same conditions as in Lemmas* 2.1 *and* 2.2; *then the IVP* (2.3) *with $\phi_i(0) = 0$, $i = 1, \dots, n$ has the following properties:*

(i) *For each $i \in \{1, \dots, n\}$, there exists $\bar{\phi}_i$ such that $\lim_{t \to \hat{t}} \phi_i(t) = \bar{\phi}_i$;*

(ii) *$(\bar{\phi}_1, \dots, \bar{\phi}_n)$ is the fixed point of the system* (2.3);

(iii) *$\phi_L > \bar{\phi}_1 > \bar{\phi}_2 > \cdots > \bar{\phi}_{n-1} > \bar{\phi}_n > \phi_R$;*

(iv) *$\bar{\phi}_i = -\bar{\phi}_{n+1-i}$, $i = 1, \dots, n$.*

Proof. By the results of Lemmas 2.1 and 2.2, (i), (ii), and (iv) are easy to check. Also we have $\phi_L \geq \bar{\phi}_1 \geq \bar{\phi}_2 \geq \cdots \geq \bar{\phi}_{n-1} \geq \bar{\phi}_n \geq \phi_R$. We need to show that all the inequalities are strict. By contradiction, suppose $\phi_L = \bar{\phi}_1$. Then we have

$$\begin{aligned} 0 &= f(\bar{\phi}_2) + g(\bar{\phi}_2) - 2g(\bar{\phi}_1) \\ &= f(\bar{\phi}_2) + g(\bar{\phi}_2) - 2g(\phi_L) \\ &= f(\bar{\phi}_2) + g(\bar{\phi}_2) - [f(\phi_L) + g(\phi_L)] \\ &= [f' + g'](\xi)(\bar{\phi}_2 - \phi_L), \end{aligned}$$

which implies $\bar{\phi}_2 = \phi_L$.

Then we would have $\bar{\phi}_i = \phi_L$, $i = 1, \dots, n$ by induction on i.

And $0 = -f(\phi_L) - 2g(\phi_L) + g(\phi_L) = -f(\phi_L) - g(\phi_L)$ by the last equation of (2.3) such that $f(\phi_L) = -g(\phi_L)$ which leads to $\phi_L = \phi_R$. This contradicts $\phi_L = -\phi_R$ since $\phi_L > 0$. Hence $\phi_L > \bar{\phi}_1$ must hold.

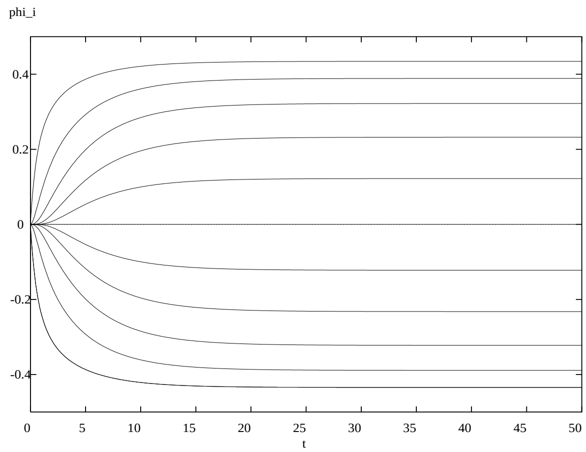


FIG. 2.1. *The isotropic case with* $H^+(\phi) = H^-(\phi) = H(\phi) = .5\cos\phi + \sin\phi$, $n = 11$, *and* $\beta_i = 0.0$.

Suppose $\bar\phi_1 = \bar\phi_2$; then $0 = f(\bar\phi_2) + g(\bar\phi_2) - 2g(\bar\phi_1)$ by (2.3). This is $0 = f(\bar\phi_1) - g(\bar\phi_1)$, which implies $\bar\phi_1 = \phi_L$. So we must have $\phi_L > \bar\phi_1 > \bar\phi_2$. By the symmetry, we have $\phi_R < \bar\phi_n < \bar\phi_{n-1}$.

Suppose i is the first index such that $\bar\phi_i = \bar\phi_{i+1}$; then

$$\begin{aligned} 0 &= f(\bar\phi_{i+1}) - f(\bar\phi_{i-1}) + g(\bar\phi_{i+1}) - 2g(\bar\phi_i) + g(\bar\phi_{i-1}) \\ &= f(\bar\phi_i) - f(\bar\phi_{i-1}) - g(\bar\phi_i) + g(\bar\phi_{i-1}) \\ &= f(\bar\phi_i) - g(\bar\phi_{i-1}) - g(\bar\phi_i) + g(\bar\phi_{i-1}) \\ &= (g' - f')(\xi)(\bar\phi_{i-1} - \bar\phi_i), \end{aligned}$$

which implies $\bar\phi_{i-1} = \bar\phi_i$, a contradiction.

Hence $\phi_L > \bar\phi_1 > \bar\phi_2 > \cdots > \bar\phi_{n-1} > \bar\phi_n > \phi_R$. ∎

In Figure 2.1 we illustrate the theory of Lemma 2.1 and Theorem 2.3 with a numerical example. Here we take $H^+(\phi) = H^-(\phi) = 0.5\cos\phi + \sin\phi$. Then $\phi_L = -\phi_R = \arctan(0.5) \approx 0.464$ and $J = (-\arctan 2, \arctan 2) \approx (-1.107, 1.107)$ for the conditions (H1) and (H2). We implemented the numerical computation by using the interactive package **XPPAUT** which was developed by B. Ermentrout. From the top to the bottom, the curves are $\phi_1(t), \ldots, \phi_{11}(t)$ $(n = 11)$, respectively. Note $\phi_6(t) \equiv 0$ is on the x-axis. The figure shows the monotonicity and symmetry of solution $(\phi_1(t), \ldots, \phi_n(t))$ along the trajectory.

In Lemmas 2.1 and 2.2 and Theorem 2.3, we have the condition $f(0) > 0$. For $f(0) < 0$, the results and the proofs are very similar. We just state Theorem 2.4 without proof.

THEOREM 2.4. *Assume f and g satisfy* (H1), (H2), *and* $f(0) < 0$; *then the IVP* (2.3) *with* $\phi_i(0) = 0$, $i = 1, \ldots, n$ *has the following properties:*

(i) *For each* $i \in \{1, \ldots, n\}$, *there exists* $\bar\phi_i$ *such that* $\lim_{t\to\hat t} \phi_i(t) = \bar\phi_i$;

(ii) $(\bar\phi_1, \ldots, \bar\phi_n)$ *is the fixed point of the system* (2.3);

(iii) $\phi_L < \bar\phi_1 < \bar\phi_2 < \cdots < \bar\phi_{n-1} < \bar\phi_n < \phi_R$;

(iv) $\bar\phi_i = -\bar\phi_{n+1-i}$, $i = 1, \ldots, n$.

We turn our attention back to the system (2.1). Notice that we have

$$\omega + H^+(\bar\phi_1) = \omega + H^-(-\bar\phi_{i-1}) + H^+(\bar\phi_i) = \omega + H^-(-\bar\phi_n), i = 2, \ldots, n-1.$$

We take $\Omega = \omega + H^-(-\bar{\phi}_{i-1}) + H^+(\bar{\phi}_i)$; then $\theta_1 = \Omega t$, $\theta_i = \Omega t + \sum_{k=1}^{i-1} \bar{\phi}_k$, $i = 2, \ldots, n+1$, is the phaselocked solution of (2.1). Before showing that this phaselocked solution is stable, we state a general stability result due to Ermentrout [10].

THEOREM 2.5 (Ermentrout, 1992). *Consider the equations*

$$d\theta_k/dt = H_k(\theta_1 - \theta_k, \ldots, \theta_M - \theta_k), \quad k = 1, \ldots, M. \tag{2.11}$$

Let $\theta_k = \Omega t + \bar{\psi}_k$ be a phaselocked solution and let

$$a_{jk} = \partial H_k(z_1, \ldots, z_M)/\partial z_j \tag{2.12}$$

evaluated at $z_j = \bar{\psi}_j - \bar{\psi}_k$. Suppose that $a_{jk} \geq 0$ and the graph of the matrix (a_{jk}) is complete. Then the phaselocked solution is orbitally asymptotically stable in the sense that there is a simple zero eigenvalue corresponding to translation in time and all other eigenvalues have negative real parts.

Due to (H1), we have $g' \pm f' > 0$ in J. Then the phaselocked solution $\theta_1 = \Omega t$, $\theta_i = \Omega t + \sum_{k=1}^{i-1} \bar{\phi}_k$, $i = 2, \ldots, n+1$, satisfies the nonnegativity assumption in Theorem 2.5. The graph of (a_{jk}) is complete since $a_{i,i+1} > 0$ and $a_{i+1,i} > 0$ for $i = 1, \ldots, n$. So we have shown that the phaselocked solution is asymptotically stable. This result is summarized in the following theorem.

THEOREM 2.6. *Under the conditions of Theorem* 2.3 *or* 2.4, $\theta_1 = \Omega t$, $\theta_i = \Omega t + \sum_{k=1}^{i-1} \bar{\phi}_k$, $i = 2, \ldots, n+1$, *is the phaselocked solution of* (2.1), *orbitally asymptotically stable in the sense that there is a simple zero eigenvalue corresponding to translation in time and other eigenvalues have negative real parts.*

As a matter of fact, in Theorem 2.6, all the n nonzero eigenvalues with negative real parts are actually the eigenvalues of the system (2.3) linearized around the equilibrium $(\bar{\phi}_1, \ldots, \bar{\phi}_n)$.

COROLLARY 2.7. *Under the conditions of Theorems* 2.3 *or* 2.4, $(\bar{\phi}_1, \ldots, \bar{\phi}_n)$ *is an asymptotically stable steady state of* (2.3) *and all the eigenvalues of the system* (2.3) *linearized around it have negative real parts.*

2.2. Isotropic case with $\beta_i = \beta \neq 0, i = 1, \ldots, n$. Throughout this section, without loss of generality, we assume $\beta < 0$. If $\beta > 0$, you can subtract the consecutive equations of (2.1) in another direction such that the frequency difference is less than zero. In this case, we still have $H^+ = H^-$, which implies that f is even and g odd such that $\phi_L = -\phi_R$. We restate (2.2) in the form

$$\begin{aligned}
&\phi_1' = \beta + f(\phi_2) + g(\phi_2) - 2g(\phi_1), \\
&\phi_i' = \beta + f(\phi_{i+1}) - f(\phi_{i-1}) + g(\phi_{i+1}) - 2g(\phi_i) + g(\phi_{i-1}), \\
&\qquad i = 2, \ldots, n-1, \\
&\phi_n' = \beta - f(\phi_{n-1}) - 2g(\phi_n) + g(\phi_{n-1}).
\end{aligned} \tag{2.13}$$

For $\beta = 0$, we have that $(\bar{\phi}_1, \ldots, \bar{\phi}_n)$ is the asymptotically stable steady state of (2.13) following Corollary 2.7. Then if $|\beta|$ is small enough, we should get an asymptotically stable steady state $\bar{\phi}_i(\beta)$, $i = 1, \ldots, n$ near $(\bar{\phi}_1, \ldots, \bar{\phi}_n)$ by the implicit function theorem. We denote the trajectory by $\phi_i(t, \beta)$, $i = 1, \ldots, n$ for the IVP (2.13) with $\phi_i(0) = 0$. By continuity, $\phi_i(t, \beta)$ should have the monotonicity and boundedness as in Lemmas 2.1 and 2.2, and $\phi_i(t, \beta) \to \bar{\phi}_i(\beta)$ as $t \to +\infty$ if $|\beta|$ is small enough. We summarize this fact in Theorem 2.8

THEOREM 2.8. *Assume that f and g satisfy* (H1) *and* (H2). *Let $|\beta|$ be small enough; then the IVP* (2.13) *with $\phi_i(0) = 0$ satisfies that*

(i) *if* $f(0) > 0$, *then*

$$\phi_L > \phi_1(t,\beta) > \cdots > \phi_n(t,\beta) > \phi_R; \tag{2.14}$$

(ii) *if* $f(0) < 0$, *then*

$$\phi_L < \phi_1(t,\beta) < \cdots < \phi_n(t,\beta) < \phi_R. \tag{2.15}$$

Also $\phi_i(t,\beta) \to \bar{\phi}_i(\beta)$ *as* $t \to +\infty$ *for* $i = 1, \ldots, n$, *where* $(\bar{\phi}_1(\beta), \ldots, \bar{\phi}_n(\beta))$ *is the asymptotically stable steady state of* (2.13) *near* $(\bar{\phi}_1, \ldots, \bar{\phi}_n)$.

