$\mathbb{L}^p$ Solutions of Reflected Backward Stochastic Differential Equations with Jumps

Song Yao*

Abstract

Given $p \in (1,2)$, we study $\mathbb{L}^p-$solutions of a reflected backward stochastic differential equation with jumps (RBSDEJ) whose generator g is Lipschitz continuous in (y,z,u). Based on a general comparison theorem as well as the optimal stopping theory for uniformly integrable processes under jump filtration, we show that such a RBSDEJ with $p-$integrable parameters admits a unique $\mathbb{L}^p$ solution via a fixed-point argument. The $Y-$component of the unique $\mathbb{L}^p$ solution can be viewed as the Snell envelope of the reflecting obstacle $\mathfrak{L}$ under $g-$evaluations, and the first time Y meets $\mathfrak{L}$ is an optimal stopping time for maximizing the $g-$evaluation of reward $\mathfrak{L}$.

Keywords: Reflected backward stochastic differential equations with jumps, $\mathbb{L}^p$ solutions, comparison theorem, optimal stopping, Snell envelope, Doob-Meyer decomposition, martingale representation theorem, fixed-point argument, $g-$evaluations.

1 Introduction

Let $p \in (1,2)$ and $T \in (0,\infty)$. In this paper, we study $\mathbb{L}^p$ solutions of a reflected backward stochastic differential equation with jumps (RBSDEJ)

$$\begin{cases} \mathfrak{L}_t \le Y_t = \xi + \displaystyle\int_t^T g(s,Y_s,Z_s,U_s)ds + K_T - K_t - \int_t^T Z_s\,dB_s - \int_{(t,T]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t \in [0,T], \\ \displaystyle\int_0^T (Y_{t-} - \mathfrak{L}_{t-})dK_t = 0 \quad (\textit{flat-off}\ \text{condition}) \end{cases} \tag{1.1}$$

over a probability space $(\Omega,\mathcal{F},P)$ on which B is a Brownian motion and $\mathfrak{p}$ is an $\mathcal{X}-$valued Poisson point process independent of B. Practically speaking, if the Brownian motion represents the noise from the financial market, then the Poisson random measure can be viewed as the randomness of insurance claims. In RBSDEJ (1.1) with generator g, terminal data ξ and obstacle $\mathfrak{L}$, a solution consists of an adapted càdlàg process Y, a locally square-integrable predictable process Z, a locally $p-$integrable predictable random fields U and an adapted càdlàg increasing process K. The role of the increasing process K is to keep Y stay above the obstacle $\mathfrak{L}$ at the minimal effort: i.e., Only when Y tends to drop below $\mathfrak{L}$, K will generates an upward momentum.

When the generator g is Lipschitz in (y,z,u) and the obstacle $\mathfrak{L}$ is a $p-$integrable adapted càdlàg process, we mainly demonstrate that for any $p-$integrable terminal data ξ, the RBSDEJ (1.1) admits a unique $\mathbb{L}^p-$solution (see Theorem 3.1).

The backward stochastic differential equation (BSDE) was introduced by Bismut [4] and later systematically developed by Pardoux and Peng [32] to a fully nonlinear version. Since then, the theory of BSDEs has grown rapidly and has been applied to various areas such as mathematical finance, theoretical economics, stochastic control and optimization, partial differential equations, differential geometry and etc (see the references in [16, 9]).

As a variation of BSDEs, the reflected BSDEs (RBSDEs) are closely related to the theory of optimal stopping: By utilizing the martingale characterization of the *Snell* envelope of the reward process in an optimal stopping problem, El Karoui et al. [14] deduced that the Snell envelope is exactly the $Y-$solution of a RBSDE with null generator. These authors then used either fixed-point argument or penalization method to show the existence and uniqueness of a $\mathbb{L}^2-$solution to a RBSDE with Lipschitz generator and square-integrable terminal data.

*Department of Mathematics, University of Pittsburgh, Pittsburgh, PA 15260; email: songyao@pitt.edu.

Main Contributions

Given $U \in \mathbb{U}^2_{\text{loc}}$, unlike the case of Brownian stochastic integrals, the Burkholder-Davis-Gundy inequality is not applicable for the $p/2-$th power of the Poisson stochastic integral $\int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$ (see e.g. Theorem VII.92 of [11]): i.e. $E\Big[\sup\limits_{t\in[0,T]}\Big(\int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\Big)^{\frac{p}{2}}\Big]$ can not be dominated by $E\Big[\Big(\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{4}}\Big]$. So to derive *a priori* $\mathbb{L}^p$ estimates for BSDEJs and RBSDEJs, we could not follow the classical argument in the proof of [6, Proposition 3.2], neither could we employ the space $\mathbb{U}^{2,p} := \Big\{U : E\Big[\Big(\int_0^T\int_{\mathcal{X}}|U_t(x)|^2\nu(dx)dt\Big)^{\frac{p}{2}}\Big] < \infty\Big\}$ or the space $\widetilde{\mathbb{U}}^{2,p} := \Big\{U : E\Big[\Big(\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{2}}\Big] < \infty\Big\}$ $\Big($Actually, one may not be able to compare $E\Big[\Big(\int_{(0,T]}\int_{\mathcal{X}}|U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{2}}\Big]$ with $E\Big[\Big(\int_0^T\int_{\mathcal{X}}|U_t(x)|^2\nu(dx)dt\Big)^{\frac{p}{2}}\Big]\Big)$.

To address these technical difficulties, we first generalize in [42] the Poisson stochastic integral for a random field $U\in\mathbb{U}^p$ by constructing a càdlàg uniformly integrable martingale $M^U_t := \int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$, whose quadratic variation $[M^U, M^U]$ is still $\int_{(0,t]}\int_{\mathcal{X}}|U_s(x)|^2 N_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$. In deriving the key $\mathbb{L}^p-$type inequality (see Lemma 3.1 of [42]) about the difference $Y = Y^1 - Y^2$ of two $p-$integrable solutions to BSDEJs with different parameters, the estimation of the variational jump part

$$\sum_s\Big(|Y_s|^p - |Y_{s-}|^p - p\big\langle|Y_{s-}|^{p-1}, \Delta Y_s\big\rangle\Big) \tag{1.2}$$

in the dynamics of $|Y|^p$ is full of analytical subtleties. By exploiting some new methods and techniques, we managed to boil down the expectation of (1.2) to the term $E\int_0^T\int_{\mathcal{X}}|U^1_t(x) - U^2_t(x)|^p\nu(dx)dt$, which justifies our choice of $\mathbb{U}^p$ over $\mathbb{U}^{2,p}$ or $\widetilde{\mathbb{U}}^{2,p}$ as the space for jump diffusion $\big($see (5.11)$-$(5.21) of [42] for details$\big)$. These method and techniques still play important roles in the main result of the present paper (see Part 2 in the proof of Theorem 3.1).

In the present paper, we start with a comparison result for generalized BSDEJs (Proposition 3.1), which directly leads to the uniqueness of $\mathbb{L}^p-$solutions to a RBSDEJ. For the existence result, we first generalize the optimal stopping theory for right-continuous reward processes of class (D) that may take unbounded negative values. Since the classic method ([13] or Appendix D of [26]) heavily relies on the non-negativity of the Snell envelope $\big($see four lines below (2.32.1) of [13] or line 16 on page 357 of [26]$\big)$, we take a different approach by proving a dynamic programming principle for the Snell envelope $\mathfrak{S}$ of a $p-$integrable càdlàg obstacle $\mathfrak{L}$ (Proposition A.3) and then deriving the martingale property of $\mathfrak{S}$ (Theorem A.1). In virtue of the Doob-Meyer decomposition, the Snell envelope $\mathfrak{S}$ can be decomposed as the difference between a $p-$integrable càdlàg martingale M and a $p-$integrable increasing càdlàg process K such that $\int_0^T(\mathfrak{S}_{t-} - \mathfrak{L}_{t-})dK_t = 0$, $P-$a.s. (see Proposition A.5 and A.6). Applying a generalized martingale representation theorem from [42] to M, we then show in Proposition 3.2 that a RBSDEJ with *simple* generator (generator that does not depend on (y,z,u)) and $p-$integrable parameters admits a unique $\mathbb{L}^p-$solution, which together with a fixed-point argument proves our main result (Theorem 3.1).

Based on our study [42] on $\mathbb{L}^p$ solutions of BSDEJs, we generalized in [41] the notion of (conditional) $g-$evaluations to the jump case with $\mathbb{L}^p$ domain, which are closely related to a large class of coherent/convex risk measures for $p-$integrable financial positions in a market with jumps. We can also derive from the comparison result (Proposition 3.1) that the $Y-$component of the unique $\mathbb{L}^p$ solution to a RBSDEJ with generator g is the Snell envelope of the obstacle $\mathfrak{L}$ under $g-$evaluations, and the first time Y meets $\mathfrak{L}$ is an optimal stopping time for maximizing the $g-$evaluation of reward $\mathfrak{L}$.

Relevant Literature

Li and Tang [40] introduced into the BSDE a jump term that is driven by a Poisson random measure independent of the Brownian motion. Hamadene and Ouknine [20] made a similar extension to RBSDEs, they showed that when the square-integrable obstacle has only inaccessible jump times, a RBSDE with Lipschitz generator and square-integrable terminal data admits a unique square-integrable solution. Then [1, 8, 18, 21, 38, 12] among others commenced an extensive study of RBSDEJs with càdlàg obstacles and the related optimal stopping under dynamic risk measures.

To match with the fact that linear BSDEs are well-posed for integrable terminal data: El Karoui et al. [16] studied the BSDE with $p-$integrable terminal data and showed that such a BSDE with Lipschitz generator admits a unique $\mathbb{L}^p-$solution. Later Briand et al. [5, 6] reduced the Lipschitz condition of the generator to a monotonicity condition, while some other researches [22, 2, 28, 29] were made on the wellposedness of RBSDEs with Lipschitz or monotonic generators in $\mathbb{L}^p$ sense.

The present paper analyzes RBSDEJs with Lipschitz generators in $\mathbb{L}^p$ sense, which generalizes [18, 21] for $\mathbb{L}^p-$solutions and also extends [22] to the jump case. Klimsiak [30] studied a similar problem of $\mathbb{L}^p-$solutions to reflected BSDEs under a general right-continuous filtration, however, they only obtained a wellposedness result for simple generators.

There are many recent developments on reflected BSDEs with jumps in various interesting directions: See e.g. [17] for a class of constrained BSDEJs which includes multi-dimensional BSDEs with oblique reflection and is thus closely related to the optimal switching problem; see [7] for RBSDEs and RBSDEJs with right upper-semicontinuous obstacles and the related optimal stopping; see [19] for a class of reflected BSDEs with non-positive jumps and upper obstacles, which provides a Feynman-Kac type formula for PDEs associated to general zero-sum controller-and-stopper games and etc.

The rest of the paper is organized as follows: After we list necessary notations in Section 1.1, Section 2 quotes some basic results for $\mathbb{L}^p$ solutions of BSDEJs with Lipschitz generator g and reviews the corresponding $g-$evaluations. In Section 3, we present the main results of our paper: the comparison theorem and the wellposedness result for $\mathbb{L}^p-$solutions to RBSDEJs with Lipschitz generators and $p-$terminal data. Section 4 discusses the optimal stopping under $g-$evaluations as an application of the comparison theorem. The proofs of our results are relegated to Section 5. We generalize the optimal stopping theory for the reward processes with unbounded negative values in the appendix.

1.1 Notation and Preliminaries

Throughout this paper, we fix a time horizon $T \in (0,\infty)$. Let $(\Omega,\mathcal{F},P)$ be a complete probability space on which a $d-$dimensional Brownian motion B is defined.

For a generic càdlàg process X, let us denote its corresponding jump process by $\Delta X_t := X_t - X_{t-}$, $t \in [0,T]$ with $X_{0-} := X_0$. Given a measurable space $(\mathcal{X},\mathcal{F}_{\mathcal{X}})$, let $\mathfrak{p}$ denote an $\mathcal{X}-$valued Poisson point process on $(\Omega,\mathcal{F},P)$ which is independent of B. For any scenario $\omega\in\Omega$, let $D_{\mathfrak{p}(\omega)}$ be the set of all jump times of the path $\mathfrak{p}(\omega)$, which is a countable subset of $(0,T]$ (see e.g. Section 1.9 of [23]). We assume that for some finite measure ν on $\big(\mathcal{X},\mathcal{F}_{\mathcal{X}}\big)$, the counting measure $N_{\mathfrak{p}}(dt,dx)$ of $\mathfrak{p}$ on $[0,T]\times\mathcal{X}$ has compensator $E\big[N_{\mathfrak{p}}(dt,dx)\big]=\nu(dx)dt$. The corresponding compensated Poisson random measure $\widetilde{N}_{\mathfrak{p}}$ will be denoted by $\widetilde{N}_{\mathfrak{p}}(dt,dx) := N_{\mathfrak{p}}(dt,dx) - \nu(dx)dt$.

