Boolean Algebra

Boolean Algebra

- In 1849 George Boole introduced a scheme for describing logical processes, now called **Boolean Algebra**
- In 1904 Edward V. Huntington formulated postulates that refined and formalized Boolean Algebra
- In the 1930's, Claude Shannon showed that Boolean Algebra could be used to analyze circuits built with switches and that they can be designed in terms of logic gates
- Boolean Algebra is the foundation of modern digital technology

Boolean Algebra

- In Boolean Algebra, logical statements are built up from:
  - **Variables:**
  - **Operators:**
- Boolean Algebra is a two-valued system, in which all variables take values on the set {0,1} and the operators (+ , ●, ‘) correspond to (OR, AND, NOT) respectively

Axioms of Boolean Algebra

- Boolean Algebra is based on a set of rules that are derived from a small number of basic assumptions called **axioms**

1a) $0 \cdot 0 =$
1b) $1 + 1 =$

2a) $1 \cdot 1 =$
2b) $0 + 0 =$

3a) $0 \cdot 1 = 1 \cdot 0 =$
3b) $1 + 0 = 0 + 1 =$

4a) $\mathrm{If\ x} = 0, then\ \bar{x} =$
4b) $\mathrm{If\ x} = 1, then\ \bar{x} =$

Huntington's Postulates

- Boolean Algebra is defined by **Huntington's postulates**. The 6 basic postulates that must be satisfied are:
  - **Closure** with respect to the operators:
  - **Identity** elements with respect to the operators:
  - **Commutativity** with respect to the operators:
  - **Distributivity**:
  - **Complements** exist for all the elements:
  - **Distinct Elements**:

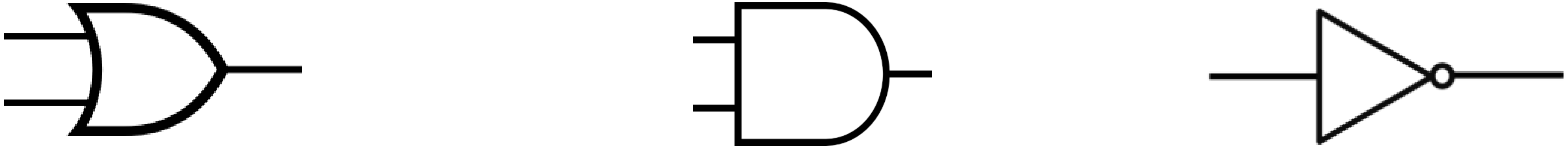
Application to Digital Circuits

- In 1938, Claude Shannon showed that a two-valued Boolean Algebra, which he called **switching algebra**, could be used to describe digital circuits:
  - Two elements: {0,1}
  - Two binary operators : + (OR) , ● AND
  - One unary operator, ‘ (NOT)

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

x	y	$x \bullet y$
0	0	0
0	1	0
1	0	0
1	1	1

x	x'
0	1
1	0



Is this really a Boolean Algebra? Does it satisfy Huntington's Postulates?

- **Closure** with respect to +, ●:
- **Identity** elements with respect to +, ●:
- **Commutativity** with respect to +, ●:
- **Distributivity**:
- **Complements** exist for all the elements:
- **Distinct Elements**:

Proofs: What are they?

- **Proofs** are a tool for establishing new results or theorems in algebra. The kinds of things we need to prove usually fall into one of several categories.

OK, so prove it! Hmmm, how do we do that?

- Here are four ways to **prove** that two expressions are equivalent
  - **Perfect Induction:**
  - **Axiomatic Proof:**
  - **Duality Principle:**
  - **Proof by contradiction:**

Example: Proof of Distributivity of ● over +, by Perfect Induction.

x	y	z	$y + z$	$x \bullet (y + z)$	$x \bullet y$	$x \bullet z$	$(x \bullet y) + (x \bullet z)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

Example: Proof of Distributivity of + over ●, by Perfect Induction.

x	y	z	$y \bullet z$	$x + (y \bullet z)$	$x + y$	$x + z$	$(x + y) \bullet (x + z)$
0	0	0					
0	0	1					
0	1	0					
0	1	1					
1	0	0					
1	0	1					
1	1	0					
1	1	1					

OK, where are we so far?

- So far, we have only shown (through 1 rigorous proof and several less formal arguments) that Huntington's postulates are satisfied. Therefore, the two valued system with the operations **(●, +, ')** as defined above is a Boolean Algebra

Why do we care?