Theorem 2.8 is not a particularly strong result. To keep the monotonicity and boundedness, $|\beta|$ has to be assumed very small. We would like to know when the monotonicity breaks down. This leads to the following theorem.

THEOREM 2.9. *Assume that* f *and* g *satisfy* (H1), (H2), *and* $f(0) > 0$. *Let* $|\beta|$ *be small enough that* $\phi_{n-1}(t) \geq \phi_R$, $t > 0$ *for the IVP* (2.13) *with* $\phi_i(0) = 0$, $i = 1, \ldots, n$. *Then we have the following properties along the trajectory:*

(i) *there is a sequence* $\{t_k\}_{k=1}^{\infty}$ *(it could be a finite sequence) such that* $0 = t_1 < t_2 < \cdots < t_k < \cdots < \hat{t}$, *and for each* k, *there is* $l_k \in \{1, \ldots, n\}$ *so that*

$$\phi_i'(t) > 0, \quad i = 1, \ldots, l_k, \quad t_k < t < t_{k+1},$$

$$\phi_j'(t) < 0, \quad j = l_k + 1, \ldots, n, \quad t_k < t < t_{k+1}, \tag{2.16}$$

$$l_{k+1} \in \{0, l_k - 1, l_k, l_k + 1, n\}, \tag{2.17}$$

$$\text{either } \phi_{l_k}'(t_{k+1}) = 0 \text{ or } \phi_{l_k+1}'(t_{k+1}) = 0 \text{ (not both)}, \tag{2.18}$$

$$\phi_L > \phi_1(t) > \cdots > \phi_{n-1}(t) > \phi_n(t) > \phi_\beta, \quad t_k < t \leq t_{k+1} \tag{2.19}$$

where $\hat{t}$ *is such that* $\phi_i'(\hat{t}) = 0$, $i = 1, \ldots, n$ *or* $\hat{t} = +\infty$, *and* $\phi_\beta \in J$ *is such that* $f(\phi_\beta) + g(\phi_\beta) = \beta$ *(note that* $\phi_\beta < \phi_R$*).*

(ii) *for each* $i \in \{1, \ldots, n\}$, *there exists* $\bar{\phi}_i$ *such that*

$$\lim_{t \to \hat{t}} \phi_i(t) = \bar{\phi}_i, \tag{2.20}$$

$$\phi_L > \bar{\phi}_1 > \cdots > \bar{\phi}_n > \phi_\beta, \tag{2.21}$$

and $(\bar{\phi}_1, \ldots, \bar{\phi}_n)$ *is a fixed point of* (2.13).

Remark. The condition $\phi_{n-1}(t) \geq \phi_R, t > 0$ means that ϕ_{n-1} cannot cross ϕ_R along the trajectory. It is weaker than the condition in Theorem 2.8 since it allows ϕ_n to cross ϕ_R. It holds when $|\beta|$ is small enough (but not as small as in Theorem 2.8) according to the results of section 2.1.

The proof of the theorem is very long. We put it in the Appendix for interested readers.

Figure 2.2 is a numerical solution illustrating Theorem 2.9. Here $H^+(\phi) = H^-(\phi) = 0.5\cos\phi + \sin\phi$ and $\beta = -0.005$. The figure shows monotonicity of solution along the trajectory.

For the case $f(0) < 0$, we have results parallel to Theorem 2.9.

THEOREM 2.10. *Assume that* f *and* g *satisfy* (H1), (H2), *and* $f(0) < 0$. *Let* $|\beta|$ *be small enough that* $\phi_2(t) \geq \phi_L$, $t > 0$ *for the IVP* (2.13) *with* $\phi_i(0) = 0$, $i = 1, \ldots, n$. *Then we have the following properties along the trajectory:*

(i) *there is a sequence* $\{t_k\}_{k=1}^{\infty}$ *(it could be a finite sequence) such that* $0 = t_1 < t_2 < \cdots < t_k < \cdots < \hat{t}$ *and for each* k, *there is* $l_k \in \{1, \ldots, n\}$ *so that*

$$\phi_i'(t) < 0, \quad i = 1, \ldots, l_k, \quad t_k < t < t_{k+1},$$

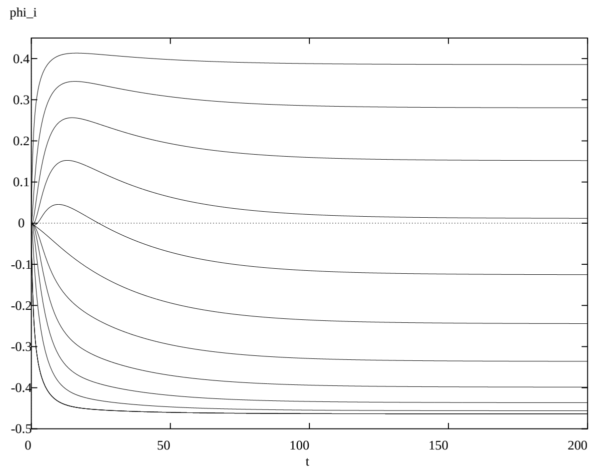


FIG. 2.2. *The isotropic case with* $H^+(\phi) = H^-(\phi) = H(\phi) = .5\cos\phi + \sin\phi$, $n = 11$, *and* $\beta_i = \beta = -0.005$.

$$\phi_j'(t) > 0, \quad j = l_k + 1, \dots, n, \quad t_k < t < t_{k+1}, \tag{2.22}$$

$$l_{k+1} \in \{0, l_k - 1, l_k, l_k + 1, n\}, \tag{2.23}$$

$$\textit{either } \phi_{l_k}'(t_{k+1}) = 0 \textit{ or } \phi_{l_k+1}'(t_{k+1}) = 0 \textit{ (not both)}, \tag{2.24}$$

$$\phi_\beta < \phi_1(t) < \cdots < \phi_{n-1}(t) < \phi_n(t) < \phi_R, \quad t_k < t \le t_{k+1}, \tag{2.25}$$

where $\hat{t}$ *is such that* $\phi_i'(\hat{t}) = 0$, $i = 1, \dots, n$ *or* $\hat{t} = +\infty$, *and* $\phi_\beta \in J$ *is such that* $-f(\phi_\beta) + g(\phi_\beta) = \beta$ *(note that* $\phi_\beta < \phi_L$*).*

(ii) *for each* $i \in \{1, \dots, n\}$, *there exists* $\bar{\phi}_i$ *such that*

$$\lim_{t \to \hat{t}} \phi_i(t) = \bar{\phi}_i, \tag{2.26}$$

$$\phi_\beta < \bar{\phi}_1 < \cdots < \bar{\phi}_n < \phi_R, \tag{2.27}$$

and $(\bar{\phi}_1, \dots, \bar{\phi}_n)$ *is a fixed point of* (2.13).

As we did in section 2.1, if we let $\theta_1(t) = \Omega t$, $\theta_i(t) = \Omega t + \sum_{k=1}^{i-1} \bar{\phi}_k$, $i = 1, \dots, n+1$, where $(\bar{\phi}_1, \dots, \bar{\phi}_n)$ is the fixed point of (2.13) which we obtained in Theorems 2.9 and 2.10, then Theorem 2.5 assures us that $(\theta_1(t), \dots, \theta_{n+1}(t))$ is an orbitally asymptotically stable phaselocked solution of (2.1).

2.3. Nonisotropic case with $\beta_i = 0, i = 1, \dots, n$. In this case, we have $H^+ \ne H^-$ which implies that f is not even and g is not odd anymore. So $\phi_L \ne -\phi_R$ in general. And we would like to restate (2.2) in the form

$$\begin{aligned} \phi_1' &= f(\phi_2) + g(\phi_2) - 2g(\phi_1), \\ \phi_i' &= f(\phi_{i+1}) - f(\phi_{i-1}) + g(\phi_{i+1}) - 2g(\phi_i) + g(\phi_{i-1}), \\ &\quad i = 2, \dots, n-1, \\ \phi_n' &= -f(\phi_{n-1}) - 2g(\phi_n) + g(\phi_{n-1}). \end{aligned} \tag{2.28}$$

THEOREM 2.11. *Assume that* f *and* g *satisfy* (H1), (H2), *and* $f(0) > |g(0)|$. *Then the IVP* (2.28) *with* $\phi_i(0) = 0$, $i = 1, \dots, n$ *has the following properties along the trajectory:*

(i) *There is a sequence* $\{t_k\}_{k=1}^{\infty}$ *(it could be a finite sequence) such that* $0 = t_1 < t_2 < \cdots < t_k < \cdots < \hat{t}$ *and for each* k*, there is* $l_k \in \{1, \ldots, n\}$ *so that*

$$\phi_i'(t) > 0, \quad i = 1, \ldots, l_k, \quad t_k < t < t_{k+1},$$

$$\phi_j'(t) < 0, \quad j = l_k + 1, \ldots, n, \quad t_k < t < t_{k+1}, \tag{2.29}$$

$$l_{k+1} \in \{0, l_k - 1, l_k, l_k + 1, n\}, \tag{2.30}$$

$$\text{either } \phi_{l_k}'(t_{k+1}) = 0 \text{ or } \phi_{l_k+1}'(t_{k+1}) = 0 \text{ (not both)}, \tag{2.31}$$

$$\phi_L > \phi_1(t) > \cdots > \phi_{n-1}(t) > \phi_n(t) > \phi_R, \quad t_k < t \le t_{k+1}, \tag{2.32}$$

where $\hat{t}$ *is such that* $\phi_i'(\hat{t}) = 0$*,* $i = 1, \ldots, n$*, or* $\hat{t} = +\infty$*.*