For any $t\in[0,T]$, we define sigma-fields

$$\mathcal{F}^B_t := \sigma\big\{B_s; s\le t\big\}, \quad \mathcal{F}^N_t := \sigma\big\{N_{\mathfrak{p}}\big((0,s],A\big); s\le t, A\in\mathcal{F}_{\mathcal{X}}\big\}, \quad \mathcal{F}_t := \sigma\big(\mathcal{F}^B_t \cup \mathcal{F}^N_t\big)$$

and augment them by all $P-$null sets of $\mathcal{F}$. Clearly, the jump filtration $\mathbf{F}=\{\mathcal{F}_t\}_{t\in[0,T]}$ satisfies the *usual hypotheses* (cf. e.g., [36]). Denote by $\mathscr{P}$ $\big($resp. $\widehat{\mathscr{P}}\big)$ the $\mathbf{F}-$progressively measurable (resp. $\mathbf{F}-$predictable) sigma-field on $[0,T]\times\Omega$, and let $\mathcal{T}$ be the set of all $\mathbf{F}-$stopping times. For any $\gamma\in\mathcal{T}$, we set $\mathcal{T}_\gamma := \{\tau\in\mathcal{T} : \tau\ge\gamma,\ P\text{-a.s.}\}$.

Recall that a $\mathscr{B}\big([0,T]\big)\otimes\mathcal{F}-$measurable process X is of class (D) if $\{X_\tau\}_{\tau\in\mathcal{T}}$ is $P-$uniformly integrable. Also, we say that an adapted process X is *quasi left-continuous* if for any increasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}$, it holds $P-$a.s. that $\varlimsup\limits_{n\to\infty} E[X_{\tau_n}|\mathcal{F}_{\tau_1}] \le E[X_{\overline{\tau}}|\mathcal{F}_{\tau_1}]$, where $\overline{\tau} := \lim\limits_{n\to\infty}\uparrow \tau_n \in \mathcal{T}$. Clearly, a quasi left-continuous process X is *left upper semi-continuous in expectation* along stopping times (l.u.s.c.e.): for any increasing sequence $\{\tau_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}$, $\varlimsup\limits_{n\to\infty} E[X_{\tau_n}] \le E[X_{\overline{\tau}}]$.

For any $i\in\mathbb{N}$, as $\big\{N_{\mathfrak{p}}((0,t],\mathcal{X})\big\}_{t\in[0,T]}$ is an $\mathbf{F}-$adapted càdlàg process, its $i-$th jump time

$$\tau^N_i := \inf\big\{t\in(0,T] : N_{\mathfrak{p}}((0,t],\mathcal{X}) \ge i\big\}$$

is a totally inaccessible $\mathbf{F}-$stopping time (see e.g. the comment below Theorem III.4 of [36]). For each $\omega\in\Omega$, $\tau^N_i(\omega)$ is actually the $i-$th smallest element in $D_{\mathfrak{p}(\omega)}$ or T. Put in another way, $\tau^N_i(\omega) := \min\big\{t\in D_{\mathfrak{p}(\omega)} : t > \tau^N_{i-1}(\omega)\big\}\wedge T$, with $\tau^N_0(\omega) := 0$ and $\min\emptyset := \infty$.

The following spaces of functions will be used in the paper:

1) For any $p\in[1,\infty)$, let $L^p_+[0,T]$ be the space of all measurable functions $\psi\colon[0,T]\to[0,\infty)$ with $\int_0^T\big(\psi(t)\big)^p dt<\infty$.

2) For $p\in(1,\infty)$, let $L^p_\nu := L^p(\mathcal{X},\mathcal{F}_{\mathcal{X}},\nu;\mathbb{R})$ be the space of all real-valued, $\mathcal{F}_{\mathcal{X}}-$measurable functions u with $\|u\|_{L^p_\nu} := \big(\int_{\mathcal{X}}|u(x)|^p\nu(dx)\big)^{\frac{1}{p}}<\infty$. For any $u_1,u_2\in L^p_\nu$, we say $u_1=u_2$ if $u_1(x)=u_2(x)$ for $\nu-$a.s. $x\in\mathcal{X}$.

3) For any sub-sigma-field $\mathcal{G}$ of $\mathcal{F}$, let $L^0(\mathcal{G})$ be the space of all real-valued, $\mathcal{G}-$measurable random variables and set

• $L^p(\mathcal{G}) := \Big\{\xi \in L^0(\mathcal{G}) : \|\xi\|_{L^p(\mathcal{G})} := \big\{E\big[|\xi|^p\big]\big\}^{\frac{1}{p}} < \infty\Big\}$ for any $p\in(1,\infty)$;

• $L^\infty(\mathcal{G}) := \Big\{\xi \in L^0(\mathcal{G}) : \|\xi\|_{L^\infty(\mathcal{G})} := \underset{\omega\in\Omega}{\mathrm{esssup}}\,|\xi(\omega)| < \infty\Big\}$.

4) Let $\mathbb{D}^0$ be the space of all real-valued, $\mathbf{F}-$adapted càdlàg processes and set $\mathbb{V}^0 := \{X\in\mathbb{D}^0 : X$ is of finite variation$\}$, let $\mathbb{K}^0$ be a subspace of $\mathbb{V}^0$ that includes all $\mathbf{F}-$predictable càdlàg increasing processes X with $X_0=0$.

5) Set $\mathbb{Z}^2_{\mathrm{loc}} := L^2_{\mathrm{loc}}\big([0,T]\times\Omega, \widehat{\mathscr{P}}, dt\times dP;\mathbb{R}^d\big)$, the space of all $\mathbb{R}^d-$valued, $\mathbf{F}-$predictable processes Z with $\int_0^T|Z_t|^2\,dt < \infty$, $P-$a.s.

6) For any $p\in[1,2)$, we let

• $\mathbb{D}^p := \Big\{X\in\mathbb{D}^0 : \|X\|_{\mathbb{D}^p} := \big\{E[X_*^p]\big\}^{\frac{1}{p}} < \infty\Big\}$, where $X_* := \underset{t\in[0,T]}{\sup}|X_t| < \infty$.

• $\mathbb{K}^p := \mathbb{K}^0\cap\mathbb{D}^p = \big\{K\in\mathbb{K}^0 : E[K_T^p] < \infty\big\}$.

• $\mathbb{Z}^{2,p} := \Big\{Z\in\mathbb{Z}^2_{\mathrm{loc}} : \|Z\|_{\mathbb{Z}^{2,p}} := \Big\{E\Big[\big(\int_0^T|Z_t|^2\,dt\big)^{\frac{p}{2}}\Big]\Big\}^{\frac{1}{p}} < \infty\Big\}$. For any $Z\in\mathbb{Z}^{2,p}$, the Burkholder-Davis-Gundy inequality implies that

$$E\bigg[\sup_{t\in[0,T]}\Big|\int_0^t Z_s dB_s\Big|^p\bigg] \le C_p E\bigg[\Big(\int_0^T \big|Z_s\big|^2 ds\Big)^{\frac{p}{2}}\bigg] < \infty \tag{1.3}$$

for some constant $C_p>0$ depending on p. So $\big\{\int_0^t Z_s dB_s\big\}_{t\in[0,T]}$ is a uniformly integrable martingale.

• $\mathbb{U}^p_{\mathrm{loc}} := L^p_{\mathrm{loc}}\big([0,T]\times\Omega\times\mathcal{X}, \widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}, dt\times dP\times\nu(dx);\mathbb{R}\big)$ be the space of all $\widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}-$measurable random fields $U : [0,T]\times\Omega\times\mathcal{X}\to\mathbb{R}$ such that $\int_0^T\int_{\mathcal{X}}|U_t(x)|^p\nu(dx)dt = \int_0^T\|U_t\|^p_{L^p_\nu}dt < \infty$, $P-$a.s.

• $\mathbb{U}^p := \Big\{U\in\mathbb{U}^p_{\mathrm{loc}} : \|U\|_{\mathbb{U}^p} := \big\{E\int_0^T\int_{\mathcal{X}}|U_t(x)|^p\nu(dx)dt\big\}^{\frac{1}{p}} < \infty\Big\} = L^p\big([0,T]\times\Omega\times\mathcal{X}, \widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}, dt\times dP\times\nu(dx);\mathbb{R}\big)$.

Given $U\in\mathbb{U}^p_{\mathrm{loc}}$ $\big($resp. $\mathbb{U}^p\big)$, it is clear that $U(t,\omega)\in L^p_\nu$ for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$. According to Section 1.2 of [42], one can define a Poisson stochastic integral of U:

$$M_t^U := \int_{(0,t]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T], \tag{1.4}$$

which is a càdlàg local martingale (resp. uniformly integrable martingale) with quadratic variation $[M^U,M^U]_t = \int_{(0,t]}\int_{\mathcal{X}}|U_s(x)|^2 N_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$. The jump process of M^U is $\Delta M_t^U(\omega) = \mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}U\big(t,\omega,\mathfrak{p}_t(\omega)\big)$, $t\in(0,T]$. For any $U\in\mathbb{U}^p$, an analogy to (5.1) of [42] shows that

$$E\Big[\Big(\int_{(t,s]}\int_{\mathcal{X}}|U_t(x)|^2 N_{\mathfrak{p}}(dt,dx)\Big)^{\frac{p}{2}}\Big] \le E\int_t^s\int_{\mathcal{X}}\big|U_t(x)\big|^p\nu(dx)dt, \quad \forall\, 0\le t<s\le T. \tag{1.5}$$

• We simply denote $\mathbb{D}^p\times\mathbb{Z}^{2,p}\times\mathbb{U}^p$ by $\mathbb{S}^p$.

As usual, we set $x^+ := x\vee 0$ for any $x\in\mathbb{R}$, and use the convention $\inf\emptyset := \infty$. For any $p\in(0,\infty)$, the following inequality will be frequently applied in this paper: for any finite subset $\{a_1,\cdots,a_n\}$ of $(0,\infty)$

$$(1\wedge n^{p-1})\sum_{i=1}^n a_i^p \le \bigg(\sum_{i=1}^n a_i\bigg)^p \le (1\vee n^{p-1})\sum_{i=1}^n a_i^p\,. \tag{1.6}$$

Also, we let c_p denote a generic constant depending only on p (in particular, c_0 stands for a generic constant depending on nothing), whose form may vary from line to line.

2 BSDEs with Jumps and Related $g-$Evaluations

From now on, we fix $p\in(1,2)$ and set $q := \frac{p}{p-1}$.

A mapping $g : [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\to\mathbb{R}$ is called a $p-$*generator* if it is $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurable. For any $\tau\in\mathcal{T}$,

$$g_\tau(t,\omega,y,z,u) := \mathbf{1}_{\{t<\tau(\omega)\}}\,g(t,\omega,y,z,u), \quad \forall\,(t,\omega,y,z,u)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu$$

is also $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurable.

Definition 2.1. *Given $p\in(1,2)$, let $\xi\in L^0(\mathcal{F}_T)$ and g be a $p-$generator. A triplet $(Y,Z,U)\in\mathbb{D}^0\times\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$ is called a solution of a backward stochastic differential equation with jumps that has terminal data ξ and generator g $\big(BSDEJ(\xi,g)$ for short$\big)$ if $\int_0^T|g(s,Y_s,Z_s,U_s)|ds<\infty$, $P-$a.s. and if it holds $P-$a.s. that*

$$Y_t=\xi+\int_t^T g(s,Y_s,Z_s,U_s)ds-\int_t^T Z_s\,dB_s-\int_{(t,T]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T].$$

We shall make the following standard assumptions on $p-$generators g:

(A1) $\int_0^T|g(t,0,0,0)|dt\in L^p(\mathcal{F}_T)$.