- Armed with these 6 postulates, we can now go on to establish other theorems that will help us analyze logic circuits

Basic Theorems of Boolean Algebra

- From axioms, we can define rules for dealing with variables, call **theorems.** For the Boolean variables x,y,z the following theorems hold

Theorem 1 (Idempotency)

a) $x + x =$

b) $x \cdot x =$

Theorem 2 (Tautology and Contradiction)

a) $x + 1 =$

b) $x \cdot 0 =$

Theorem 3 (Involution)

$\bar{\bar{x}} =$

Theorem 4 (Associativity)

a) $(x + y) + z = x + (y + z)$

b) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Theorem 5: DeMorgan's Law

DeMorgan's Law (a) $\overline{(x + y)} = \bar{x} \cdot \bar{y}$

Proof: By Perfect Induction

x	y	$(x+y)$	$\overline{(x+y)}$	$\overline{x}$	$\overline{y}$	$\overline{x} \bullet \overline{y}$
0	0					
0	1					
1	0					
1	1					

DeMorgan's Law (b) $\overline{(x \cdot y)} = \bar{x} + \bar{y}$

Proof:

Theorem 6 (Absorption)

Absorption a) $x + (x \cdot y) = x$

Axiomatic Proof:

Absorption b) $x \cdot (x + y) = x$

Proof:

Other Useful Theorems

Theorem 7 (Common Identities)

a) $x + (\bar{x} \cdot y) = x + y$

b) $x \cdot (\bar{x} + y) = x \cdot y$

Theorem 8 (Consensus)

a) $(x \cdot y) + (y \cdot z) + (\bar{x} \cdot z) = (x \cdot y) + (\bar{x} \cdot z)$

b) $(x + y) \cdot (y + z) \cdot (\bar{x} + z) = (x + y) \cdot (\bar{x} + z)$

Try to prove these theorems on your own. Note that each pair of equations are related through the duality principle

So now we have all these theorems. What good are they?

Example 1: Find the compliment of the logical function *f*

$$f = \bar{x} \cdot (\bar{y} + \bar{z}) \cdot (x + y + \bar{z})$$

Solution:

So now we have all these theorems. What good are they?

Example 2: Simplify the previous expression

$$\bar{f} = x + (y \cdot z) + (\bar{x} \cdot \bar{y} \cdot z)$$

Solution:

Precedence of Boolean Operators

To avoid confusion or incorrect evaluation, the operators in Boolean expressions are applied according to the following order of precedence:

1.
2.
3.
4.

Also, it is conventional to omit the ● symbol for AND where convenient. As a result, many Boolean expressions can be written in a compact form, often eliminating extraneous parentheses.

Reference Sheet: Boolean Algebra Postulates and Theorems

Postulate 1: Closure

Postulate 2: Identity

a) $x + 0 = x$
b) $x \cdot 1 = x$

Postulate 3: Commutativity

a) $x + y = y + x$
b) $x \cdot y = y \cdot x$

Postulate 4: Distributivity

a) $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$
b) $x + (y \cdot z) = (x + y) \cdot (x + z)$

Postulate 5: Complement

a) $x + \bar{x} = 1$
b) $x \cdot \bar{x} = 0$

Theorem 1 (Idempotency)

a) $x + x = x$
b) $x \cdot x = x$

Theorem 2 (Tautology and Contradiction)

a) $x + 1 = 1$
b) $x \cdot 0 = 0$

Theorem 3 (Involution)

$$\bar{\bar{x}} = x$$

Theorem 4 (Associativity)

a) $(x + y) + z = x + (y + z)$
b) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

Theorem 5 (DeMorgan's Law)

a) $\overline{(x + y)} = \bar{x} \cdot \bar{y}$
b) $\overline{(x \cdot y)} = \bar{x} + \bar{y}$

Theorem 6 (Absorption)

a) $x + (x \cdot y) = x$
b) $x \cdot (x + y) = x$

Theorem 7 (Common Identities)

a) $x + (\bar{x} \cdot y) = x + y$
b) $x \cdot (\bar{x} + y) = x \cdot y$

Theorem 8 (Consensus)

a) $(x \cdot y) + (y \cdot z) + (\bar{x} \cdot z) = (x \cdot y) + (\bar{x} \cdot z)$
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