(ii) *For each* $i \in \{1, \ldots, n\}$*, there exists* $\bar{\phi}_i$ *such that*

$$\lim_{t \to \hat{t}} \phi_i(t) = \bar{\phi}_i, \tag{2.33}$$

$$\phi_L > \bar{\phi}_1 > \cdots > \bar{\phi}_n > \phi_R, \tag{2.34}$$

and $(\bar{\phi}_1, \ldots, \bar{\phi}_n)$ *is a fixed point of* (2.28)*.*

Proof. Note that $f(0) > |g(0)|$; then

$$\begin{aligned} \phi_1'(0) &= f(0) - g(0) > 0, \\ \phi_i'(0) &= 0, \quad i = 2, \ldots, n-1, \\ \phi_n'(0) &= -f(0) - g(0) < 0. \end{aligned} \tag{2.35}$$

Then by (2.28) and (2.35), we have

$$\begin{aligned} \phi_2''(0) &= [g'(0) - f'(0)][f(0) - g(0)] > 0, \\ \phi_{n-1}''(0) &= [g'(0) + f'(0)][-f(0) - g(0)] < 0. \end{aligned} \tag{2.36}$$

By induction on i, we can get that for $i = 3, \ldots, m-1$

$$\begin{aligned} \phi_i^{(k)}(0) &= 0, \quad k = 1, \ldots, i-1, \\ \phi_i^{(i)}(0) &= [g'(0) - f'(0)]^{i-1}[f(0) - g(0)] > 0, \\ \phi_{n-i+1}^{(k)}(0) &= 0, \quad k = 1, \ldots, i-1, \\ \phi_{n-i+1}^{(i)}(0) &= [g'(0) - f'(0)]^{i-1}[-f(0) + g(0)] < 0 \end{aligned} \tag{2.37}$$

whenever $n = 2m - 1$ or $2m - 2$. And when $n = 2m - 1$, we have extra terms

$$\phi_m^{(k)}(0) = 0, \quad k = 1, \ldots, m. \tag{2.38}$$

Assume $\phi_m^{(m+1)}(0) \neq 0$ (otherwise we can figure out $\phi_m^{(M)}(0) \neq 0$ and $\phi_m^{(k)}(0) = 0, k = 1, \ldots, M-1$).

Without loss of generality, we assume $\phi_m^{m+1}(0) > 0$ when $n = 2m - 1$.

Then we have $t_1 = 0, l_1 = m - 1$ when $n = 2m - 2$, and $t_1 = 0, l_1 = m$ when $n = 2m - 1$. And the rest of the proof just mimics all the steps of proving Theorem 2.9 ∎

Again in Figure 2.3 we show the results of Theorem 2.11. Here $H^+(\phi) = H(\phi)$ and $H^-(\phi) = 0.2H(\phi)$ where $H(\phi) = 0.5\cos\phi + \sin\phi$. And $\phi_L = -\phi_R = \arctan(0.5)$ and $J = (-\arctan 2, \arctan 2)$. The monotonicity of the solution along the trajectory can be seen from the figure. Also we see that the solution converges to a fixed point.

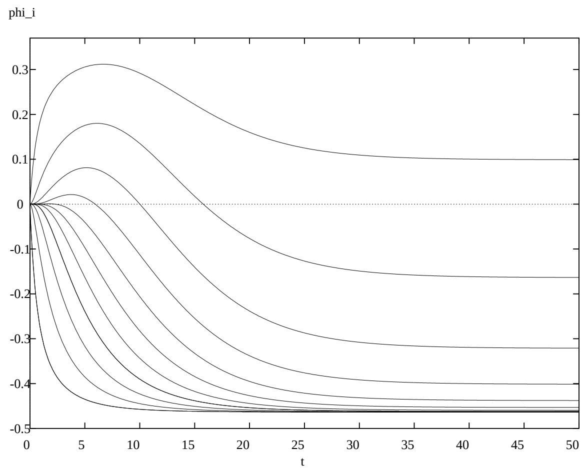


FIG. 2.3. *The nonisotropic case with* $H^+(\phi) = H(\phi), H^-(\phi) = 0.2H(\phi), H(\phi) = .5\cos\phi + \sin\phi$, $n = 11$, *and* $\beta_i = \beta = 0$.

THEOREM 2.12. *Assume that* f *and* g *satisfy* (H1), (H2), *and* $f(0) < -|g(0)|$. *Then the IVP* (2.28) *with* $\phi_i(0) = 0$, $i = 1, \dots, n$ *has the following properties along the trajectory:*

(i) *there is a sequence* $\{t_k\}_{k=1}^{\infty}$ *(it could be a finite sequence) such that* $0 = t_1 < t_2 < \cdots < t_k < \cdots < \hat{t}$ *and for each* k, *there is* $l_k \in \{1, \dots, n\}$ *so that*

$$\phi_i'(t) < 0, \quad i = 1, \dots, l_k, \quad t_k < t < t_{k+1},$$

$$\phi_j'(t) > 0, \quad j = l_k + 1, \dots, n, \quad t_k < t < t_{k+1}, \tag{2.39}$$

$$l_{k+1} \in \{0, l_k - 1, l_k, l_k + 1, n\}, \tag{2.40}$$

$$\text{either } \phi_{l_k}'(t_{k+1}) = 0 \text{ or } \phi_{l_k+1}'(t_{k+1}) = 0 \text{ (not both)}, \tag{2.41}$$

$$\phi_L < \phi_1(t) < \cdots < \phi_{n-1}(t) < \phi_n(t) < \phi_R, t_k < t \le t_{k+1}, \tag{2.42}$$

where $\hat{t}$ *is such that* $\phi_i'(\hat{t}) = 0$, $i = 1, \dots, n$ *or* $\hat{t} = +\infty$.

(ii) *for each* $i \in \{1, \dots, n\}$, *there exists* $\bar{\phi}_i$ *such that*

$$\lim_{t \to \hat{t}} \phi_i(t) = \bar{\phi}_i, \tag{2.43}$$

$$\phi_L < \bar{\phi}_1 < \cdots < \bar{\phi}_n < \phi_R, \tag{2.44}$$

and $(\bar{\phi}_1, \dots, \bar{\phi}_n)$ *is a fixed point of* (2.28).

2.4. Nonisotropic case with $\beta_i = \beta \neq 0, i = 1, \dots, n$. Throughout this section, without loss of generality, we assume $\beta < 0$. If $\beta > 0$, you can subtract the consecutive equations of (2.1) in another direction such that the frequency difference is less than zero. In this case, like in section 2.2, we have $H^+ \neq H^-$ which implies f is not even and g not odd such that $\phi_L \neq -\phi_R$. And we would like to restate (2.2) in the form

$$\begin{aligned} \phi_1' &= \beta + f(\phi_2) + g(\phi_2) - 2g(\phi_1), \\ \phi_i' &= \beta + f(\phi_{i+1}) - f(\phi_{i-1}) + g(\phi_{i+1}) - 2g(\phi_i) + g(\phi_{i-1}), \\ &\quad i = 2, \dots, n-1, \\ \phi_n' &= \beta - f(\phi_{n-1}) - 2g(\phi_n) + g(\phi_{n-1}). \end{aligned} \tag{2.45}$$

THEOREM 2.13. *Assume that f and g satisfy* (H1), (H2), *and* $f(0) > |g(0)|$. *Let* $|\beta|$ *be small enough that* $\phi_{n-1}(t) \geq \phi_R$, $t > 0$ *for the IVP* (2.45) *with* $\phi_i(0) = 0$, $i = 1, \ldots, n$. *Then we have the following properties along the trajectory:*

(i) *there is a sequence* $\{t_k\}_{k=1}^{\infty}$ *(it could be a finite sequence) such that* $0 = t_1 < t_2 < \cdots < t_k < \cdots < \hat{t}$ *and for each* k, *there is* $l_k \in \{1, \ldots, n\}$ *so that*

$$\phi_i'(t) > 0, \quad i = 1, \ldots, l_k, \quad t_k < t < t_{k+1},$$

$$\phi_j'(t) < 0, \quad j = l_k + 1, \ldots, n, \quad t_k < t < t_{k+1}, \tag{2.46}$$

$$l_{k+1} \in \{0, l_k - 1, l_k, l_k + 1, n\}, \tag{2.47}$$

$$\text{either } \phi_{l_k}'(t_{k+1}) = 0 \text{ or } \phi_{l_k+1}'(t_{k+1}) = 0 \text{ (not both)}, \tag{2.48}$$

$$\phi_L > \phi_1(t) > \cdots > \phi_{n-1}(t) > \phi_n(t) > \phi_\beta, \quad t_k < t \leq t_{k+1}, \tag{2.49}$$

where $\hat{t}$ *is such that* $\phi_i'(\hat{t}) = 0$, $i = 1, \ldots, n$ *or* $\hat{t} = +\infty$, *and* $\phi_\beta \in J$ *is such that* $f(\phi_\beta) + g(\phi_\beta) = \beta$ *(note that* $\phi_\beta < \phi_R$*).*

(ii) *for each* $i \in \{1, \ldots, n\}$, *there exists* $\bar{\phi}_i$ *such that*

$$\lim_{t \to \hat{t}} \phi_i(t) = \bar{\phi}_i, \tag{2.50}$$

$$\phi_L > \bar{\phi}_1 > \cdots > \bar{\phi}_n > \phi_\beta, \tag{2.51}$$

and $(\bar{\phi}_1, \ldots, \bar{\phi}_n)$ *is a fixed point of* (2.13).

Figures 2.4(a) and 2.4(b) are numerical illustrations of Theorem 2.13. Here $H^+(\phi) = H(\phi)$ and $H^-(\phi) = 0.2H(\phi)$, where $H(\phi) = 0.5\cos\phi + \sin\phi$. And $\phi_L = -\phi_R = \arctan(0.5)$ and $J = (-\arctan 2, \arctan 2)$. Note that in Fig. 2.4(b), $\phi_{n-1}(t)$ crosses ϕ_R somewhere so that the monotonicity is destroyed. However, the trajectory still converges to a fixed point. Hence the monotonicity is not necessary for the convergence of the solution. In Fig. 2.4(a) the monotonicity is preserved since the $|\beta|$ is so small that $\phi_{n-1}(t)$ does not cross ϕ_R.