(A2) There exist two $[0,\infty)-$valued, $\mathscr{B}[0,T]\otimes\mathcal{F}_T-$measurable processes β, Λ with $\int_0^T\big(\beta_t^q\vee\Lambda_t^2\big)dt\in L^\infty(\mathcal{F}_T)$ such that for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$

$$\big|g(t,\omega,y_1,z_1,u)-g(t,\omega,y_2,z_2,u)\big|\le\beta(t,\omega)|y_1-y_2|+\Lambda(t,\omega)|z_1-z_2|,\ \ \forall\,(y_1,z_1),\ (y_2,z_2)\in\mathbb{R}\times\mathbb{R}^d,\ \ \forall\,u\in L^p_\nu.$$

(A3) There exists a function $\mathfrak{h}\colon[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu\to L^q_\nu$ such that

(i) $\mathfrak{h}$ is $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}(L^p_\nu)\otimes\mathscr{B}(L^p_\nu)/\mathscr{B}(L^q_\nu)-$measurable;

(ii) There exist $\kappa_1\in(-1,0]$ and $\kappa_2\ge-\kappa_1$ such that for any $(t,\omega,y,z,u_1,u_2,x)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu\times\mathcal{X}$

$$\kappa_1\le\big(\mathfrak{h}(t,\omega,y,z,u_1,u_2)\big)(x)\le\kappa_2;$$

(iii) It holds for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$ that

$$g(t,\omega,y,z,u_1)-g(t,\omega,y,z,u_2)\le\int_{\mathcal{X}}\big(u_1(x)-u_2(x)\big)\cdot\big(\mathfrak{h}(t,\omega,y,z,u_1,u_2)\big)(x)\,\nu(dx),\ \ \forall\,(y,z,u_1,u_2)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu\times L^p_\nu.$$

Remark 2.1. *Let $p\in(1,2)$ and let g be a $p-$generator.*

(1) (A2) and (A3) imply

(A2') *There exist two $[0,\infty)-$valued, $\mathscr{B}[0,T]\otimes\mathcal{F}_T-$measurable processes β, Λ with $\int_0^T\big(\beta_t^q\vee\Lambda_t^2\big)dt\in L^\infty(\mathcal{F}_T)$ such that for $dt\times dP-$a.s. $(t,\omega)\in[0,T]\times\Omega$*

$$\big|g(t,\omega,y_1,z_1,u_1)-g(t,\omega,y_2,z_2,u_2)\big|\le\beta(t,\omega)\big(|y_1-y_2|+\|u_1-u_2\|_{L^p_\nu}\big)+\Lambda(t,\omega)|z_1-z_2|,\ \ \forall\,(y_i,z_i,u_i)\in\mathbb{R}\times\mathbb{R}^d\times L^p_\nu,\ i=1,2.$$

(2) If g satisfies (A2') and $\int_0^T|g(t,0,0,0)|dt<\infty$, $P-$a.s., then for any $(Y,Z,U)\in\mathbb{D}^1\times\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$, one has $\int_0^T|g(s,Y_s,Z_s,U_s)|ds<\infty$, $P-$a.s.

(3) We need the assumption "$\kappa_1>-1$" in (A3) (ii) for the comparison theorem of generalized BSDEJs (Proposition 3.1). Actually, it is necessary for the Doléans-Dade exponential $\mathscr{E}.(M)$ in (5.8) to be a strictly positive martingale (see e.g. [27]), which then allows us to apply Girsanov Theorem to change probability in the proof of Proposition 3.1.

We simply set $\widehat{C}:=\big\|\int_0^T(1\vee\beta_t^q\vee\Lambda_t^2)dt\big\|_{L^\infty(\mathcal{F}_T)}$. For $\mathbb{L}^p$ solutions of BSDEs with jumps, let us first quote a wellposedness result, the corresponding martingale representation theorem as well as a strict comparison theorem from Theorem 4.1, Corollary 4.1, Corollary 2.1, Lemma 3.1 of [42] and Theorem 2.2 of [41].

Theorem 2.1. *Given $p\in(1,2)$, Let g be a $p-$generator satisfying (A1) and (A2'). For any $\xi\in L^p(\mathcal{F}_T)$, the $BSDEJ(\xi,g)$ admits a unique solution $\big(Y^{\xi,g},Z^{\xi,g},U^{\xi,g}\big)\in\mathbb{S}^p$. In particular, for any $\tau\in\mathcal{T}$ and $\xi\in L^p(\mathcal{F}_\tau)$, the unique solution $\big(Y^{\xi,g_\tau},Z^{\xi,g_\tau},U^{\xi,g_\tau}\big)$ of the $BSDEJ(\xi,g_\tau)$ in $\mathbb{S}^p$ satisfies that $P\big\{Y_t^{\xi,g_\tau}=Y_{\tau\wedge t}^{\xi,g_\tau},\,t\in[0,T]\big\}=1$ and that $\big(Z_t^{\xi,g_\tau},U_t^{\xi,g_\tau}\big)=\mathbf{1}_{\{t\le\tau\}}\big(Z_t^{\xi,g_\tau},U_t^{\xi,g_\tau}\big)$, $dt\times dP-$a.s.*

Corollary 2.1. *Let $p\in(1,2)$. For any $\xi\in L^p(\mathcal{F}_T)$, there exists a unique pair $(Z,U)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$ such that $P-$a.s.*

$$E[\xi|\mathcal{F}_t]=E[\xi]+\int_0^t Z_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T].$$

Proposition 2.1. *Let $p\in(1,2)$. For $i=1,2$, let $\xi_i\in L^0(\mathcal{F}_T)$, g^i be a $p-$generator, and (Y^i,Z^i,U^i) be a solution of $BSDEJ\,(\xi_i,g^i)$ such that $Y^1-Y^2\in\mathbb{D}^p$. If g^i satisfies (A2'), then*

$$\big\|Y^1-Y^2\big\|^p_{\mathbb{D}^p}+\big\|Z^1-Z^2\big\|^p_{\mathbb{Z}^{2,p}}+\big\|U^1-U^2\big\|^p_{\mathbb{U}^p}\le\mathcal{C}E\bigg[|\xi_1-\xi_2|^p+\Big(\int_0^T\big|g^1(t,Y_t^2,Z_t^2,U_t^2)-g^2(t,Y_t^2,Z_t^2,U_t^2)\big|dt\Big)^p\bigg]\qquad(2.1)$$

for some constant $\mathcal{C}$ depending on T, $\nu(\mathcal{X})$, p and $\widehat{C}$.

Theorem 2.2. *Let $p\in(1,2)$, $\tau\in\mathcal{T}$ and $\gamma\in\mathcal{T}_\tau$. For $i=1,2$, let $\xi_i\in L^0(\mathcal{F}_T)$, let g^i be a $p-$generator, and let (Y^i,Z^i,U^i) be a solution of $BSDEJ(\xi_i,g^i)$ such that $Y^1_\gamma\le Y^2_\gamma$, $P-$a.s. and that $Y^1-Y^2\in\mathbb{D}^p$. For either $i=1$ or $i=2$, if g^i satisfies (A2), (A3), and if $g^1(t,Y^{3-i}_t,Z^{3-i}_t,U^{3-i}_t)\le g^2(t,Y^{3-i}_t,Z^{3-i}_t,U^{3-i}_t)$, $dt\times dP-$a.s. on $\rrbracket\tau,\gamma\llbracket$, then it holds $P-$a.s. that $Y^1_t\le Y^2_t$ for any $t\in[\tau,\gamma]$. If one further has $Y^1_\tau=Y^2_\tau$, $P-$a.s., then*
(i) it holds $P-$a.s. that $Y^1_t=Y^2_t$ for any $t\in[\tau,\gamma]$;
(ii) it holds $dt\times dP-$a.s. on $\rrbracket\tau,\gamma\rrbracket$ that $(Z^1_t,U^1_t)=(Z^2_t,U^2_t)$ and $g^1(t,Y^i_t,Z^i_t,U^i_t)=g^2(t,Y^i_t,Z^i_t,U^i_t)$, $i=1,2$.

The wellposedness result of BSDEs with jumps in $\mathbb{L}^p$ sense (Theorem 2.1) gives rise to a nonlinear expectation, called $g-$*evaluations* with $\mathbb{L}^p$ domains, which generalizes the one introduced in [33] and [34]:

Definition 2.2. *Given $p\in(1,2)$, let g be a $p-$generator satisfying (A1), (A2'), and let $\tau\in\mathcal{T}$, $\gamma\in\mathcal{T}_\tau$. Define $g-$evaluation $\mathcal{E}^g_{\tau,\gamma}: L^p(\mathcal{F}_\gamma)\to L^p(\mathcal{F}_\tau)$ by*

$$\mathcal{E}^g_{\tau,\gamma}[\xi]:=Y^{\xi,g_\gamma}_\tau,\quad \forall\,\xi\in L^p(\mathcal{F}_\gamma).$$

When $\gamma=T$, we call $\mathcal{E}_g[\xi|\mathcal{F}_\tau]:=\mathcal{E}^g_{\tau,T}[\xi]$ the (conditional) $g-$expectation of $\xi\in L^p(\mathcal{F}_T)$ at time τ. According to Lemma 3.1 of [41], if g further satisfies that $dt\times dP-$a.s.

$$g(t,y,0,0)=0,\quad \forall\,y\in\mathbb{R},\tag{2.2}$$

then it holds for any $\xi\in L^p(\mathcal{F}_\gamma)$ that $\mathcal{E}^g_{\tau,\gamma}[\xi]=\mathcal{E}_g[\xi|\mathcal{F}_\tau]$, $P-$a.s. In particular, when $g\equiv0$, the $g-$evaluation degenerates to the classic linear expectation: $\mathcal{E}^g_{\tau,\gamma}[\xi]=E[\xi|\mathcal{F}_\tau]$, $P-$a.s. for any $\xi\in L^p(\mathcal{F}_\gamma)$.

Let $p\in(1,2)$ and let g be a $p-$generator satisfying (A1) and (A2'). We know from [41] that the $g-$evaluations with $\mathbb{L}^p$ domains possess the following basic properties (cf. [35]): Let $\tau\in\mathcal{T}$, $\gamma\in\mathcal{T}_\tau$ and $\xi\in L^p(\mathcal{F}_\gamma)$.
(g1) "Strict Monotonicity": If g further satisfies (A3), then for any $\eta\in L^p(\mathcal{F}_\gamma)$ with $\xi\le\eta$, $P-$a.s. one has $\mathcal{E}^g_{\tau,\gamma}[\xi]\le\mathcal{E}^g_{\tau,\gamma}[\eta]$, $P-$a.s.; Moreover, if it further holds that $\mathcal{E}^g_{\tau,\gamma}[\xi]=\mathcal{E}^g_{\tau,\gamma}[\eta]$, $P-$a.s., then $\xi=\eta$, $P-$a.s.
(g2) "Constant Preserving": Under (2.2), if ξ is $\mathcal{F}_\tau-$measurable, then $\mathcal{E}^g_{\tau,\gamma}[\xi]=\xi$, $P-$a.s.
(g3) "Time Consistency": For any $\zeta\in\mathcal{T}$ with $\tau\le\zeta\le\gamma$, $P-$a.s., it holds $P-$a.s. that $\mathcal{E}^g_{\tau,\zeta}\big[\mathcal{E}^g_{\zeta,\gamma}[\xi]\big]=\mathcal{E}^g_{\tau,\gamma}[\xi]$.
(g4) "Zero$-$One Law": For any $A\in\mathcal{F}_\tau$, we have $\mathbf{1}_A\mathcal{E}^g_{\tau,\gamma}[\mathbf{1}_A\xi]=\mathbf{1}_A\mathcal{E}^g_{\tau,\gamma}[\xi]$, $P-$a.s.; In addition, if $g(t,0,0,0)=0$, $dt\times dP-$a.s., then $\mathcal{E}^g_{\tau,\gamma}[\mathbf{1}_A\xi]=\mathbf{1}_A\mathcal{E}^g_{\tau,\gamma}[\xi]$, $P-$a.s.
(g5) "Translation Invariance": If g is independent of y, then $\mathcal{E}^g_{\tau,\gamma}[\xi+\eta]=\mathcal{E}^g_{\tau,\gamma}[\xi]+\eta$, $P-$a.s. for any $\eta\in L^p(\mathcal{F}_\tau)$.

We can define the corresponding $g-$martingales with $\mathbb{L}^p$ domains under jump filtration as usual: A real-valued, $\mathbf{F}-$adapted process X is called a $g-$submartingale (resp. $g-$supermartingale or $g-$martingale) if for any $0\le t\le s\le T$, $E[|X_s|^p]<\infty$ and

$$\mathcal{E}^g_{t,s}[X_s]\ge(\text{resp. }\le\text{ or }=)\,X_t,\quad P-\text{a.s.}$$

The $g-$martingales retain many classic martingale properties such as "upcrossing inequality", "optional sampling theorem", "Doob-Meyer decomposition" and etc, which relate the $g-$evaluations with $\mathbb{L}^p$ domains to risk measures with $\mathbb{L}^p$ domains in mathematical markets with jumps, see [41] for details.

3 Reflected BSDEs with Jumps

When saying a parameter pair $(\xi,\mathfrak{L})$, we mean a random variable $\xi\in L^0(\mathcal{F}_T)$ and a real-valued, $\mathbf{F}-$adapted càdlàg process $\mathfrak{L}$ such that $\mathfrak{L}_T\le\xi$, $P-$a.s.