THEOREM 2.14. *Assume that f and g satisfy* (H1), (H2), *and* $f(0) < -|g(0)|$. *Let* $|\beta|$ *be small enough that* $\phi_2(t) \geq \phi_L$, $t > 0$ *for the IVP* (2.45) *with* $\phi_i(0) = 0$, $i = 1, \ldots, n$. *Then we have the following properties along the trajectory:*

(i) *there is a sequence* $\{t_k\}_{k=1}^{\infty}$ *(it could be a finite sequence) such that* $0 = t_1 < t_2 < \cdots < t_k < \cdots < \hat{t}$ *and for each* k, *there is* $l_k \in \{1, \ldots, n\}$ *so that*

$$\phi_i'(t) < 0, \quad i = 1, \ldots, l_k, \quad t_k < t < t_{k+1},$$

$$\phi_j'(t) > 0, \quad j = l_k + 1, \ldots, n, \quad t_k < t < t_{k+1}, \tag{2.52}$$

$$l_{k+1} \in \{0, l_k - 1, l_k, l_k + 1, n\}, \tag{2.53}$$

$$\text{either } \phi_{l_k}'(t_{k+1}) = 0 \text{ or } \phi_{l_k+1}'(t_{k+1}) = 0 \text{ (not both)}, \tag{2.54}$$

$$\phi_\beta < \phi_1(t) < \cdots < \phi_{n-1}(t) < \phi_n(t) < \phi_R, \quad t_k < t \leq t_{k+1}, \tag{2.55}$$

where $\hat{t}$ *is such that* $\phi_i'(\hat{t}) = 0$, $i = 1, \ldots, n$ *or* $\hat{t} = +\infty$, *and* $\phi_\beta \in J$ *is such that* $-f(\phi_\beta) + g(\phi_\beta) = \beta$ *(note that* $\phi_\beta < \phi_L$*).*

(ii) *for each* $i \in \{1, \ldots, n\}$, *there exists* $\bar{\phi}_i$ *such that*

$$\lim_{t \to \hat{t}} \phi_i(t) = \bar{\phi}_i, \tag{2.56}$$

$$\phi_\beta < \bar{\phi}_1 < \cdots < \bar{\phi}_n < \phi_R, \tag{2.57}$$

and $(\bar{\phi}_1, \ldots, \bar{\phi}_n)$ *is a fixed point of* (2.45).

By Theorem 2.5, the fixed points $(\bar{\phi}_1, \ldots, \bar{\phi}_n)$ from the two theorems above are asymptotically stable steady state of (2.45).

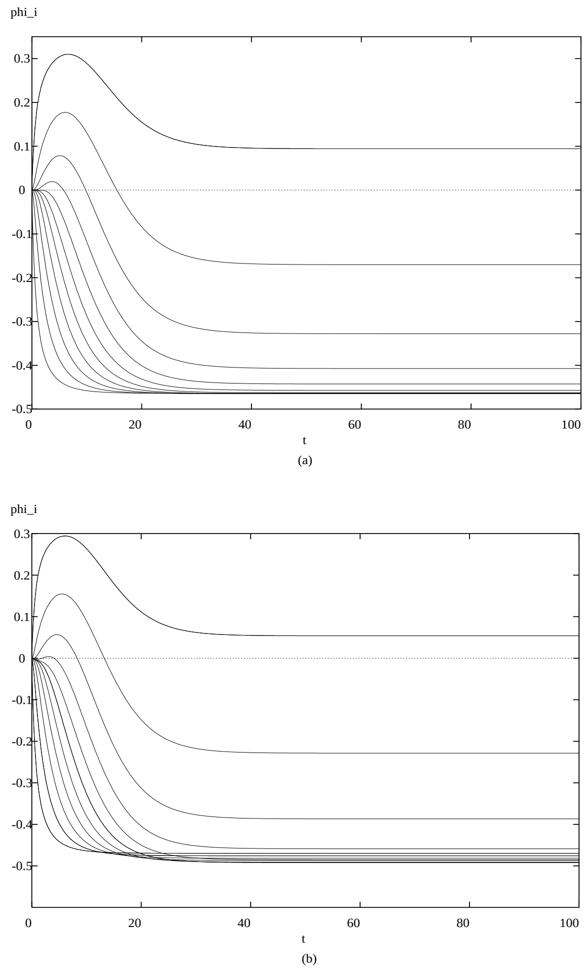


FIG. 2.4. *The nonisotropic case with* $H^+(\phi) = H(\phi), H^-(\phi) = 0.2H(\phi), H(\phi) = .5\cos\phi + \sin\phi$, $n = 11$, (a) $\beta_i = \beta = -0.0005$, (b) $\beta_i = \beta = -0.005$.

3. Arrays of oscillators. In this section, we consider a two-dimensional array of coupled oscillators. The equations to be considered have the form

$$\begin{aligned} \theta'_{ij} &= \omega_{ij} + H^{+X}(\theta_{i+1,j} - \theta_{ij}) + H^{-X}(\theta_{i-1,j} - \theta_{ij}) \\ &\quad + H^{+Y}(\theta_{i,j+1} - \theta_{ij}) + H^{-Y}(\theta_{i,j-1} - \theta_{ij}), \\ &\quad i, j = 1, \ldots, n+1, \end{aligned} \tag{3.1}$$

where H^{+X}, H^{+Y}, H^{-X}, and H^{-Y} are smooth 2π-periodic functions of the arguments and ω_{ij} is the frequency for each oscillator.

Note that in (3.1), each oscillator is coupled with its four nearest neighbors. The term H^{-X} (respectively, H^{+X} or H^{-Y} or H^{+Y}) is ignored for $i = 1$ (respectively, $i = n+1$ or $j = 1$ or $j = n+1$). We take

$$\begin{aligned} \phi_{ij} &= \theta_{i+1,j} - \theta_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n+1, \\ \psi_{ij} &= \theta_{i,j+1} - \theta_{ij}, \quad i = 1, \ldots, n+1, \quad j = 1, \ldots, n, \end{aligned}$$

$$\alpha_{ij} = \omega_{i+1,j} - \omega_{ij}, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n+1,$$
$$\beta_{ij} = \omega_{i,j+1} - \omega_{ij}, \quad i = 1, \ldots, n+1, \quad j = 1, \ldots, n$$

and define the functions f, g, p, and q as

$$\begin{aligned} f(\phi) + g(\phi) &= H^{+X}(\phi), \\ f(\phi) - g(\phi) &= H^{-X}(-\phi), \\ p(\psi) + q(\psi) &= H^{+Y}(\psi), \\ p(\psi) - q(\psi) &= H^{-Y}(-\psi). \end{aligned} \tag{3.2}$$

Then in (3.1), if we subtract the (i,j)th equation from the $(i+1,j)$th one and the $(i,j+1)$th one, respectively, we have

$$\begin{aligned} \phi'_{ij} = {} & \alpha_{ij} + f(\phi_{i+1,j}) - f(\phi_{i-1,j}) + g(\phi_{i+1,j}) - 2g(\phi_{ij}) + g(\phi_{i-1,j}) \\ & + p(\psi_{i+1,j}) + p(\psi_{i+1,j-1}) - p(\psi_{ij}) - p(\psi_{i,j-1}) \\ & + q(\psi_{i+1,j}) - q(\psi_{i+1,j-1}) - q(\psi_{ij}) + q(\psi_{i,j-1}), \\ & i = 1, \ldots, n, j = 1, \ldots, n+1, \\ \psi'_{ij} = {} & \beta_{ij} + p(\psi_{i,j+1}) - p(\psi_{i,j-1}) + q(\psi_{i,j+1}) - 2q(\psi_{ij}) + q(\psi_{i,j-1}) \\ & + f(\phi_{i,j+1}) + f(\phi_{i-1,j+1}) - f(\phi_{ij}) - f(\phi_{i-1,j}) \\ & + g(\phi_{i,j+1}) - g(\phi_{i-1,j+1}) - g(\phi_{ij}) + g(\phi_{i-1,j}) \\ & i = 1, \ldots, n+1, j = 1, \ldots, n. \end{aligned} \tag{3.3}$$

Note that the index (i,j) for ϕ_{ij} should satisfy $1 \le i \le n$ and $1 \le j \le n+1$ and the index (i,j) for ψ_{ij} should satisfy $1 \le i \le n+1$, $1 \le j \le n$. Hence if (i,j) is out of range for ϕ_{ij} or ψ_{ij}, the corresponding terms on the right-hand sides of (3.3) are ignored.

Again we define several constants related to f, g, p, and q. We define

- ϕ_L as $f(\phi_L) = g(\phi_L)$, i.e., $H^{-X}(-\phi_L) = 0$;
- ϕ_R as $f(\phi_R) = -g(\phi_R)$, i.e., $H^{+X}(\phi_R) = 0$;
- ψ_L as $p(\psi_L) = q(\psi_L)$, i.e., $H^{-Y}(-\psi_L) = 0$;
- ψ_R as $p(\psi_R) = -q(\psi_R)$, i.e., $H^{+Y}(\psi_R) = 0$.

We assume some hypotheses on f, g, p, and q in sufficiently large intervals J_X and J_Y around $\phi = 0$ and $\psi = 0$, respectively:

(HX1) *$g'(\phi) > |f'(\phi)|$ for $\phi \in J_X$;*

(HX2) *there exists a unique ϕ_L (respectively, ϕ_R) to $f = g$ (respectively, $f = -g$) for $\phi \in J_X$;*

(HY1) *$q'(\psi) > |p'(\psi)|$ for $\psi \in J_Y$;*

(HY2) *there exists a unique ψ_L (respectively, ψ_R) to $p = q$ (respectively, $p = -q$) for $\psi \in J_Y$.*

Note that (HX1), (HX2), (HY1), and (HY2) are the assumptions extended from the chain model.

Our goal is to apply the results obtained from the chain model to this array model. In order to achieve this task, let us first consider a very special system of equations:

$$\begin{aligned} \phi'_{ij} &= F_{ij}(\Phi) + G_{ij}(\Psi), \quad i = 1, \ldots, n, \quad j = 1, \ldots, n+1, \\ \psi'_{ij} &= P_{ij}(\Psi) + Q_{ij}(\Phi), \quad i = 1, \ldots, n+1, \quad j = 1, \ldots, n, \end{aligned} \tag{3.4}$$

where

$$\Phi = (\phi_{ij})_{n\times(n+1)} = [\Phi_1, \dots, \Phi_{n+1}]$$

with

$$\Phi_j = \begin{bmatrix} \phi_{1j} \\ \vdots \\ \phi_{nj} \end{bmatrix}, \quad j = 1, \dots, n+1$$

and

$$\Psi = (\psi_{ij})_{(n+1)\times n} = \begin{bmatrix} \Psi_1 \\ \vdots \\ \Psi_{n+1} \end{bmatrix}$$

with

$$\Psi_i = [\psi_{i1}, \dots, \psi_{in}], \quad i = 1, \dots, n+1$$

and F_{ij}, P_{ij}, G_{ij}, and Q_{ij} satisfy the following assumptions:

(i) *if* $\Phi_1 = \Phi_2 = \cdots = \Phi_{n+1}$ *(i.e.,* ϕ_{ij} *is independent of the index* j*), then*

$$F_{i1}(\Phi) = F_{i2}(\Phi) = \cdots = F_{i,n+1}(\Phi), \quad i = 1, \dots, n, \tag{3.5}$$

$$Q_{ij}(\Phi) = 0, i = 1, \dots, n+1, \quad j = 1, \dots, n; \tag{3.6}$$

(ii) *if* $\Psi_1 = \Psi_2 = \cdots = \Psi_{n+1}$ *(i.e.,* ψ_{ij} *is independent of the index* i*), then*

$$P_{1j}(\Psi) = P_{2j}(\Psi) = \cdots = P_{n+1,j}(\Psi), \quad j = 1, \dots, n, \tag{3.7}$$

$$G_{ij}(\Psi) = 0, \quad i = 1, \dots, n, \quad j = 1, \dots, n+1. \tag{3.8}$$

Remark. The special form of (3.4) is a generalization of the system (3.3). We will see this later. The conditions on F_{ij}, G_{ij}, P_{ij}, and Q_{ij} reflect a homogeneity requirement for the two-dimensional domain. That is, the phase lags between left and right neighbors are the same for each row. Similarly, the lags between top and bottom neighbors are the same for each column.

LEMMA 3.1. *The set* $S = \{(\Phi, \Psi) | \Phi_1 = \Phi_2 = \cdots = \Phi_{n+1}$ *and* $\Psi_1 = \Psi_2 = \cdots = \Psi_{n+1}\}$ *is an invariant set for the system* (3.4).

Proof. We only need to show that if $(\Phi(0), \Psi(0)) \in S$, then $\Phi_1'(0) = \cdots = \Phi_{n+1}'(0)$ and $\Psi_1'(0) = \cdots = \Psi_{n+1}'(0)$, i.e.,

$$\phi_{i1}'(0) = \phi_{i2}'(0) = \cdots = \phi_{i,n+1}'(0) \quad \text{for} \quad i = 1, \dots, n, \tag{3.9}$$

$$\psi_{1j}'(0) = \psi_{2j}'(0) = \cdots = \psi_{n+1,j}'(0) \quad \text{for} \quad j = 1, \dots, n. \tag{3.10}$$

By (3.4), (3.5), and (3.8), we have that for each $i \in \{1, \dots, n\}$,

$$\begin{aligned} \phi_{ij}'(0) &= F_{ij}(\Phi(0)) + G_{ij}(\Psi(0)) \\ &= F_{ik}(\Phi(0)) + G_{ik}(\Psi(0)) \\ &= \phi_{ik}'(0). \end{aligned}$$

Hence (3.9) is proven. Also, we can prove (3.10) in the same way. ∎

LEMMA 3.2. *In the system* (3.3), *if we assume*

$$\alpha_{ij} = \alpha_i \text{ and } \beta_{ij} = \beta_j \tag{3.11}$$

then (3.3) *is a system of the type* (3.4).

Proof. (3.3) is a special case of (3.4) where

$$\begin{aligned} F_{ij}(\Phi) &= \alpha_{ij} + f(\phi_{i+1,j}) - f(\phi_{i-1,j}) + g(\phi_{i+1,j}) - 2g(\phi_{ij}) + g(\phi_{i-1,j}), \\ G_{ij}(\Psi) &= p(\psi_{i+1,j}) + p(\psi_{i+1,j-1}) - p(\psi_{ij}) - p(\psi_{i,j-1}) \\ &\quad + q(\psi_{i+1,j}) - q(\psi_{i+1,j-1}) - q(\psi_{ij}) + q(\psi_{i,j-1}), \\ P_{ij}(\Psi) &= \beta_{ij} + p(\psi_{i,j+1}) - p(\psi_{i,j-1}) + q(\psi_{i,j+1}) - 2q(\psi_{ij}) + q(\psi_{i,j-1}), \\ Q_{ij}(\Phi) &= f(\phi_{i,j+1}) + f(\phi_{i-1,j+1}) - f(\phi_{ij}) - f(\phi_{i-1,j}) \\ &\quad + g(\phi_{i,j+1}) - g(\phi_{i-1,j+1}) - g(\phi_{ij}) + g(\phi_{i-1,j}). \end{aligned}$$

Since we have (3.11), α_{ij} is independent of j and β_{ij} is independent of i. Then if ϕ_{ij} is independent of j and ψ_{ij} is independent of i, (3.5)–(3.8) are satisfied. The proof is completed. ∎

Remark. (3.11) means that the distribution of intrinsic frequencies is a sum of two stripe distributions: one with constant frequencies along each row, and another with constant frequencies along each column. Hence ω_{ij} is in the form of $\omega_{ij} = \omega_i^X + \omega_j^Y$.

LEMMA 3.3. *If the system* (3.3) *satisfies* (3.11), *then the IVP* (3.3) *with*

$$\phi_{ij}(0) = 0, \quad i = 1, \dots, n, \quad j = 1, \dots, n+1, \tag{3.12}$$

$$\psi_{ij}(0) = 0, \quad i = 1, \dots, n+1, \quad j = 1, \dots, n \tag{3.13}$$

has the following identity property:

$$\phi_{i1}(t) = \phi_{i2}(t) = \cdots = \phi_{i,n+1}(t), \quad i = 1, \dots, n, \tag{3.14}$$

$$\psi_{1j}(t) = \psi_{2j}(t) = \cdots = \psi_{n+1,j}(t), \quad j = 1, \dots, n \tag{3.15}$$

for $t \geq 0$.

Proof. This is an immediate result of Lemmas 3.1 and 3.2. ∎

Hence the IVP (3.3), (3.12), and (3.13) satisfying (3.11) is reduced to two independent systems of chain model, i.e.,

$$\begin{aligned} \phi_1' &= \alpha_1 + f(\phi_2) + g(\phi_2) - 2g(\phi_1), \\ \phi_i' &= \alpha_i + f(\phi_{i+1}) - f(\phi_{i-1}) + g(\phi_{i+1}) - 2g(\phi_i) + g(\phi_{i-1}), \\ &\quad i = 2, \dots, n-1, \\ \phi_n' &= \alpha_n - f(\phi_{n-1}) - 2g(\phi_n) + g(\phi_{n-1}) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \psi_1' &= \beta_1 + p(\psi_2) + q(\psi_2) - 2q(\psi_1), \\ \psi_j' &= \beta_j + p(\psi_{j+1}) - p(\psi_{j-1}) + q(\psi_{j+1}) - 2q(\psi_j) + q(\psi_{j-1}), \\ &\quad j = 2, \dots, n-1, \\ \psi_n' &= \beta_n - p(\psi_{n-1}) - 2q(\psi_n) + q(\phi_{n-1}), \end{aligned} \tag{3.17}$$

where $\phi_i = \phi_{i1} = \cdots = \phi_{i,n+1}$ and $\psi_j = \psi_{1j} = \cdots = \psi_{n+1,j}$.

Note that both (3.16) and (3.17) are exactly in the form of (2.2).

THEOREM 3.4. *If the trajectories of the IVP* (3.16) *with* $\phi_i(0) = 0$ *and the IVP with* (3.17) *with* $\psi_j(0) = 0$ *converge to the fixed point* $(\bar{\phi}_1, \dots, \bar{\phi}_n)$ *of* (3.16) *and the fixed point* $(\bar{\psi}_1, \dots, \bar{\psi}_n)$ *of* (3.17) *respectively, then the trajectory of the IVP* (3.3) *with* (3.12) *and* (3.13) *goes to* $((\bar{\phi}_{ij})_{n\times(n+1)}, (\bar{\psi}_{ij})_{(n+1)\times n})$ *which is the fixed point of* (3.3), *where*

$$\bar{\phi}_{ij} = \bar{\phi}_i, \quad i = 1, \dots, n, \quad j = 1, \dots, n+1$$

and

$$\bar{\psi}_{ij} = \bar{\psi}_j, \quad i = 1, \dots, n+1, \quad j = 1, \dots, n.$$

Also, $\Omega \equiv \omega_{ij} + H^{+X}(\bar{\phi}_{ij}) + H^{-X}(-\bar{\phi}_{i-1,j}) + H^{+Y}(\bar{\psi}_{ij}) + H^{-Y}(-\bar{\psi}_{i,j-1})$ *(*$i, j = 1, \dots, n+1$*) is the locked frequency of* (3.1).

Proof. This is a straightforward result of Lemma 3.3. □

Now if we let $\theta_{1,1}(t) = \Omega t$, $\theta_{ij}(t) = \Omega t + \sum_{k=1}^{i-1} \bar{\phi}_k + \sum_{k=1}^{j-1} \bar{\psi}_k$, $\{\theta_{ij}(t)\}$ is the phaselocked solution of (3.1). And it is orbitally asymptotically stable by Theorem 2.5.

Therefore, all the results which we obtained in section 2 can be extended to this system.

Remark 3. If the condition (3.11) is not satisfied, we will not achieve the reduction. But if $\omega_{ij} = \omega_i^X + \omega_j^Y + o(\varepsilon)$ for small ε, we still get a stable phaselocked solution by the implicit function theorem.

Remark 4. The reduction technique could be applied to three-dimensional arrays of oscillators as long as ω_{ijk} is in the form of $\omega_{ijk} = \omega_i^X + \omega_j^Y + \omega_k^Z$. And the array models could be reduced to three independent chain models.

The following are some numerical results for the two-dimensional arrays of oscillators (3.1) and the reduced chains (3.16) and (3.17). For all cases, $\omega_{ij} = \omega_i^X + \omega_j^Y$ is assumed. A basic function $H(\phi) = 0.5\cos\phi + \sin\phi$ is assumed.

Example 1. Let $H^{+X} = H^{-X} = H^{+Y} = H^{-Y} = H$ and $\omega_{ij} \equiv \omega > 0$. Then (HX1), (HX2), (HY1), and (HY2) are satisfied with $J_X = J_Y = (-\arctan 2, \arctan 2)$ and $\phi_L = -\phi_R = \psi_L = -\psi_R = \arctan(0.5)$. Also, $f = p$ and $g = q$. Since $\omega_{ij} \equiv \omega$, the condition (3.11) holds so that the array system (3.3) can be reduced to the two chain systems (3.16) and (3.17) by Lemma 3.3. And (3.16) and (3.17) have asymptotically stable equilibria following the results in section 2.1. Then (3.3) has an asymptotically stable equilibrium. Noting that $f = p$, $g = q$, and $\alpha_i = \beta_j = 0$, the solutions of (3.16) and (3.17) are the identical. So we only study the solution $\bar{\phi}_i$ of (3.16). Figure 3.1(a) is the plot for $\bar{\phi}_i$ where $(i/(n+1), \bar{\phi}_i)$ are the coordinates. We can see that there is a wave traveling outward in both directions from the midpoint of the chain [5, 11]. The wave speed is almost constant except near the middle. By Theorem 3.4, $\bar{\phi}_{ij} = \bar{\phi}_i$ and $\bar{\psi}_{ij} = \bar{\psi}_j$. Then for the array, we have a wave traveling outward from the midpoint of the array. Figure 3.1b shows this observation by plotting the relative phases. As we mentioned in the introduction, with isotropic "synaptic coupling" target patterns are the generic phaselocked behavior. (See the remarks at the end of this section for a discussion about other stable patterns.)

Example 2. Let $H^{+X} = H^{+Y} = 1.5H$, $H^{-X} = H^{-Y} = 0.5H$, and $\omega_{ij} \equiv \omega > 0$. Then (HX1), (HX2), (HY1), and (HY2) hold with $J_X = J_Y = (-\arctan 2, \arctan 2)$ and $\phi_L = -\phi_R = \psi_L = -\psi_R = \arctan(0.5)$. Also, $f = p$ and $g = q$. The reduction from an array to two chains is then obtained. These two chains are identical according to our choice of coupling functions. Figure 3.2 shows the results for the reduced chains and the array. There is a wave traveling from the left of chain to the right. Thus there is a wave traveling from the southwest corner to the northeast corner of the array.