Definition 3.1. *Given $p\in(1,2)$, let $(\xi,\mathfrak{L})$ be a parameter pair and let g be a $p-$generator. A quadruplet $(Y,Z,U,K)\in\mathbb{D}^0\times\mathbb{Z}^2_{\text{loc}}\times\mathbb{U}^p_{\text{loc}}\times\mathbb{K}^0$ is called a solution of a reflected backward stochastic differential equation with jumps that has terminal data ξ, generator g and obstacle $\mathfrak{L}$ ($RBSDEJ(\xi,g,\mathfrak{L})$ for short) if $\int_0^T\big|g(s,Y_s,Z_s,U_s)\big|ds<\infty$, $P-$a.s. and if (1.1) holds $P-$a.s.*

Remark 3.1. *Given $p\in(1,2)$, let g be a $p-$generator satisfying (A2') and $\int_0^T|g(t,0,0,0)|dt<\infty$, $P-$a.s. Then it holds for any $(Y,Z,U)\in\mathbb{D}^1\times\mathbb{Z}^2_{\text{loc}}\times\mathbb{U}^p_{\text{loc}}$ that $\int_0^T|g(s,Y_s,Z_s,U_s)|ds<\infty$, $P-$a.s.*

Remark 3.2. *Given $p\in(1,2)$, let $(\xi,\mathfrak{L})$ be a parameter pair, let g be a $p-$generator, and let (Y,Z,U,K) be a solution of RBSDEJ$(\xi,g,\mathfrak{L})$. We denote by K^c (resp. K^d) the continuous part (resp. purely discontinuous part) of K.*
(1) The process Y has two jumps sources: the jump times of the stochastic integral M^U are totally inaccessible, while the jumps of the $\mathbf{F}-$predictable càdlàg increasing process K are exhausted by a sequence $\{\zeta_n\}_{n\in\mathbb{N}}$ of $\mathbf{F}-$predictable stopping times $\big($i.e. $\{(t,\omega)\in[0,T]\times\Omega:\Delta K^d_t(\omega)>0\}$ is a union of graphs $[\![\zeta_n]\!]$ and these graphs are disjoint on $(0,T)$, see e.g. "Complements to Chapter IV" of [11] or Proposition I.2.24 of [24] for details$\big)$. In particular, one can deduce that for $P-$a.s. $\omega\in\Omega$

$$\mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}\Delta K^d_t(\omega)=0 \quad \text{and thus} \quad \Delta Y_t(\omega)=\mathbf{1}_{\{t\in D_{\mathfrak{p}(\omega)}\}}U\big(t,\omega,\mathfrak{p}_t(\omega)\big)-\mathbf{1}_{\{t\notin D_{\mathfrak{p}(\omega)}\}}\Delta K^d_t(\omega), \quad \forall\, t\in[0,T]. \tag{3.1}$$

(2) According to Remark 2.2 of [18] or Remark 2.1 of [21], the "flat-off condition" is equivalent to

$$P\Big\{\int_0^T(Y_s-\mathfrak{L}_s)dK^c_s=0 \text{ and } \Delta K^d_\tau=\mathbf{1}_{\{Y_{\tau-}=\mathfrak{L}_{\tau-}\}}(\mathfrak{L}_{\tau-}-Y_\tau)^+ \text{ for any } \mathbf{F}-\text{predictable stopping time } \tau\Big\}=1. \tag{3.2}$$

The second part in (3.2) characterizes the $\mathbf{F}-$predictable jump times of Y: For $P-$a.s. $\omega\in\Omega$, if $\Delta Y_\tau(\omega)=-\Delta K^d_\tau(\omega)<0$ for some $\mathbf{F}-$predictable stopping time τ, then (3.2) implies that $\mathfrak{L}_{\tau-}(\omega)>Y_\tau(\omega)\ge\mathfrak{L}_\tau(\omega)$, or $\Delta\mathfrak{L}_\tau(\omega)<0$. Roughly speaking, the $\mathbf{F}-$predictable jumps of Y stem from the $\mathbf{F}-$predictable negative jumps of $\mathfrak{L}$.
(3) Since the càdlàg increasing processes K and the Poisson stochastic integral M^U jump countably many times along their $P-$a.s. paths, so does process Y: $\big\{t\in[0,T]:Y_{t-}(\omega)\ne Y_t(\omega)\big\}$ is a countable subset of $[0,T]$ for $P-$a.s. $\omega\in\Omega$.

To demonstrate the wellposedness of reflected BSDEs with jumps in $\mathbb{L}^p$ sense, we start with a comparison result for generalized BSDEJs over stochastic intervals.

Proposition 3.1. *Let $p\in(1,2)$, $\tau\in\mathcal{T}$ and $\gamma\in\mathcal{T}_\tau$. For $i=1,2$, let g^i be a $p-$generator and let $(Y^i,Z^i,U^i,V^i)\in\mathbb{D}^0\times\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}\times\mathbb{V}^0$ such that $Y^1-Y^2\in\mathbb{D}^p$ and that $P-$a.s.*

$$Y^i_t=Y^i_\gamma+\int_t^\gamma g^i(s,Y^i_s,Z^i_s,U^i_s)ds+V^i_\gamma-V^i_t-\int_t^\gamma Z^i_s dB_s-\int_{(t,\gamma]}\int_{\mathcal{X}}U^i_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad \forall\, t\in[\tau,\gamma]. \tag{3.3}$$

Assume also that $Y^1_\gamma\le Y^2_\gamma$, $P-$a.s. and that $P-$a.s.

$$\int_t^s\mathbf{1}_{\{Y^1_{r-}>Y^2_{r-}\}}(dV^1_r-dV^2_r)\le 0, \quad \forall\, t,s\in[\tau,\gamma] \text{ with } t<s. \tag{3.4}$$

For either $i=1$ or $i=2$, if g^i satisfies (A2), (A3), and if $g^1(t,Y^{3-i}_t,Z^{3-i}_t,U^{3-i}_t)\le g^2(t,Y^{3-i}_t,Z^{3-i}_t,U^{3-i}_t)$, $dt\times dP-$a.s. on $]\!]\tau,\gamma[\![$, then it holds $\mathbb{P}-$a.s. that $Y^1_t\le Y^2_t$ for any $t\in[\tau,\gamma]$.

Corollary 3.1. *Let $p\in(1,2)$, $\tau\in\mathcal{T}$ and $\gamma\in\mathcal{T}_\tau$. For $i=1,2$, let $(\xi_i,\mathfrak{L}^i)$ be a parameter pair, let g^i be a $p-$generator, and let (Y^i,Z^i,U^i,K^i) be a solution of RBSDEJ$(\xi_i,g^i,\mathfrak{L}^i)$ such that $Y^1-Y^2\in\mathbb{D}^p$ and that $P\big\{Y^1_\gamma\le Y^2_\gamma\big\}=P\big\{\mathfrak{L}^1_t\le\mathfrak{L}^2_t,\ \forall\, t\in(\tau,\gamma)\big\}=1$. For either $i=1$ or $i=2$, if g^i satisfies (A2), (A3), and if $g^1(t,Y^{3-i}_t,Z^{3-i}_t,U^{3-i}_t)\le g^2(t,Y^{3-i}_t,Z^{3-i}_t,U^{3-i}_t)$, $dt\times dP-$a.s. on $]\!]\tau,\gamma[\![$, then it holds $P-$a.s. that $Y^1_t\le Y^2_t$ for any $t\in[\tau,\gamma]$.*

Corollary 3.1 directly implies the uniqueness of the $\mathbb{L}^p-$solution to a RBSDEJ. As to the existence result, we adapt [15]'s Snell envelope approach by first extending the optimal stopping theory for right-continuous processes of class (D) and with unbounded negative values (see the appendix). Then a Doob-Meyer decomposition and a flat-off condition of the Snell envelope of $\mathfrak{L}$ (Proposition A.5 and A.6) as well as a generalized martingale representation theorem from [42] give rise to the following wellposedness result for RBSDEJs with simple generators in $\mathbb{L}^p$ sense.

Proposition 3.2. *Given $p\in(1,2)$, let $(\xi,\mathfrak{L})$ be a parameter pair with $(\xi,\mathfrak{L}^+)\in L^p(\mathcal{F}_T)\times\mathbb{D}^p$ and let $\mathfrak{g}$ be a $\mathbf{F}-$progressively measurable process with $\int_0^T|\mathfrak{g}_t|dt\in L^p(\mathcal{F}_T)$. Then RBSDEJ $(\xi,\mathfrak{g},\mathfrak{L})$ admits a unique solution $(Y,Z,U,K)\in\mathbb{S}^p\times\mathbb{K}^p$. Moreover, the process K is continuous if one of the following conditions holds:*
(L1) $\Delta\mathfrak{L}_\tau\ge 0$ for any $\mathbf{F}-$predictable stopping time τ;
(L2) $\mathfrak{L}\in\mathbb{D}^p$ and $\mathfrak{L}$ is l.u.s.c.e.

With Proposition 3.2, we can then employ a fixed-point argument (or a Picard iterative approximation) to derive the main result of the present paper.

Theorem 3.1. *Given $p\in(1,2)$, let $(\xi,\mathfrak{L})$ be a parameter pair with $(\xi,\mathfrak{L}^+)\in L^p(\mathcal{F}_T)\times\mathbb{D}^p$ and let g be a $p-$generator satisfying (A1) and (A2'). Then RBSDEJ $(\xi,g,\mathfrak{L})$ admits a unique solution $(Y,Z,U,K)\in\mathbb{S}^p\times\mathbb{K}^p$. Moreover, the process K is continuous if either (L1) or (L2) holds.*

4 Optimal Stopping under $g-$Evaluations

We derived the wellposedness result (Proposition 3.2) of RBSDEJs with simple generators from the optimal stopping of uniformly integrable reward processes under jump filtration. Conversely, the general wellposedness result (Theorem 3.1) as well as the comparison theorem (Proposition 3.1) give rise to the optimal stopping under $g-$evaluations: Let $\mathcal{R}$ be a $p-$integrable reward process consisting of a running reward $\mathfrak{L}$ and a terminal reward ξ. The $Y-$component of the unique $\mathbb{L}^p$ solution to RBSDEJ $(\xi, g, \mathfrak{L})$ is exactly the Snell envelope of $\mathcal{R}$ in the optimal stopping problem under $g-$evaluations:

Proposition 4.1. *Given $p \in (1,2)$, let $(\xi, \mathfrak{L})$ be a parameter pair such that $(\xi, \mathfrak{L}^+) \in L^p(\mathcal{F}_T) \times \mathbb{D}^p$, let g be a $p-$generator satisfying (A1)−(A3) and let (Y, Z, U, K) be the unique solution of RBSDEJ $(\xi, g, \mathfrak{L})$ in $\mathbb{S}^p \times \mathbb{K}^p$.*
(1) Y is a $g-$supermartingale: for any $\gamma \in \mathcal{T}$ and $\zeta \in \mathcal{T}_\gamma$, $Y_\gamma \ge \mathcal{E}^g_{\gamma,\zeta}\big[Y_\zeta\big]$, $P-$a.s.; For any $n \in \mathbb{N}$, Y is a $g-$martingale up to time $\tau_n(0) := \{t \in [0,T]: Y_t \le \mathcal{R}_t + 1/n\} \in \mathcal{T}$: for any $\gamma, \zeta \in \mathcal{T}$ with $0 \le \gamma \le \zeta \le \tau_n(0)$, $Y_\gamma = \mathcal{E}^g_{\gamma,\zeta}\big[Y_\zeta\big]$, $P-$a.s.
(2) Y is the Snell envelope of the reward process $\mathcal{R}_t := \mathbf{1}_{\{t<T\}}\mathfrak{L}_t + \mathbf{1}_{\{t=T\}}\xi$, $t \in [0,T]$ under the $g-$evaluations:

$$Y_\gamma = \underset{\zeta \in \mathcal{T}_\gamma}{\operatorname{esssup}}\, \mathcal{E}^g_{\gamma,\zeta}\big[\mathcal{R}_\zeta\big], \quad P-a.s. \text{ for any } \gamma \in \mathcal{T}. \tag{4.1}$$

(3) If process K is continuous, then for any $\gamma \in \mathcal{T}$, $\tau_*(\gamma) := \inf\{t \in [\gamma, T] : Y_t = \mathcal{R}_t\}$ (resp. $\widehat{\tau}(\gamma) := \inf\{t \in (\gamma, T] : K_t > K_\gamma\}$) is the minimal (resp. maximal) optimal stopping time for $\underset{\zeta \in \mathcal{T}_\gamma}{\operatorname{esssup}}\, \mathcal{E}^g_{\gamma,\zeta}\big[\mathcal{R}_\zeta\big]$, and Y is a $g-$martingale up to time $\widehat{\tau}(0)$.

This result generalizes Theorem 3.3 and Theorem 3.7 of [38] to the $\mathbb{L}^p$ case.