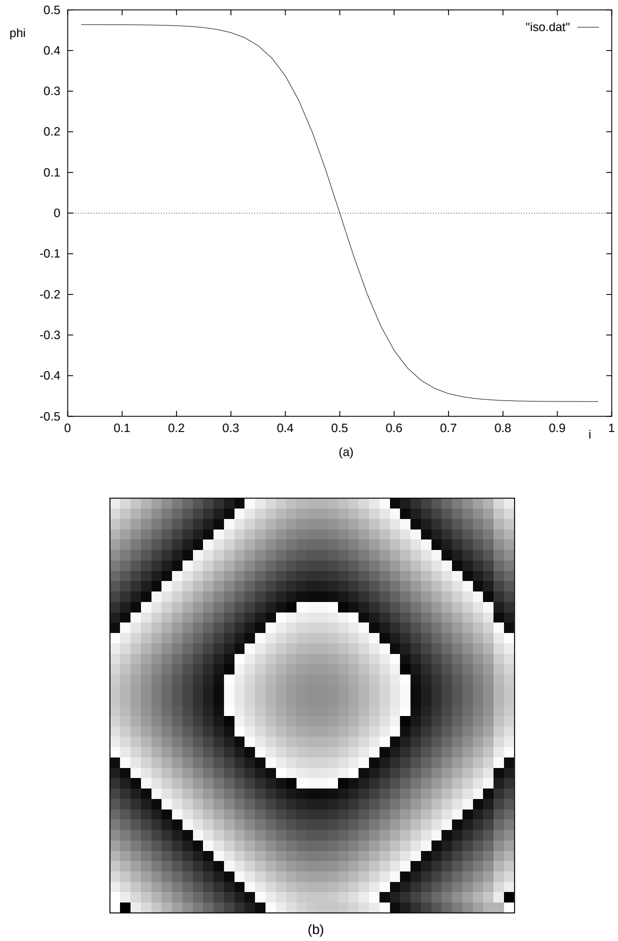


FIG. 3.1. $n+1=40$. (a) *Phase lags of the reduced chains. There is a wave traveling outward in both directions from the midpoint of the chain. The wave speed is almost constant except near the middle.* (b) *Relative phases of the array. There is a wave traveling outward from the midpoint of the array.*

Example 3. The coupling functions are the same as in Example 2. We choose $\omega_{ij} = 2\omega + 0.1[1 - i/(n+1)] + 0.1[1 - j/(n+1)]$ which is in the form $\omega_{ij} = \omega_i^X + \omega_j^Y$, where $\omega_i^X = \omega + 0.1[1 - i/(n+1)]$ and $\omega_j^X = \omega + 0.1[1 - j/(n+1)]$. Then the solutions of the two chain systems (3.16) and (3.17) are the same. Figure 3.3 shows the numerical solutions for the reduced chains and the array.

Example 4. In this example, we show how the size of the chain can apparently affect the qualitative features of the phases in one- and two-dimensional arrays. In Fig. 3.4(a), we show the results of a simulation with a 50×50 array of oscillators with no frequency gradient and all of the interactions functions identical and given by $H(\phi) = \sin\phi + 0.05\cos\phi + 0.8$. The phases give the appearance of a circularly symmetric target pattern, quite different from the rectangular-looking patterns of Figure 3.1. This effect can be understood by looking at the behavior of the chain. In Figure 3.4b, the phase-shifts between successive oscillators are shown for a chain with $n = 50$ and $n = 500$ oscillators. In the case of $n = 50$ the phase-difference

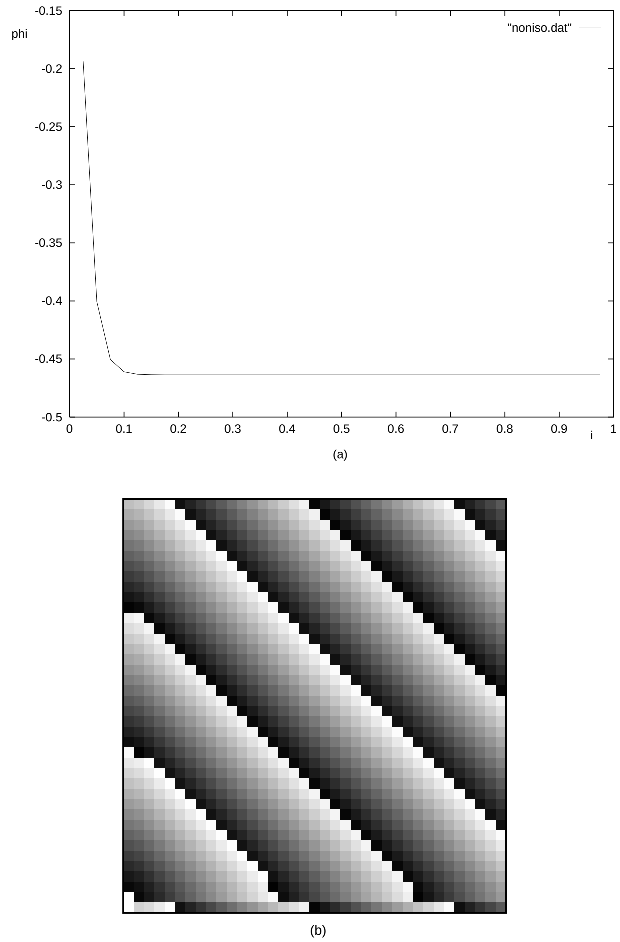


FIG. 3.2. $n+1=40$. (a) *Phase lags of the reduced chains. There is a wave traveling from the left of chain to the right.* (b) *Relative phases of the array. There is a wave traveling from the southwest corner to the northeast corner.*

is nearly a straight line so that the relative phases (which are the "integral" of the phase differences) are quadratic. Since the results of this section show that the array behaves like two orthogonal chains, it is now clear why the relative phases in the square array have apparently circular contours; the relative phase along any axes of the array are nearly quadratic. This is actually an artifact of the chain size. For, as n increases, Figure 3.4(b) shows that the phase differences become piecewise constant and so the relative phases will be linear and, in the array, will look like Figure 3.1. This is also what is predicted by the continuum theory in [5]. However, due to the small size of the cosine coefficient, n must be very large before there is qualitative similarity to the continuum approximation.

3.1. Some remarks on the stability of the patterns. In one-dimensional chains with "synaptic coupling" the traveling wave solutions described in section 2 appear to be the only stable solutions. That is, no matter what the initial conditions, solutions converge to the monotone solutions described in section 2. On the other hand, if the one-dimensional chain has a ring geometry so that the two ends are

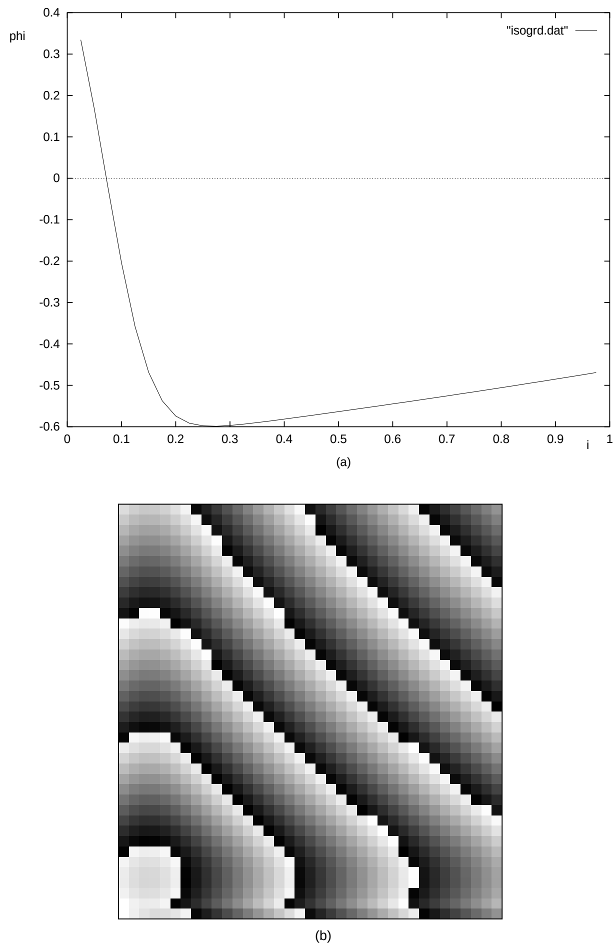


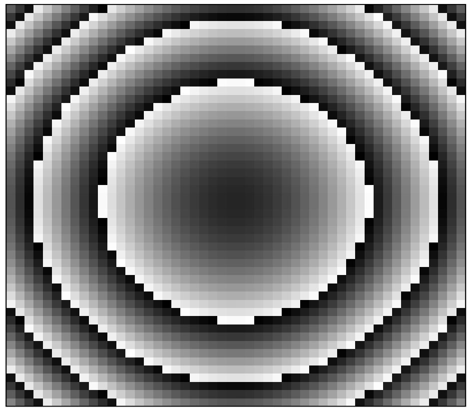
FIG. 3.3. $n + 1 = 40$. (a) *Phase lags of the reduced chains.* (b) *Relative phases of the array.*

identified, then, there are several stable solutions that correspond to synchrony and traveling waves. Thus, the domain of attraction of any given solution varies and does not constitute the entire phase space. In particular, the larger the chain, the more different types of stable solutions are possible.

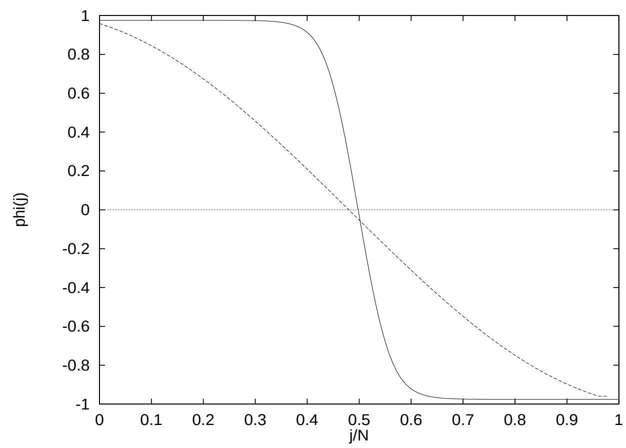
In two-dimensional systems, everything gets worse; there are many stable phase-locked patterns possible and a characterization of all of them remains a topic of current research. Finding domains of stability is even harder. Consider an $N \times N$ array where the coupling functions $H^{\pm X}, H^{\pm Y}$ are of the form

$$H(\phi) = \lambda \cos \phi + \sin \phi.$$

When $\lambda = 0$ one stable phaselocked solution is synchrony. As λ increases away from 0, the resulting phaselocked solution perturbs to the target-like patterns that we have discussed here. For $\lambda = 0$ Paullet and Ermentrout [9] have proven that there are also stable solutions analogous to spiral waves. Since these are stable, they persist for small λ and thus represent another phaselocked solution distinct from the target patterns described in this paper. Random initial data (rather than initial data identically 0)



(a)



(b)

FIG. 3.4. *Relative phases in an array with almost circular symmetry and their analogue in a chain.* (a) *Relative phase for a* 50×50 *array.* (b) *A chain of length* 50 *and* 500 *showing how the almost quadratic behavior of the phase shifts for* $n = 50$ *becomes the piecewise linear phases for* $n = 500$ *as is predicted by the continuum equations.*

converge on phaselocked solutions, but sometimes they are not targets but rather are related to the spiral patterns. For small arrays, random initial data converge mainly to the target patterns but on larger arrays (e.g., 40×40) the tendency is to converge to series of broken spiral-like patterns. Thus, target patterns are "homotopes" of synchrony and have essentially the same global stability behavior. They are not unique phaselocked patterns, unlike their analogue in one dimension.