5 Proofs

Proof of Remark 3.1: Let $(Y, Z, U) \in \mathbb{D}^1 \times \mathbb{Z}^2_{\text{loc}} \times \mathbb{U}^p_{\text{loc}}$. Fix $n \in \mathbb{N}$ and define

$$\tau_n := \inf\left\{t \in [0,T] : \int_0^t |g(s,0,0,0)| ds + \int_0^t \big|Z_s\big|^2 ds + \int_0^t \int_{\mathcal{X}} |U_s(x)|^p \nu(dx) ds > n\right\} \wedge T \in \mathcal{T}.$$

Hölder's inequality and (A2') imply that

$$\begin{aligned} E\int_0^{\tau_n} \big|g(t, Y_t, Z_t, U_t)\big| dt &\le E\int_0^{\tau_n} \big(\big|g(t,0,0,0)\big| + \beta_t |Y_t| + \Lambda_t |Z_t| + \beta_t \|U_t\|_{L^p_\nu}\big) dt \\ &\le n + E\bigg[\sup_{t \in [0,\tau_n]} |Y_t| \cdot \int_0^{\tau_n} \beta_t dt\bigg] + \Big(E\int_0^{\tau_n} \Lambda_t^2 dt\Big)^{\frac{1}{2}} \Big(E\int_0^{\tau_n} |Z_t|^2 dt\Big)^{\frac{1}{2}} + \Big(E\int_0^{\tau_n} \beta_t^q dt\Big)^{\frac{1}{q}} \Big(E\int_0^{\tau_n} \|U_t\|^p_{L^p_\nu} dt\Big)^{\frac{1}{p}} \\ &\le n + \overline{\beta} E[Y_*] + \widehat{C}^{\frac{1}{2}} n^{\frac{1}{2}} + \widehat{C}^{\frac{1}{q}} n^{\frac{1}{p}} < \infty. \end{aligned}$$

So $\int_0^{\tau_n} \big|g(t, Y_t, Z_t, U_t)\big| dt < \infty$ except on a $P-$null set $\mathcal{N}_n$. Since $\int_0^T |g(t,0,0,0)| dt < \infty$, $P-$a.s. and since $(Z, U) \in \mathbb{Z}^2_{\text{loc}} \times \mathbb{U}^p_{\text{loc}}$, one can find a $P-$null set $\mathcal{N}_0$ such that for any $\omega \in \mathcal{N}_0^c$, $\tau_{\mathfrak{n}}(\omega) = T$ for some $\mathfrak{n} = \mathfrak{n}(\omega) \in \mathbb{N}$. Now, for any $\omega \in \underset{n \in \mathbb{N} \cup \{0\}}{\cap} \mathcal{N}_n^c$, one can deduce that $\int_0^T \big|g(t, \omega, Y_t(\omega), Z_t(\omega), U_t(\omega))\big| dt = \int_0^{\tau_{\mathfrak{n}}(\omega)} \big|g(t, \omega, Y_t(\omega), Z_t(\omega), U_t(\omega))\big| dt < \infty$. □

Proof of Remark 3.2: 1) We just prove (3.1): Let $\{\zeta_n\}_{n \in \mathbb{N}}$ be the sequence of $\mathbf{F}-$predictable stopping times that exhausts the jumps of process K and set $\mathcal{N}_i := \underset{n \in \mathbb{N}}{\cup} \{\tau_i^N = \zeta_n\}$, which is a $P-$null set. For all $\omega \in \Omega$ except on $\underset{n \in \mathbb{N}}{\cup} \mathcal{N}_i$, since the path $K.(\omega)$ only jumps at $t \in \Big(\underset{i \in \mathbb{N}}{\cup} \{\zeta_n(\omega)\}\Big)$ and since $\Big(\underset{i \in \mathbb{N}}{\cup} \{\zeta_n(\omega)\}\Big) \cap \Big(\underset{i \in \mathbb{N}}{\cup} \{\tau_i^N(\omega)\}\Big) = \emptyset$, we see that the path $K.(\omega)$ does not jump at $t \in D_{\mathfrak{p}(\omega)} = \Big(\underset{i \in \mathbb{N}}{\cup} \{\tau_i^N(\omega)\}\Big)$. Thus it holds for any $t \in [0,T]$ that $\mathbf{1}_{\{t \in D_{\mathfrak{p}(\omega)}\}} \Delta K_t^d(\omega) = 0$ and thus that

$$\Delta Y_t(\omega) = \mathbf{1}_{\{t \in D_{\mathfrak{p}(\omega)}\}} U\big(t, \omega, \mathfrak{p}_t(\omega)\big) - \mathbf{1}_{\{t \notin D_{\mathfrak{p}(\omega)}\}} \Delta K_t^d(\omega).$$

2) Now, we show the equivalence of (3.2) to the flat-off condition. It holds for any $\omega \in \Omega$ except on a $P-$null set $\mathcal{N}$ that (1.1) holds and the path $Y.(\omega) - \mathfrak{L}.(\omega)$ is càdlàg. Fix $\omega \in \mathcal{N}^c$, $\int_0^T \big(Y_{s-}(\omega) - \mathfrak{L}_{s-}(\omega)\big) dK_s(\omega) = 0$ is equivalent to

$$\int_0^T \big(Y_{s-}(\omega) - \mathfrak{L}_{s-}(\omega)\big) dK^c_s(\omega) = 0 \tag{5.1}$$

plus

$$0=\sum_{t\in[0,T]}\big(Y_{s-}(\omega)-\mathfrak{L}_{s-}(\omega)\big)\Delta K^d_s(\omega)=\sum_{n\in\mathbb{N}}\Big(Y\big(\zeta_n(\omega)-,\omega\big)-\mathfrak{L}\big(\zeta_n(\omega)-,\omega\big)\Big)\Delta K^d\big(\zeta_n(\omega),\omega\big), \tag{5.2}$$

where $\{\zeta_n\}_{n\in\mathbb{N}}$ is sequence of $\mathbf{F}-$predictable stopping times that exhausts the jumps of process K.

As the path $Y_\cdot(\omega)-\mathfrak{L}_\cdot(\omega)$ is càdlàg, the continuity of path $K^c_\cdot(\omega)$ implies that (5.1) is amount to

$$\int_0^T\big(Y_s(\omega)-\mathfrak{L}_s(\omega)\big)dK^c_s(\omega)=0. \tag{5.3}$$

Clearly, the equality (5.2) holds if

$$\Big(Y\big(\tau(\omega)-,\omega\big)-\mathfrak{L}\big(\tau(\omega)-,\omega\big)\Big)\Delta K^d\big(\tau(\omega),\omega\big)=0 \text{ holds for any } \mathbf{F}-\text{predictable stopping time } \tau. \tag{5.4}$$

Conversely, given an $\mathbf{F}-$predictable stopping time τ, there exists a $m=m(\omega)\in\mathbb{N}$ such that $\tau(\omega)=\zeta_m(\omega)$. Then we can deduce from (5.2) that

$$\begin{aligned}0 &\le \Big(Y\big(\tau(\omega)-,\omega\big)-\mathfrak{L}\big(\tau(\omega)-,\omega\big)\Big)\Delta K^d\big(\tau(\omega),\omega\big)=\Big(Y\big(\zeta_m(\omega)-,\omega\big)-\mathfrak{L}\big(\zeta_m(\omega)-,\omega\big)\Big)\Delta K^d\big(\zeta_m(\omega),\omega\big)\\ &\le \sum_{n\in\mathbb{N}}\Big(Y\big(\zeta_n(\omega)-,\omega\big)-\mathfrak{L}\big(\zeta_n(\omega)-,\omega\big)\Big)\Delta K^d\big(\zeta_n(\omega),\omega\big)=0.\end{aligned}$$

So (5.2) is equivalent to (5.4).

Now, let τ be an $\mathbf{F}-$predictable stopping time. It is easy to see that "$\Big(Y\big(\tau(\omega)-,\omega\big)-\mathfrak{L}\big(\tau(\omega)-,\omega\big)\Big)\Delta K^d\big(\tau(\omega),\omega\big)=0$" amounts to

$$\Delta K^d\big(\tau(\omega),\omega\big)=\mathbf{1}_{\{Y(\tau(\omega)-,\omega)=\mathfrak{L}(\tau(\omega)-,\omega)\}}\Delta K^d\big(\tau(\omega),\omega\big). \tag{5.5}$$

Since the stochastic integral M^U does not jump at $\mathbf{F}-$predictable stopping times, one has $\Delta Y\big(\tau(\omega),\omega\big)=-\Delta K^d\big(\tau(\omega),\omega\big)\le 0$. Then (5.5) is equivalent to

$$\Delta K^d\big(\tau(\omega),\omega\big)=\mathbf{1}_{\{Y(\tau(\omega)-,\omega)=\mathfrak{L}(\tau(\omega)-,\omega)\}}\Big(-\Delta Y\big(\tau(\omega),\omega\big)\Big)^+=\mathbf{1}_{\{Y(\tau(\omega)-,\omega)=\mathfrak{L}(\tau(\omega)-,\omega)\}}\Big(\mathfrak{L}\big(\tau(\omega)-,\omega\big)-Y\big(\tau(\omega),\omega\big)\Big)^+. \quad \square$$

Proof of Proposition 3.1: Without loss of generality, we suppose that g^1 satisfies (A2), (A3) and that

$$g^1(t,Y^2_t,Z^2_t,U^2_t)\le g^2(t,Y^2_t,Z^2_t,U^2_t), \quad dt\times dP-\text{a.s. on }]\!]\tau,\gamma[\![\,. \tag{5.6}$$

1) Set $(Y,Z,U):=(Y^1-Y^2,Z^1-Z^2,U^1-U^2)$ and consider the following $\mathbf{F}-$progressively measurable processes:

$$\begin{aligned}a_t &:= \mathbf{1}_{\{Y_t\neq 0\}}\frac{g^1(t,Y^1_t,Z^1_t,U^1_t)-g^1(t,Y^2_t,Z^1_t,U^1_t)}{Y_t}, \quad \Theta_t:=e^{\int_0^t a^+_s ds}>0, \quad \text{and}\\ \mathfrak{b}_t &:= \mathbf{1}_{\{Z_t\neq 0\}}\frac{g^1(t,Y^2_t,Z^1_t,U^1_t)-g^1(t,Y^2_t,Z^2_t,U^1_t)}{|Z_t|^2}Z_t, \quad \forall\, t\in[0,T].\end{aligned}$$

By (A2), it holds $dt\times dP-$a.s. that

$$|a_t|\le\beta_t \quad \text{and} \quad |\mathfrak{b}_t|\le\Lambda_t. \tag{5.7}$$

Set $\mathfrak{H}_t:=\mathfrak{h}(t,Y^2_t,Z^2_t,U^1_t,U^2_t)$, $t\in[0,T]$. Using similar arguments to those in the proof of [41, Theorem 2.2], we know that $M_t:=\int_0^t\mathfrak{b}_s dB_s+\int_{(0,t]}\int_{\mathcal{X}}\mathfrak{H}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)$, $t\in[0,T]$ is a BMO martingale and its Doléans-Dade exponential

$$\mathscr{E}_t(M):=e^{M_t-\frac{1}{2}\langle M^c\rangle_t}\prod_{0<s\le t}(1+\Delta M_s)e^{-\Delta M_s}>0, \quad t\in[0,T] \tag{5.8}$$

is thus a uniformly integrable martingale, where M^c denotes the continuous part of M.

Define a probability measure Q by $\frac{dQ}{dP}:=\mathscr{E}_T(M)$, which satisfies $\frac{dQ}{dP}\big|_{\mathcal{F}_t}:=\mathscr{E}_t(M)$, $\forall\, t\in[0,T]$. The Girsanov's Theorem (e.g. [24, 36]) shows that $B^Q_t:=B_t-\int_0^t\mathfrak{b}_s ds$, $t\in[0,T]$ is a $Q-$Brownian motion and $\widetilde{N}^Q_{\mathfrak{p}}(t,A):=\widetilde{N}_{\mathfrak{p}}(t,A)-\int_{(0,t]}\int_{\mathcal{X}}\mathfrak{H}_s(x)\nu(dx)ds$, $t\in[0,T]$, $A\in\mathcal{F}_{\mathcal{X}}$ is a $Q-$compensated Poisson random measure. By (5.7),

$$\Theta_*\le e^{\int_0^T\beta_t dt}\le e^{\widehat{C}}, \quad P-\text{a.s. and thus } Q-\text{a.s.} \tag{5.9}$$

2) Since $\mathbb{E}[|Y_\tau|^p] \le \|Y\|^p_{\mathbb{D}^p} < \infty$, Corollary 2.1 implies that there exists a unique pair $(\mathcal{Z},\mathcal{U}) \in \mathbb{Z}^{2,p}\times\mathbb{U}^p$ such that $P\big\{E[Y_\tau|\mathcal{F}_t] = E[Y_\tau] + \int_0^t \mathcal{Z}_s dB_s + \int_{(0,t]}\int_{\mathcal{X}} \mathcal{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\ t\in[0,T]\big\} = 1$. This together with (3.3) shows that $P-$a.s.

$$
\begin{aligned}
\mathcal{Y}_t \,:=\, & \mathbb{E}[Y_\tau|\mathcal{F}_{\tau\wedge t}] + Y_{\gamma\wedge t} - Y_{\tau\wedge t} = \mathbb{E}[Y_\tau|\mathcal{F}_{\tau\wedge t}] + Y_{\tau\vee(\gamma\wedge t)} - Y_\tau = \mathbb{E}[Y_\tau] + \int_0^{\tau\wedge t} \mathcal{Z}_s dB_s + \int_{(0,\tau\wedge t]}\int_{\mathcal{X}} \mathcal{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) \\
& - \int_\tau^{\tau\vee(\gamma\wedge t)} \mathfrak{g}_s ds - V^1_{\tau\vee(\gamma\wedge t)} + V^1_\tau + V^2_{\tau\vee(\gamma\wedge t)} - V^2_\tau + \int_\tau^{\tau\vee(\gamma\wedge t)} Z_s dB_s + \int_{(\tau,\tau\vee(\gamma\wedge t)]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx) \\
= \, & \mathbb{E}[Y_\tau] - \int_0^t \mathbf{1}_{\{\tau<s\le\gamma\}}\mathfrak{g}_s ds - \int_0^t \mathbf{1}_{\{\tau<s\le\gamma\}}(dV^1_s - dV^2_s) + \int_0^t \big(\mathbf{1}_{\{s\le\tau\}}\mathcal{Z}_s + \mathbf{1}_{\{\tau<s\le\gamma\}} Z_s\big) dB_s \\
& + \int_{(0,t]}\int_{\mathcal{X}} \big(\mathbf{1}_{\{s\le\tau\}}\mathcal{U}_s(x) + \mathbf{1}_{\{\tau<s\le\gamma\}} U_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad t\in[0,T],
\end{aligned}
$$

with $\mathfrak{g}_s := g^1(s,Y^1_s,Z^1_s,U^1_s) - g^2(s,Y^2_s,Z^2_s,U^2_s)$. Similar to Remark 3.2 (3), the càdlàg processes V^1, V^2 of finite variation and the Poisson stochastic integral $\big\{\int_{(0,t]}\int_{\mathcal{X}} \big(\mathbf{1}_{\{s\le\tau\}}\mathcal{U}_s(x) + \mathbf{1}_{\{\tau<s\le\gamma\}} U_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx)\big\}_{t\in[0,T]}$ jump countably many times along their $P-$a.s. paths, so does the $\mathbf{F}-$adapted càdlàg process $\mathcal{Y}$. Namely, there exists a $P-$null set $\mathcal{N}$ such that for any $\omega\in\mathcal{N}^c$,

$$
\big\{t\in[0,T]: \mathcal{Y}_{t-}(\omega) \ne \mathcal{Y}_t(\omega)\big\} \text{ is a countable subset of } [0,T]. \tag{5.10}
$$

Applying Tanaka-type formula (see e.g. Theorem IV.68 of [36]) to process $\mathcal{Y}^+$ yields that $P-$a.s.

$$
\begin{aligned}
\mathcal{Y}^+_t = & \big(\mathbb{E}[Y_\tau]\big)^+ - \int_0^t \mathbf{1}_{\{\mathcal{Y}_{s-}>0\}}\mathbf{1}_{\{\tau<s\le\gamma\}}\mathfrak{g}_s ds - \int_0^t \mathbf{1}_{\{\mathcal{Y}_{s-}>0\}}\mathbf{1}_{\{\tau<s\le\gamma\}}(dV^1_s - dV^2_s) + \Gamma_t + \frac{1}{2}\mathcal{O}_t \\
& + \int_0^t \mathbf{1}_{\{\mathcal{Y}_{s-}>0\}}\big(\mathbf{1}_{\{s\le\tau\}}\mathcal{Z}_s + \mathbf{1}_{\{\tau<s\le\gamma\}} Z_s\big) dB_s + \int_{(0,t]}\int_{\mathcal{X}} \mathbf{1}_{\{\mathcal{Y}_{s-}>0\}}\big(\mathbf{1}_{\{s\le\tau\}}\mathcal{U}_s(x) + \mathbf{1}_{\{\tau<s\le\gamma\}} U_s(x)\big)\widetilde{N}_{\mathfrak{p}}(ds,dx),\ t\in[0,T],
\end{aligned}
$$

where $\Gamma_t := \sum_{s\in(0,t]}\big(\mathbf{1}_{\{\mathcal{Y}_{s-}>0\}}\mathcal{Y}^-_s + \mathbf{1}_{\{\mathcal{Y}_{s-}\le 0\}}\mathcal{Y}^+_s\big)$, $s\in[0,T]$ and $\big\{\mathcal{O}_t\big\}_{t\in[0,T]}$ is an $\mathbf{F}-$adapted continuous increasing process known as the "local time" of process $\mathcal{Y}$ at 0. By (5.10), Γ is an $\mathbf{F}-$adapted càdlàg increasing process with countably many jumps.

3) Now, fix $t\in[0,T]$ and $n\in\mathbb{N}$. We define $\gamma_n := \inf\big\{s\in[\tau,T]: \int_\tau^s |Z_r|^2 dr + \int_\tau^s\int_{\mathcal{X}} |U_r(x)|^p \nu(dx)dr > n\big\}\wedge\gamma \in \mathcal{T}_\tau$ and set $\zeta_n := (\tau\vee t)\wedge\gamma_n$. Applying integration-by-parts formula (see e.g. Corollary II.2 of [36]) to process $\Theta\mathcal{Y}^+$ over $[\zeta_n,\gamma_n] = [\tau,\gamma_n]\cap[t,T]$ yields that

$$
\begin{aligned}
\Theta_{\zeta_n}\mathcal{Y}^+_{\zeta_n} = \, & \Theta_{\gamma_n}\mathcal{Y}^+_{\gamma_n} + \int_{\zeta_n}^{\gamma_n} \Theta_r\big(\mathbf{1}_{\{\mathcal{Y}_{r-}>0\}}\mathfrak{g}_r - a^+_r\mathcal{Y}^+_r\big)dr + \int_{\zeta_n}^{\gamma_n} \mathbf{1}_{\{\mathcal{Y}_{r-}>0\}}\Theta_r(dV^1_r - dV^2_r) - \int_{\zeta_n}^{\gamma_n}\Theta_r d\Gamma_r - \frac{1}{2}\int_{\zeta_n}^{\gamma_n}\Theta_r d\mathcal{O}_r \\
& - \int_{\zeta_n}^{\gamma_n} \mathbf{1}_{\{\mathcal{Y}_{r-}>0\}}\Theta_r Z_r dB_r - \int_{(\zeta_n,\gamma_n]}\int_{\mathcal{X}} \mathbf{1}_{\{\mathcal{Y}_{r-}>0\}}\Theta_r U_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx), \quad P-\text{a.s.}
\end{aligned}
$$

Since $\mathcal{Y}_{\tau\vee(\gamma\wedge t)} = \mathbb{E}[Y_\tau|\mathcal{F}_\tau] + Y_{\tau\vee(\gamma\wedge t)} - Y_\tau = Y_{\tau\vee(\gamma\wedge t)}$ for any $t\in[0,T]$ (i.e. $\mathcal{Y}_t = Y_t$ for any $t\in[\tau,\gamma]$), we see from (3.4) that $P-$a.s.

$$
\Theta_{\zeta_n} Y^+_{\zeta_n} \le \Theta_{\gamma_n} Y^+_{\gamma_n} + \int_{\zeta_n}^{\gamma_n} \Theta_r\big(\mathbf{1}_{\{Y_{r-}>0\}}\mathfrak{g}_r - a^+_r Y^+_r\big)dr - \int_{\zeta_n}^{\gamma_n} \mathbf{1}_{\{Y_{r-}>0\}}\Theta_r Z_r dB_r - \int_{(\zeta_n,\gamma_n]}\int_{\mathcal{X}} \mathbf{1}_{\{Y_{r-}>0\}}\Theta_r U_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx).
$$

By (A3) (iii) and (5.6), it holds $dr\times dP-$a.s. on $]\!]\tau,\gamma[\![$ that

$$
\mathbf{1}_{\{Y_{r-}>0\}}\mathfrak{g}_r = \mathbf{1}_{\{Y_{r-}>0\}}\big(a_r Y_r + \mathfrak{b}_r Z_r + g^1\big(r,Y^2_r,Z^2_r,U^1_r\big) - g^2\big(r,Y^2_r,Z^2_r,U^2_r\big)\big) \le \mathbf{1}_{\{Y_{r-}>0\}}\Big(a_r Y_r + \mathfrak{b}_r Z_r + \int_{\mathcal{X}} \mathfrak{H}_r(x) U_r(x)\nu(dx)\Big).
$$

Also, (5.10) implies that for $P-$a.s. $\omega\in\Omega$

$$
\mathbf{1}_{\{Y_{r-}(\omega)>0\}} a_r(\omega)Y_r(\omega) = \mathbf{1}_{\{Y_r(\omega)>0\}} a_r(\omega)Y_r(\omega) = \mathbf{1}_{\{Y_r(\omega)>0\}} a_r(\omega)Y^+_r(\omega) \le a^+_r(\omega)Y^+_r(\omega) \ \text{ holds for a.e. } r\in\big(\tau(\omega),\gamma(\omega)\big).
$$

Combining the above three inequalities yields that

$$
\Theta_{\zeta_n} Y^+_{\zeta_n} \le \Theta_{\gamma_n} Y^+_{\gamma_n} - \big(\mathcal{M}^n_{\gamma_n} - \mathcal{M}^n_{\zeta_n} + \mathscr{M}^n_{\gamma_n} - \mathscr{M}^n_{\zeta_n}\big), \quad P-\text{a.s. and thus } Q-\text{a.s.}, \tag{5.11}
$$

where $\mathcal{M}^n_t := \int_0^t \mathbf{1}_{\{r\in(\tau,\gamma_n]\}}\mathbf{1}_{\{Y_{r-}>0\}}\Theta_r Z_r dB^Q_r$ and $\mathscr{M}^n_t := \int_{(0,t]}\int_{\mathcal{X}} \mathbf{1}_{\{r\in(\tau,\gamma_n]\}}\mathbf{1}_{\{Y_{r-}>0\}}\Theta_r U_r(x)\widetilde{N}^Q_{\mathfrak{p}}(dr,dx)$, $t\in[0,T]$.

The Burkholder-Davis-Gundy inequality, (1.5) and (5.9) show that

$$E_Q\bigg[\sup_{t\in[0,T]}\big|\mathcal{M}^n_t\big|^p+\sup_{t\in[0,T]}\big|\mathscr{M}^n_t\big|^p\bigg]\le c_p E_Q\bigg[\Big(\int_\tau^{\gamma_n}|\Theta_r|^2|Z_r|^2dr\Big)^{\frac p2}+\Big(\int_{(\tau,\gamma_n]}\int_{\mathcal{X}}|\Theta_r|^2|U_r(x)|^2N_{\mathfrak{p}}(dr,dx)\Big)^{\frac p2}\bigg]$$
$$\le c_p e^{p\widehat{C}}E_Q\bigg[\Big(\int_\tau^{\gamma_n}|Z_r|^2dr\Big)^{\frac p2}+\int_\tau^{\gamma_n}\int_{\mathcal{X}}|U_r(x)|^p\nu(dx)dr\bigg]\le c_p e^{p\widehat{C}}\big(n^{\frac p2}+n\big)<\infty,$$

thus $\mathcal{M}^n$ and $\mathscr{M}^n$ are two uniformly integrable $Q-$martingales. Taking the conditional expectation $E_Q[\ |\mathcal{F}_{\zeta_n}]$ in (5.11) yields that $Q-$a.s.

$$\Theta_{\zeta_n}Y^+_{\zeta_n}\le E_Q\big[\Theta_{\gamma_n}Y^+_{\gamma_n}|\mathcal{F}_{\zeta_n}\big]=\mathbf{1}_{\{\gamma_n<(\tau\vee t)\wedge\gamma\}}E_Q\big[\Theta_{\gamma_n}Y^+_{\gamma_n}|\mathcal{F}_{\gamma_n}\big]+\mathbf{1}_{\{\gamma_n\ge(\tau\vee t)\wedge\gamma\}}E_Q\big[\Theta_{\gamma_n}Y^+_{\gamma_n}|\mathcal{F}_{(\tau\vee t)\wedge\gamma}\big]:=\eta^n_1+\eta^n_2. \quad (5.12)$$

As $(Z,U)\in\mathbb{Z}^2_{\rm loc}\times\mathbb{U}^p_{\rm loc}$, one has $\int_0^T\big(|Z_r|^2+\|U_r\|^p_{L^p_\nu}\big)dr<\infty$, $P-$a.s. and thus $Q-$a.s. So for $Q-$a.s. $\omega\in\Omega$ there exists a $N_\omega\in\mathbb{N}$ such that

$$\text{for any } n\ge N_\omega,\ \gamma_n(\omega)=\gamma(\omega) \text{ and thus } \eta^n_1(\omega)=0. \quad (5.13)$$

It follows that $\lim_{n\to\infty}\eta^n_1=0$, $Q-$a.s. On the other hand, the first equality in (5.13) also shows that $\lim_{n\to\infty}\Theta_{\gamma_n}Y^+_{\gamma_n}=\Theta_\gamma Y^+_\gamma$ and $\lim_{n\to\infty}\Theta_{\zeta_n}Y^+_{\zeta_n}=\Theta_{(\tau\vee t)\wedge\gamma}Y^+_{(\tau\vee t)\wedge\gamma}$, $Q-$a.s. even though the process Y may not be left-continuous. For any $n\in\mathbb{N}$, (5.9) shows that $\Theta_{\gamma_n}Y^+_{\gamma_n}\le e^{\widehat{C}}Y_*$, $P-$a.s. Since a slight extension of [37, Proposition A.1 (a)] shows that $E\big[\mathscr{E}^q_T(M)\big]<\infty$, we can deduce from Hölder's inequality that $E_Q[Y_*]=E[\mathscr{E}_T(M)Y_*]\le\|\mathscr{E}_T(M)\|_{L^q(\mathcal{F}_T)}\|Y\|_{\mathbb{D}^p}<\infty$.