Appendix. Proof of Theorem 2.9.

(i) We prove it by applying induction on $k \in N$. Let $k = 1$. Note that $|\beta|$ is small; we have $f(0) + \beta > 0$. Then

$$\begin{aligned} \phi_1'(0) &= f(0) + \beta > 0, \\ \phi_i'(0) &= \beta < 0, \quad i = 2, \dots, n-1, \\ \phi_n'(0) &= -f(0) + \beta < 0. \end{aligned} \tag{A.1}$$

Then by (2.13) and (A.1), we have

$$\begin{aligned} \phi_2''(0) &= g'(0)f(0), \\ \phi_{n-1}''(0) &= -g'(0)f(0). \end{aligned} \tag{A.2}$$

By induction on i, we can get that for $i = 3, \ldots, m-1$

$$\begin{aligned} \phi_i'(0) = \beta, \phi_i^{(k)}(0) = 0, \quad k = 2, \ldots, i-1, \\ \phi_i^{(i)}(0) = [g'(0)]^{i-1} f(0) > 0, \\ \phi_{n-i+1}'(0) = \beta, \phi_{n-i+1}^{(k)}(0) = 0, \quad k = 2, \ldots, i-1, \\ \phi_{n-i+1}^{(i)}(0) = -[g'(0)]^{i-1} f(0) < 0, \end{aligned} \tag{A.3}$$

where $n = 2m-1$ or $n = 2m-2$ and

$$\phi_m'(0) = \beta, \phi_m''(0) = \cdots = \phi_m^{(m)}(0) = 0, \tag{A.4}$$

where $n = 2m - 1$.

Hence by (A.1)–(A.4) and the fact that $\phi_i(0) = 0, i = 1, \ldots, n$, one can apply the Taylor's formula to $\phi_i(t)$ and $\phi_i'(t)$. Then we have

$$\phi_1'(t) > 0, \phi_i'(t) < 0, \quad i = 2, \ldots, n \tag{A.5}$$

and

$$\phi_L > \phi_1(t) > \cdots > \phi_n(t) > \phi_\beta \tag{A.6}$$

in $(0, \delta)$ for $\delta > 0$ small enough. Therefore $t_1 = 0$ and $l_1 = 1$.

CLAIM 1. *From $t = 0$, as long as* (A.5) *holds, we always have* (A.6).

Suppose that there is some first place t^* such that $\phi_L = \phi_1(t^*) \geq \phi_2(t^*) \geq \cdots \geq \phi_n(t^*) \geq \phi_\beta$. Then

$$\begin{aligned} \phi_1'(t^*) &= \beta + f(\phi_2(t^*)) + g(\phi_2(t^*)) - 2g(\phi_L), \\ &= \beta + f(\phi_2(t^*)) + g(\phi_2(t^*)) - f(\phi_L) - g(\phi_L), \\ &= \beta + [f'(\xi) + g'(\xi)](\phi_2(t^*) - \phi_L), \\ &\leq \beta. \end{aligned}$$

This is a contradiction since $\phi_1'(t^*) > 0$.

Now suppose that there is a first place t^* such that for some $i \in \{1, \ldots, n-2\}$

$$\phi_L > \phi_1 > \phi_2 > \cdots > \phi_i = \phi_{i+1} \geq \cdots \geq \phi_{n-1} \geq \phi_n \geq \phi_\beta$$

at t^*. Then at this point t^*,

$$\begin{aligned} \phi_i' &= \beta + f(\phi_i) - f(\phi_{i-1}) + g(\phi_i) - 2g(\phi_i) + g(\phi_{i-1}) \\ &= \beta + f(\phi_i) - g(\phi_i) - f(\phi_{i-1}) + g(\phi_{i-1}) \\ &= \beta + [g' - f'](\xi_1)(\phi_{i-1} - \phi_i) \\ &> \beta \end{aligned}$$

and

$$\begin{aligned} \phi_{i+1}' &= \beta + f(\phi_{i+2}) - f(\phi_i) + g(\phi_{i+2}) - 2g(\phi_i) + g(\phi_i) \\ &= \beta + f(\phi_{i+2}) - f(\phi_i) + g(\phi_{i+2}) - g(\phi_i) \\ &= \beta + [g' - f'](\xi_2)(\phi_{i+2} - \phi_i) \\ &\leq \beta, \end{aligned}$$

so $\phi_i'(t^*) > \phi_{i+1}'(t^*)$. Therefore in a small neighborhood $(t^* - \delta, t^*)$ of t^* $(t^* > 0)$, we have $\phi_{i+1}(t) > \phi_i(t)$ since $\phi_i(t^*) = \phi_{i+1}(t^*)$. This leads to a contradiction.

Hence we can conclude that $\phi_L > \phi_1(t) > \phi_2(t) > \cdots > \phi_{n-1}(t) \geq \phi_n(t) \geq \phi_\beta$.

Suppose $\phi_L > \phi_1(t) > \phi_2(t) > \cdots > \phi_{n-1}(t) \geq \phi_n(t) = \phi_\beta$ at a first place t^*; then

$$\begin{aligned}\phi_n' &= \beta - f(\phi_{n-1}) + g(\phi_{n-1}) - 2g(\phi_\beta)\\ &= f(\phi_\beta) - g(\phi_\beta) - f(\phi_{n-1}) + g(\phi_{n-1})\\ &= [g' - f'](\xi)(\phi_{n-1} - \phi_\beta)\\ &\geq 0.\end{aligned}$$

This is a contradiction since we have $\phi_n'(t) < 0$ so far.

Hence $\phi_L > \phi_1(t) > \phi_2(t) > \cdots > \phi_{n-1}(t) \geq \phi_n(t) > \phi_\beta$.

Now suppose $\phi_L > \phi_1(t) > \phi_2(t) > \cdots > \phi_{n-1}(t) = \phi_n(t) > \phi_\beta$ at a first place t^*; then at t^*

$$\begin{aligned}\phi_n' &= \beta - f(\phi_n) + g(\phi_{n-1}) - 2g(\phi_n)\\ &= \beta - [f(\phi_{n-1}) + g(\phi_{n-1})]\end{aligned}$$

and

$$\begin{aligned}\phi_{n-1}' &= \beta + f(\phi_n) - f(\phi_{n-2}) + g(\phi_n) - 2g(\phi_{n-1}) + g(\phi_{n-2})\\ &= \beta + f(\phi_n) - f(\phi_{n-2}) - g(\phi_n) + g(\phi_{n-2})\\ &= \beta + [g' - f'](\xi_1)(\phi_{n-2} - \phi_n).\\ &> \beta.\end{aligned}$$

Since $\phi_{n-1}(t) \geq \phi_R$ for $t > 0$ by the assumption of the theorem,

$$\begin{aligned}f(\phi_{n-1}) + g(\phi_{n-1}) &= f(\phi_{n-1}) + g(\phi_{n-1}) - [f(\phi_R) + g(\phi_R)]\\ &= [f' + g'](\xi_2)(\phi_{n-1} - \phi_R)\\ &\geq 0\end{aligned}$$

at t^*. So $\phi_n'(t^*) < \phi_{n-1}'(t^*)$. Hence in a small neighborhood $(t^* - \delta, t^*)$ of t^*, we have $\phi_n(t) > \phi_{n-1}(t)$ which is a contradiction. Therefore Claim 1 is proven.

Suppose (A.5) breaks down at some first place $t_2 > 0$ (otherwise the proof is finished with $\hat{t} = +\infty$) and $\phi_i'(t_2) \neq 0$ for some $i \in \{1, \ldots, n\}$ (otherwise the proof is finished with $\hat{t} = t_2$). Then we have six cases to consider.

CASE 1. *There is some $l > 2$ such that $\phi_1'(t_2) \geq 0$, $\phi_{l-1}'(t_2) < 0$, $\phi_l'(t_2) = 0$, and $\phi_i'(t_2) \leq 0$ for $i \in \{2, \ldots, n\} - \{l-1, l\}$.*

CASE 2. *There is some $l \in \{3, \ldots, n-1\}$ such that $\phi_1'(t_2) \geq 0$, $\phi_i'(t_2) = 0$ for $i = 2, \ldots, l$ and $\phi_{l+1}'(t_2) < 0$, $l \in \{3, \ldots, n-1\}$.*

CASE 3. *$\phi_1'(t_2) = \phi_2'(t_2) = 0$ and $\phi_3'(t_2) < 0$, $\phi_i'(t_2) \leq 0$, $i = 4, \ldots, n$.*

CASE 4. *$\phi_1'(t_2) = 0$ and $\phi_i'(t_2) < 0$, $i = 2, \ldots, n$.*

CASE 5. *$\phi_1'(t_2) > 0$, $\phi_2'(t_2) = 0$ and $\phi_i'(t_2) < 0$, $i = 3, \ldots, n$.*

CASE 6. *$\phi_1'(t_2) > 0$ and $\phi_i'(t_2) = 0$, $i = 2, \ldots, n$.*

Assume Case 1 is true. Then we have

$$\begin{aligned}\phi_l'(t_2 - \varepsilon) = \beta &+ f(\phi_{l+1}(t_2 - \varepsilon)) - f(\phi_{l-1}(t_2 - \varepsilon))\\ &+ g(\phi_{l+1}(t_2 - \varepsilon)) - 2g(\phi_l(t_2 - \varepsilon)) + g(\phi_{l-1}(t_2 - \varepsilon))\end{aligned}$$

$$\begin{aligned}
&= \phi_l'(t_2) - f'(\phi_{l+1}(t_2))\phi_{l+1}'(t_2)\varepsilon + f'(\phi_{l-1}(t_2))\phi_{l-1}'(t_2)\varepsilon \\
&\quad - g'(\phi_{l+1}(t_2))\phi_{l+1}'(t_2)\varepsilon + 2g'(\phi_l(t_2))\phi_l'(t_2)\varepsilon \\
&\quad - g'(\phi_{l-1}(t_2))\phi_{l-1}'(t_2)\varepsilon + o(\varepsilon^2) \\
&= -\phi_{l+1}'(t_2)[g' + f'](\phi_{l+1}(t_2))\varepsilon \\
&\quad - \phi_{l-1}'(t_2)[g' - f'](\phi_{l-1}(t_2))\varepsilon + o(\varepsilon^2) \\
&> 0
\end{aligned}$$

for $\varepsilon > 0$ small enough (since $g' \pm f' > 0$ in J). This is a contradiction! So *Case* 1 *is eliminated in our concern.*

Assume Case 2 is true. Then

$$\begin{aligned}
\phi_l'(t_2 - \varepsilon) &= -\phi_{l+1}'(t_2)[g' + f'](\phi_{l+1}(t_2))\varepsilon + o(\varepsilon^2) \\
&> 0
\end{aligned}$$

for $\varepsilon > 0$ small enough. This is a contradiction! So *Case* 2 *is also eliminated.*

Case 3 *can be eliminated in the same way as Case* 2.