As $n\to\infty$ in (5.12), a conditional-expectation version of dominated convergence theorem and (5.13) yield that

$$0\le\Theta_{(\tau\vee t)\wedge\gamma}Y^+_{(\tau\vee t)\wedge\gamma}\le\lim_{n\to\infty}E_Q\big[\Theta_{\zeta_n}Y^+_{\zeta_n}|\mathcal{F}_{(\tau\vee t)\wedge\gamma}\big]=E_Q\big[\Theta_\gamma Y^+_\gamma|\mathcal{F}_{(\tau\vee t)\wedge\gamma}\big]=0,\quad Q-\text{a.s. and thus } P-\text{a.s.}$$

It follows that $Y^+_{(\tau\vee t)\wedge\gamma}=0$ or $Y^1_{(\tau\vee t)\wedge\gamma}\le Y^2_{(\tau\vee t)\wedge\gamma}$, $P-$a.s. By the right continuity of processes Y^1 and Y^2, it holds $P-$a.s. that $Y^1_t\le Y^2_t$ for any $t\in[\tau,\gamma]$. □

Proof of Corollary 3.1: Since $P\big\{\mathfrak{L}^1_t\le\mathfrak{L}^2_t,\ \forall\, t\in(\tau,\zeta)\big\}=1$, implies that $P\big\{\mathfrak{L}^1_{t-}\le\mathfrak{L}^2_{t-},\ \forall\, t\in(\tau,\zeta]\big\}=1$ we can deduce from the flat-off condition of reflected BSDEs that $P-$a.s.

$$0\le\int_t^s\mathbf{1}_{\{Y^1_{r-}>Y^2_{r-}\}}dK^1_r=\int_t^s\mathbf{1}_{\{\mathfrak{L}^1_{r-}=Y^1_{r-}>Y^2_{r-}\}}dK^1_r\le\int_t^s\mathbf{1}_{\{\mathfrak{L}^1_{r-}>\mathfrak{L}^2_{r-}\}}dK^1_r=0,\quad\forall\, t,s\in[\tau,\zeta] \text{ with } t<s.$$

It follows that $P-$a.s.

$$\int_t^s\mathbf{1}_{\{Y^1_{r-}>Y^2_{r-}\}}(dK^1_r-dK^2_r)=-\int_t^s\mathbf{1}_{\{Y^1_{r-}>Y^2_{r-}\}}dK^2_r\le0,\quad\forall\, t,s\in[\tau,\zeta] \text{ with } t<s.$$

Then applying Proposition 3.1 with $V^i=K^i$, $i=1,2$ leads to the conclusion. □

Proof of Proposition 3.2: 1) Let (Y^i,Z^i,U^i,K^i), $i=1,2$ be two $\mathbb{L}^p-$solutions of RBSDEJ $(\xi,\mathfrak{g},\mathfrak{L})$. Applying Corollary 3.1 with parameters $(\xi_1,g^1,\mathfrak{L}^1)=(\xi_2,g^2,\mathfrak{L}^2)=(\xi,\mathfrak{g},\mathfrak{L})$ over period $[\![\tau,\gamma]\!]=[0,T]$, we directly obtain that $P\{Y^1_t=Y^2_t,\ \forall\, t\in[0,T]\}=1$. So it suffices to show that there exists a $\mathbb{L}^p-$solution to the RBSDEJ $(\xi,\mathfrak{g},\mathfrak{L})$.

In light of Theorem 2.1, the BSDEJ $(\xi,\mathfrak{g})$ admits a unique solution $(\mathcal{Y},\mathcal{Z},\mathcal{U})\in\mathbb{S}^p$. Set $\widetilde{\mathfrak{L}}:=\mathfrak{L}\vee(\mathcal{Y}-1)$.

1a) We first show that process $\{\mathcal{Y}_t+\int_0^t\mathfrak{g}_sds\}_{t\in[0,T]}$ is the Snell envelope of a real-valued, $\mathbf{F}-$adapted càdlàg process

$$\mathcal{R}_t:=\int_0^t\mathfrak{g}_sds+\big(\mathcal{Y}_t\wedge\widetilde{\mathfrak{L}}_t\big)\mathbf{1}_{\{t<T\}}+\xi\mathbf{1}_{\{t=T\}},\quad t\in[0,T]. \quad (5.14)$$

Since $\mathcal{Y}_t-1\le\mathcal{Y}_t\wedge\widetilde{\mathfrak{L}}_t\le\widetilde{\mathfrak{L}}_t\le\mathfrak{L}^+_t\vee\big(|\mathcal{Y}_t|+1\big)$, $P-$a.s. for any $t\in[0,T)$, one can deduce from the right-continuity of processes $\mathfrak{L}$ and $\mathcal{Y}$ that $\mathcal{R}_*\le1+\int_0^T|\mathfrak{g}_s|ds+\mathcal{Y}_*+\mathfrak{L}^+_*+|\xi|$, $P-$a.s. It follows from (1.6) that

$$E\big[\mathcal{R}^p_*\big]\le5^{p-1}E\bigg[1+\Big(\int_0^T|\mathfrak{g}_s|ds\Big)^p+\mathcal{Y}^p_*+(\mathfrak{L}^+_*)^p+|\xi|^p\bigg]<\infty,\ \text{thus } \mathcal{R}\in\mathbb{D}^p. \quad (5.15)$$

Let $t\in[0,T]$. For any $\rho\in\mathcal{T}_t$, since $\mathcal{Y}_\rho\geq\big(\mathcal{Y}_\rho\wedge\widetilde{\mathfrak{L}}_\rho\big)\mathbf{1}_{\{\rho<T\}}+\xi\mathbf{1}_{\{\rho=T\}}=\mathcal{R}_\rho-\int_0^\rho\mathfrak{g}_sds$, $P-$a.s., it holds $P-$a.s. that

$$\mathcal{Y}_t+\int_0^t\mathfrak{g}_sds=\mathcal{Y}_\rho+\int_0^\rho\mathfrak{g}_sds-\big(M_\rho+M_t-\mathcal{M}_\rho+\mathcal{M}_t\big)\geq\mathcal{R}_\rho-\big(M_\rho+M_t-\mathcal{M}_\rho+\mathcal{M}_t\big). \tag{5.16}$$

where $M_s:=\int_0^s\mathcal{Z}_rdB_r$ and $\mathcal{M}_s:=\int_{(0,s]}\int_{\mathcal{X}}\mathcal{U}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx)$, $s\in[0,T]$ are two uniformly integrable martingales by (1.3) and (1.4). Taking conditional expectation $E[\cdot|\mathcal{F}_t]$ in (5.16) yields

$$\mathcal{Y}_t+\int_0^t\mathfrak{g}_sds\geq E[\mathcal{R}_\rho|\mathcal{F}_t],\quad P-\text{a.s.} \tag{5.17}$$

In light of the Debut Theorem (see e.g. Theorem IV.50 of [10]), $\tau_t:=\inf\big\{s\in[t,T]:\mathcal{Y}_s-\mathfrak{L}_s\leq 0\big\}\wedge T$ defines a $\mathcal{T}_t-$stopping time. The right-continuity of process $\mathcal{Y}-\mathfrak{L}$ shows that for $P-$a.s. $\omega\in\{\tau_t<T\}$, one has $\mathcal{Y}\big(\tau_t(\omega),\omega\big)\leq\mathfrak{L}\big(\tau_t(\omega),\omega\big)$. Then it holds $P-$a.s. that

$$\begin{aligned}\mathcal{Y}_t+\int_0^t\mathfrak{g}_sds &= \mathcal{Y}_{\tau_t}+\int_0^{\tau_t}\mathfrak{g}_sds-\int_t^{\tau_t}\mathcal{Z}_sdB_s-\int_{(t,\tau_t]}\int_{\mathcal{X}}\mathcal{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx)\\ &= \big(\mathcal{Y}_{\tau_t}\wedge\widetilde{\mathfrak{L}}_{\tau_t}\big)\mathbf{1}_{\{\tau_t<T\}}+\xi\mathbf{1}_{\{\tau_t=T\}}+\int_0^{\tau_t}\mathfrak{g}_sds-\int_t^{\tau_t}\mathcal{Z}_sdB_s-\int_{(t,\tau_t]}\int_{\mathcal{X}}\mathcal{U}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx).\end{aligned}$$

Taking conditional expectation $E[\cdot|\mathcal{F}_t]$ yields that $\mathcal{Y}_t+\int_0^t\mathfrak{g}_sds=E\big[\mathcal{R}_{\tau_t}|\mathcal{F}_t\big]$, $P-$a.s., which together with (5.17) shows

$$\mathcal{Y}_t+\int_0^t\mathfrak{g}_sds=\underset{\rho\in\mathcal{T}_t}{\operatorname{esssup}}\,E[\mathcal{R}_\rho|\mathcal{F}_t],\quad P-\text{a.s.} \tag{5.18}$$

1b) Next, we denote by $\mathfrak{S}$ the Snell envelope of the real-valued, $\mathbf{F}-$adapted càdlàg process

$$\mathfrak{R}_t:=\int_0^t\mathfrak{g}_sds+\widetilde{\mathfrak{L}}_t\mathbf{1}_{\{t<T\}}+\xi\mathbf{1}_{\{t=T\}},\quad t\in[0,T] \tag{5.19}$$

and set $Y_t:=\mathfrak{S}_t-\int_0^t\mathfrak{g}_sds$, $t\in[0,T]$. By (A.5), it holds for any $t\in[0,T]$ that

$$Y_t=\underset{\rho\in\mathcal{T}_t}{\operatorname{esssup}}\,E[\mathfrak{R}_\rho|\mathcal{F}_t]-\int_0^t\mathfrak{g}_sds\geq\mathfrak{R}_t-\int_0^t\mathfrak{g}_sds=\widetilde{\mathfrak{L}}_t\mathbf{1}_{\{t<T\}}+\xi\mathbf{1}_{\{t=T\}}\geq\mathfrak{L}_t,\quad P-\text{a.s.}$$

It follows from the right-continuity of processes Y and $\mathfrak{L}$ that $P\{Y_t\geq\mathfrak{L}_t,\ \forall\,t\in[0,T]\}=1$.

Similar to (5.15), one can show that $E\big[\mathfrak{R}_*^p\big]\leq 5^{p-1}E\Big[1+\big(\int_0^T|\mathfrak{g}_s|ds\big)^p+\mathcal{Y}_*^p+(\mathfrak{L}_*^+)^p+|\xi|^p\Big]<\infty$, namely, $\mathfrak{R}\in\mathbb{D}^p$. In light of Proposition A.5 and Proposition A.6, there exist a martingale $M\in\mathbb{D}^p$ and a process $K\in\mathbb{K}^p$ such that $P-$a.s.

$$Y_t+\int_0^t\mathfrak{g}_sds=\mathfrak{S}_t=M_t-K_t,\quad t\in[0,T], \tag{5.20}$$

and that

$$\int_0^T\mathbf{1}_{\{Y_{t-}>\widetilde{\mathfrak{L}}_{t-}\}}dK_t=\int_0^T\mathbf{1}_{\{Y_{t-}+\int_0^t\mathfrak{g}_sds>\mathfrak{R}_{t-}\}}dK_t=0,\quad P-\text{a.s.} \tag{5.21}$$

The equality (5.20) together with (1.6) shows that $E[Y_*^p]\leq 3^{p-1}E\Big[\big(\int_0^T|\mathfrak{g}_t|dt\big)^p+M_*^p+K_T^p\Big]<\infty$, i.e. $Y\in\mathbb{D}^p$.

As $M_T\in L^p(\mathcal{F}_T)$, a martingale representation theorem, Corollary 2.1, implies that there exists a unique pair $(Z,U)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$ such that $P-$a.s.

$$M_t=E[M_T|\mathcal{F}_t]=E[M_T]+\int_0^tZ_sdB_s+\int_{(0,t]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T]. \tag{5.22}$$

which together with (5.20) leads to that $P-$a.s.

$$Y_t=\xi+\int_t^T\mathfrak{g}_sds+K_T-K_t-\int_t^TZ_sdB_s-\int_{(t,T]}\int_{\mathcal{X}}U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx),\quad t\in[0,T]. \tag{5.23}$$

Since $\mathfrak{R}_t \ge \mathcal{R}_t$, $\forall\, t\in[0,T]$, (5.18) shows that $P\{Y_t \ge \mathcal{Y}_t,\ \forall\, t\in[0,T]\}=1$. Then we can deduce from (5.21) that

$$0 \le \int_0^T \mathbf{1}_{\{Y_{t-} > \mathfrak{L}_{t-}\}} dK_t = \int_0^T \mathbf{1}_{\{Y_{t-} > \widetilde{\mathfrak{L}}_{t-}\}} dK_t = 0, \quad P-\text{a.s.}$$

It follows that $\int_0^T (Y_{t-} - \mathfrak{L}_{t-})\, dK_t = 0$, $P-$a.s. Therefore, (Y,Z,U,K) is a solution of RBSDEJ $(\xi,\mathfrak{g},\mathfrak{L})$.