Hence we have Cases 4–6 left.

If case 4 is true, then

$$\begin{aligned}
\phi_1'(t_2 + \varepsilon) &= \phi_2'(t_2)[g' + f'](\phi_2(t_2))\varepsilon + o(\varepsilon^2) \\
&< 0
\end{aligned}$$

for small $\varepsilon > 0$. Then $l_2 = 0$ such that $l_2 = l_1 - 1$.

If Case 5 is true, then we have that for small $\varepsilon > 0$, either $\phi_2'(t) > 0$ in $(t_2, t_2 + \varepsilon)$ or $\phi_2'(t) < 0$ in $(t_2, t_2 + \varepsilon)$ (note that $\phi_2'(t) \equiv 0$ in $(t_2, t_2 + \varepsilon)$ cannot be true). Hence $l_2 = 2$, i.e., $l_2 = l_1 + 1$ or $l_2 = 1$, i.e., $l_2 = l_1$.

If Case 6 is true, then we can show that

$$\begin{aligned}
&\phi_2''(t_2) > 0, \\
&\phi_i^{(j)}(t_2) \;=\; 0, \quad j = 2, \ldots, i-1, \\
&\phi_i^{(i)}(t_2) \;>\; 0, \quad i = 3, \ldots, n
\end{aligned}$$

such that $\phi_i'(t) > 0$ $(i = 2, \ldots, n)$ in $(t_2, t_2 + \varepsilon)$ for ε small enough. Then $l_2 = n$.

And for Cases 4–6, we can prove by using the same techniques as above that

$$\phi_L > \phi_1(t_2) > \cdots > \phi_n(t_2) > \phi_\beta.$$

So we are done with $k = 1$.

Now suppose (2.16)–(2.19) hold for $1, 2, \ldots, k-1$ with $l_1, \ldots, l_k$, and $t_1 < t_2 < \ldots < t_k$.

Then for $t \in (t_k, t_k + \delta)$ ($\delta > 0$ is small)

$$\begin{aligned}
&\phi_i'(t) > 0, \quad i = 1, \ldots, l_k, \\
&\phi_j'(t) < 0, \quad j = l_k + 1, \ldots, n.
\end{aligned}$$

CLAIM 2. *From t_k, as long as*

$$\begin{aligned}
&\phi_i'(t) > 0, \quad i = 1, \ldots, l_k, \\
&\phi_j'(t) < 0, \quad j = l_k + 1, \ldots, n,
\end{aligned} \tag{A.7}$$

we always have (A.6).

The proof is similar to Claim 1; we just ignore it here.

Suppose (A.7) breaks down at a first place $t_{k+1} > t_k$ and $\phi_i'(t_{k+1}) \neq 0$ for some $i \in \{1, \ldots, n\}$; then several cases should be considered carefully.

CASE 1. *There is* $l < l_k$ *such that* $\phi_l'(t_{k+1}) \leq 0$, $\phi_{l+1}'(t_{k+1}) > 0$, $\phi_i'(t_{k+1}) \geq 0$ *for* $i \in \{1, \ldots, l_k\} - \{l, l+1\}$, *and* $\phi_j'(t_{k+1}) = 0$ *for* $j = l_k + 1, \ldots, n$.

CASE 2. *There is* $l > l_k + 1$ *such that* $\phi_i'(t_{k+1}) \geq 0$, $i = 1, \ldots, l_k$, $\phi_{l-1}'(t_{k+1}) < 0$, $\phi_l'(t_{k+1}) = 0$, *and* $\phi_j'(t_{k+1}) \leq 0$ *for* $j \in \{l_k + 1, \ldots, n\} - \{l-1, l\}$.

CASE 3. *There is some* $l \in \{2, \ldots, l_k - 1\}$ *such that* $\phi_i'(t_{k+1}) \geq 0$ *for* $i \in \{1, \ldots, l-2\}$, $\phi_{l-1}'(t_{k+1}) > 0, \phi_j'(t_{k+1}) = 0$ *for* $j \in \{l, \ldots, l_k\}$, *and* $\phi_j'(t_{k+1}) \leq 0$ *for* $j \in \{l_k + 1, \ldots, n\}$.

CASE 4. *There is* $l \in \{l_k + 2, \ldots, n-1\}$ *such that* $\phi_j'(t_{k+1}) \geq 0$ *for* $i \in \{1, \ldots, l_k\}$, $\phi_j'(t_{k+1}) = 0$ *for* $j \in \{l_k + 1, \ldots, l\}$, $\phi_{l+1}'(t_{k+1}) < 0$, *and* $\phi_j'(t_{k+1}) \leq 0$ *for* $j \in \{l + 2, \ldots, n\}$.

CASE 5. $\phi_i'(t_{k+1}) \geq 0$ *for* $i \in \{1, \ldots, l_k - 2\}$, $\phi_{l_k - 1}'(t_{k+1}) > 0, \phi_{l_k}'(t_{k+1}) = \phi_{l_k+1}'(t_{k+1}) = 0$, *and* $\phi_i'(t_{k+1}) \leq 0$ *for* $i \in \{l_k + 2, \ldots, n\}$.

CASE 6. $\phi_i'(t_{k+1}) \geq 0$, $i \in \{1, \ldots, l_k - 1\}$, $\phi_{l_k}'(t_{k+1}) = \phi_{l_k+1}'(t_{k+1}) = 0$, $\phi_{l_k+2}'(t_{k+1}) < 0$, $\phi_j'(t_{k+1}) \leq 0$ *for* $j \in \{l_k + 3, \ldots, n\}$.

CASE 7. $\phi_i'(t_{k+1}) > 0$ *for* $i \in \{1, \ldots, l_k - 1\}$, $\phi_{l_k}'(t_{k+1}) = 0$, $\phi_j'(t_{k+1}) < 0$ *for* $j \in \{l_k + 1, \ldots, n\}$.

CASE 8. $\phi_i'(t_{k+1}) > 0$ *for* $i \in \{1, \ldots, l_k\}$, $\phi_{l_k+1}'(t_{k+1}) = 0$, *and* $\phi_j'(t_{k+1}) < 0$ *for* $j \in \{l_k + 2, \ldots, n\}$.

CASE 9. $\phi_i'(t_{k+1}) = 0$ *for* $i \in \{1, \ldots, l_k\}$, *and* $\phi_j'(t_{k+1}) < 0$ *for* $j \in \{l_k + 1, \ldots, n\}$.

CASE 10. $\phi_i'(t_{k+1}) > 0$ *for* $i \in \{1, \ldots, l_k\}$, *and* $\phi_j'(t_{k+1}) = 0$ *for* $j \in \{l_k + 1, \ldots, n\}$.

By the techniques which we used in the case of $k = 1$, Cases 1–6 can be eliminated. Hence only Cases 7–10 are possible.

If Case 7 is true, then we have that for $\varepsilon > 0$ small enough, either $\phi_{l_k}'(t) > 0$ in $(t_{k+1}, t_{k+1} + \varepsilon)$ or $\phi_{l_k}' < 0$ in $(t_{k+1}, t_{k+1} + \varepsilon)$. Then $l_{k+1} = l_k$ or $l_{k+1} = l_k - 1$.

If Case 8 is true, then for $\varepsilon > 0$ small enough, either $\phi_{l_k+1}'(t) > 0$ in $(t_{k+1}, t_{k+1} + \varepsilon)$ or $\phi_{l_k+1}' < 0$ in $(t_{k+1}, t_{k+1} + \varepsilon)$. Then $l_{k+1} = l_k + 1$ or $l_{k+1} = l_k$.

If Case 9 is true, then for $\varepsilon > 0$ small enough, we can prove that $\phi_i'(t) < 0$ in $(t_{k+1}, t_{k+1} + \varepsilon)$ for $i = 1, \ldots, n$. Then $l_{k+1} = 0$.

If Case 10 is true, then for $\varepsilon > 0$ small enough, we can prove that $\phi_i'(t) > 0, i = 1, \ldots, n$; then $l_{k+1} = n$.

And for Cases 7–10, we can show that $\phi_L > \phi_1(t_{k+1}) > \cdots > \phi_n(t_{k+1}) > \phi_\beta$ always holds. Hence the proof is completed for this part.

(ii).

CLAIM 3. *Both* $\phi_1'(t)$ *and* $\phi_n'(t)$ *can change sign at most once. And if* $\phi_1'(t)$ *changes sign once,* $\phi_n'(t)$ *never changes sign. If* $\phi_n'(t)$ *changes sign once,* $\phi_1'(t)$ *never changes sign. That is,*

(a) *if* $\phi_i'(t) < 0$, $i = 1, \ldots, n$ *for* $t \in (t_k, t_k + \varepsilon)$, *then* $\phi_i'(t) < 0$, $i = 1, \ldots, n$ *for* $t \in (t_k, \hat{t})$;

(b) *if* $\phi_i'(t) > 0$, $i = 1, \ldots, n$ *for* $t \in (t_k, t_k + \varepsilon)$, *then* $\phi_i'(t) > 0$, $i = 1, \ldots, n$ *for* $t \in (t_k, \hat{t})$.

Claim 3 can be shown by contradiction. We ignore the proof here.

Hence by Claim 3, without loss of generality, we assume $\phi_1'(t)$ never changes sign; then we always have that $\phi_1'(t) > 0$ for $0 < t < \hat{t}$. So $\phi_1(t)$ increases as t increases. Since $\phi_L > \phi_1(t) > \cdots > \phi_n(t) > \phi_\beta$, $\phi_1(t)$ is bounded above by ϕ_L such that

$\lim_{t\to\hat{t}} \phi_1(t) = \bar{\phi}_1$ for some $\bar{\phi}_1 \in [\phi_\beta, \phi_L]$. Also we have $\lim_{t\to\hat{t}} \phi_1'(t) = 0$ such that

$$0 = \beta + \lim_{t\to\hat{t}}[f+g](\phi_2(t)) - 2g(\bar{\phi}_1).$$

Then $\lim_{t\to\hat{t}} \phi_2(t) = \lim_{t\to\hat{t}}[f+g]^{-1}[f+g](\phi_2(t))$ exists. Let $\bar{\phi}_2 = \lim_{t\to\hat{t}} \phi_2(t)$. By the boundedness, $\phi_L \geq \bar{\phi}_2 \geq \phi_\beta$.

By induction we can show $\lim_{t\to\hat{t}} \phi_i(t) = \bar{\phi}_i$, where $\bar{\phi}_i \in [\phi_\beta, \phi_L]$. Since $\phi_L > \phi_1(t) > \cdots > \phi_n(t) > \phi_\beta$ for $t > 0$, then $\phi_L \geq \bar{\phi}_1 \geq \cdots \geq \bar{\phi}_n \geq \phi_\beta$.

If we apply the argument in the proof of Theorem 2.3, we also can show $\phi_L > \bar{\phi}_1 > \cdots > \bar{\phi}_n > \phi_\beta$.

Remark. If we recall the proof of Claims 1 and 2, we need to assume $\phi_{n-1} \geq \phi_R$ along the trajectory. If this condition breaks down somewhere, the monotonicity may be destroyed.

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