2) *Next, let us show the continuity of process K under (L1) or (L2).*

2a) Assume first that (L1) holds and carry on all notations/results of Part (1). For any $\mathbf{F}-$predictable stopping time τ, the flat-off condition in RBSDE $(\xi,g,\mathfrak{L})$ and (3.2) imply that $0 \le \Delta K^d_\tau = \mathbf{1}_{\{Y_{\tau-}=\mathfrak{L}_{\tau-}\}}(\mathfrak{L}_{\tau-} - Y_\tau)^+ \le \mathbf{1}_{\{Y_{\tau-}=\mathfrak{L}_{\tau-}\}}(\mathfrak{L}_{\tau-} - \mathfrak{L}_\tau)^+ = 0$, $P-$a.s. Then one can deduce from Remark 3.2 (1) that $P\{\Delta K^d_t = 0,\ \forall\, t\in[0,T]\}=1$ or K is a continuous process.

2b) Then, let us assume that $\mathfrak{L}$ satisfies (L2). We have to adjust the arguments in Part (1) by resetting $\widetilde{\mathfrak{L}} = \mathfrak{L}$: we still set process $\mathcal{R}$ as in (5.14). Since $-|\mathcal{Y}_t| - |\mathfrak{L}_t| \le \mathcal{Y}_t \wedge \widetilde{\mathfrak{L}}_t \le \mathfrak{L}_t$, $P-$a.s. for any $t\in[0,T)$, the right-continuity of $\mathfrak{L}$ and $\mathcal{Y}$ implies that $\mathcal{R}_* \le \int_0^T |\mathfrak{g}_s| ds + \mathcal{Y}_* + \mathfrak{L}_* + |\xi|$, $P-$a.s. Analogous to (5.15), one can deduce from (1.6) and $\mathfrak{L}\in\mathbb{D}^p$ that $\mathcal{R}\in\mathbb{D}^p$. Then following the same arguments in Part (1a), we can again show that $\{\mathcal{Y}_t + \int_0^t \mathfrak{g}_s ds\}_{t\in[0,T]}$ is the Snell envelope of $\mathcal{R}$.

Still set process $\mathfrak{R}$ as in (5.19), which is clearly of $\mathbb{D}^p$. Let $\mathfrak{S}$ denote the Snell envelope of $\mathfrak{R}$ and set $Y_t := \mathfrak{S}_t - \int_0^t \mathfrak{g}_s ds$, $t\in[0,T]$. By Proposition A.5 and Proposition A.6, one can still find a martingale $M\in\mathbb{D}^p$ and a process $K\in\mathbb{K}^p$ such that (5.20) holds and that

$$\int_0^T \mathbf{1}_{\{Y_{t-} > \mathfrak{L}_{t-}\}} dK_t = \int_0^T \mathbf{1}_{\{Y_{t-} > \widetilde{\mathfrak{L}}_{t-}\}} dK_t = \int_0^T \mathbf{1}_{\{Y_{t-} + \int_0^t \mathfrak{g}_s ds > \mathfrak{R}_{t-}\}} dK_t = 0, \quad P-\text{a.s.}$$

The equality (5.20) and (1.6) shows $Y\in\mathbb{D}^p$.

As $M_T \in L^p(\mathcal{F}_T)$, Corollary 2.1 implies that there exists a unique pair $(Z,U)\in\mathbb{Z}^{2,p}\times\mathbb{U}^p$ such that (5.22) holds, which together with (5.20) leads to (5.23). Hence (Y,Z,U,K) solves RBSDEJ $(\xi,\mathfrak{g},\mathfrak{L})$. Since $\xi \ge \mathfrak{L}_T$, $P-$a.s. , we can deduce from the l.u.s.c.e. of $\mathfrak{L}$ that the process $\mathfrak{R}_t = \int_0^t \mathfrak{g}_s ds + \mathfrak{L}_t \mathbf{1}_{\{t<T\}} + \xi \mathbf{1}_{\{t=T\}}$, $t\in\mathbb{R}$ is also l.u.s.c.e. By Example A.1, process K is continuous. □

Proof of Theorem 3.1 1): By the Burkholder-Davis-Gundy inequality, there exists a constant $\kappa>0$ such that

$$E[M_*] \le \kappa E\big\{[M,M]_T^{1/2}\big\} \tag{5.24}$$

for any càdlàg local martingale M. We set constants $\wp_1 := \big(\frac{q}{4}\big)^{\frac{1}{q}}$, $\wp_2 := (\nu(\mathcal{X})T)^{\frac{1}{p}} 2^{\frac{2}{p}+1} \kappa p^{1-\frac{1}{p}} \wp_1^{-1}$, $\wp_3 := T+1-\frac{p}{2}+\wp_2^{-p}\nu(\mathcal{X})T$, $\wp_4 := \frac{2}{p-1}\big(16\kappa^2 p^2 \wp_3 + \frac{p}{2}\big)$, $\wp_5 := \frac{2}{p-1}(1+\wp_2)^{2-p}\big(2^{2+p}\kappa^p p^{p-1}\wp_1^{-p}\wp_3 + 1\big)$ and $\wp_6 := (4p\wp_3 + \wp_4 + \wp_5)^{-\frac{1}{p}}$. Define processes

$$a_t := \frac{2\beta_t^q + \widehat{C}^{\frac{q}{2}-1}\Lambda_t^2}{q\wp_6^{p+q}(\wp_4+\wp_5)} \quad \text{and} \quad A_t := p\int_0^t a_s ds, \quad t\in[0,T].$$

Then $C_A := \|A_T\|_{L^\infty(\mathcal{F}_T)} \le \dfrac{(p-1)(4p\wp_3+\wp_4+\wp_5)}{\wp_6^q(\wp_4+\wp_5)}\big(2\widehat{C} + \widehat{C}^{\frac{q}{2}}\big)$.

Let us introduce the following norm on $\mathbb{S}^p$:

$$\big\|(Y,Z,U)\big\|_\sharp := \left\{E\left[T\sup_{t\in[0,T]}\big(e^{A_t}|Y_t|^p\big) + \Big(\int_0^T e^{\frac{2}{p}A_t}|Z_t|^2\, dt\Big)^{\frac{p}{2}} + \int_0^T\int_{\mathcal{X}} e^{A_t}|U_t(x)|^p \nu(dx)dt\right]\right\}^{\frac{1}{p}}, \ \forall\,(Y,Z,U)\in\mathbb{S}^p.$$

Fix $(Y,Z,U)\in\mathbb{S}^p$. The $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}\big(L^p_\nu\big)/\mathscr{B}(\mathbb{R})-$measurability of generator g, the $\mathbf{F}-$predictability of processes (Y,Z) as well as the $\widehat{\mathscr{P}}\otimes\mathcal{F}_{\mathcal{X}}-$measurability of random field U implies that process $\mathfrak{g}_t := g(t,Y_t,Z_t,U_t)$, $t\in[0,T]$ is $\mathbf{F}-$progressively measurable. Similar to (5.81) of [42], we can deduce from (A2'), (1.6) and Hölder's inequality that

$$E\bigg[\Big(\int_0^T |\mathfrak{g}_t| dt\Big)^p\bigg] \le 4^{p-1}E\bigg[\Big(\int_0^T \big|g(t,0,0,0)\big| dt\Big)^p\bigg] + 4^{p-1}\Big(\widehat{C}^{p-1}T\|Y\|^p_{\mathbb{D}^p} + \widehat{C}^{\frac{p}{2}}\|Z\|^p_{\mathbb{Z}^{2,p}} + \widehat{C}^{p-1}\|U\|^p_{\mathbb{U}^p}\Big) < \infty.$$

As $\mathfrak{g}$ clearly satisfies (A2'), Proposition 3.2 shows that the RBSDEJ $(\xi,\mathfrak{g},\mathfrak{L})$ admits a unique solution $(\mathcal{Y},\mathcal{Z},\mathcal{U},\mathcal{K})\in\mathbb{S}^p\times\mathbb{K}^p$.

We set $\Psi(Y,Z,U):=(\mathcal{Y},\mathcal{Z},\mathcal{U})$. To see that Ψ defines a contraction map on $\mathbb{S}^p$ under the norm $\|\cdot\|_\sharp$, let $\big(\widetilde{Y},\widetilde{Z},\widetilde{U}\big)$ be another triplet in $\mathbb{S}^p$ and let $\big(\widetilde{\mathcal{Y}},\widetilde{\mathcal{Z}},\widetilde{\mathcal{U}},\widetilde{\mathcal{K}}\big)$ be the unique solution to the RBSDEJ $\big(\xi,\widetilde{\mathfrak{g}},\mathfrak{L}\big)$ with $\widetilde{\mathfrak{g}}_t:=g\big(t,\widetilde{Y}_t,\widetilde{Z}_t,\widetilde{U}_t\big)$, $t\in[0,T]$, so $\Psi\big(\widetilde{Y},\widetilde{Z},\widetilde{U}\big)=\big(\widetilde{\mathcal{Y}},\widetilde{\mathcal{Z}},\widetilde{\mathcal{U}}\big)$. For simplicity, we denote $(\mathscr{Y},\mathscr{Z},\mathscr{U}):=\big(\mathcal{Y}-\widetilde{\mathcal{Y}},\mathcal{Z}-\widetilde{\mathcal{Z}},\mathcal{U}-\widetilde{\mathcal{U}}\big)$.

2) Given $\varepsilon\in(0,1]$, the function $\varphi_\varepsilon(x):=\big(|x|^2+\varepsilon\big)^{\frac12}$, $x\in\mathbb{R}$ has the following derivatives of its $p-$th power:

$$D\,\varphi^p_\varepsilon(x)=p\,\varphi^{p-2}_\varepsilon(x)\,x\quad\text{and}\quad D^2\,\varphi^p_\varepsilon(x)=p\,\varphi^{p-2}_\varepsilon(x)+p(p-2)\varphi^{p-4}_\varepsilon(x)\,x^2\ge p(p-1)\varphi^{p-2}_\varepsilon(x).\tag{5.25}$$

Let $(t,\varepsilon)\in[0,T]\times(0,1]$. Applying Itô's formula to process $e^{A_s}\varphi^p_\varepsilon(\mathscr{Y}_s)$ over the interval $[t,T]$, we see from Remark 3.2 (3) that $P-$a.s.

$$\begin{aligned}&e^{A_t}\varphi^p_\varepsilon(\mathscr{Y}_t)+\frac12\int_t^T e^{A_s}D^2\varphi^p_\varepsilon(\mathscr{Y}_s)|\mathscr{Z}_s|^2ds+\sum_{s\in(t,T]}e^{A_s}\Big(\varphi^p_\varepsilon(\mathscr{Y}_s)-\varphi^p_\varepsilon(\mathscr{Y}_{s-})-D\varphi^p_\varepsilon(\mathscr{Y}_{s-})\Delta\mathscr{Y}_s\Big)\\&\quad=e^{A_T}\varepsilon^{\frac p2}+p\int_t^T e^{A_s}\big[\varphi^{p-2}_\varepsilon(\mathscr{Y}_s)\mathscr{Y}_s(\mathfrak{g}_s-\widetilde{\mathfrak{g}}_s)-a_s\varphi^p_\varepsilon(\mathscr{Y}_s)\big]ds+p\int_t^T e^{A_s}\varphi^{p-2}_\varepsilon(\mathscr{Y}_{s-})\mathscr{Y}_{s-}\big(d\mathcal{K}_s-d\widetilde{\mathcal{K}}_s\big)\\&\quad\quad-p\big(\mathscr{M}_T-\mathscr{M}_t+\mathfrak{M}_T-\mathfrak{M}_t\big),\end{aligned}\tag{5.26}$$

where $\mathscr{M}_s:=\mathscr{M}^\varepsilon_s=\int_0^s e^{A_r}\varphi^{p-2}_\varepsilon(\mathscr{Y}_{r-})\mathscr{Y}_{r-}\mathscr{Z}_r dB_r$ and $\mathfrak{M}_s:=\mathfrak{M}^\varepsilon_s=\int_{(0,s]}\int_{\mathcal{X}}e^{A_r}\varphi^{p-2}_\varepsilon(\mathscr{Y}_{r-})\mathscr{Y}_{r-}\mathscr{U}_r(x)\widetilde{N}_{\mathfrak{p}}(dr,dx)$, $\forall\,s\in[0,T]$. Similar to (5.10) of [42], we can deduce from Taylor's Expansion Theorem and (5.25) that

$$\varphi^p_\varepsilon\big(\mathscr{Y}_s\big)-\varphi^p_\varepsilon\big(\mathscr{Y}_{s-}\big)-D\varphi^p_\varepsilon\big(\mathscr{Y}_{s-}\big)\Delta\mathscr{Y}_s\ge p(p-1)|\Delta\mathscr{Y}_s|^2\int_0^1(1-\alpha)\varphi^{p-2}_\varepsilon(\mathscr{Y}_{s-}+\alpha\Delta\mathscr{Y}_s)d\alpha.$$

When $|\mathscr{Y}_{s-}|\le\wp_2|\Delta\mathscr{Y}_s|$, one has $\varphi^{p-2}_\varepsilon(\mathscr{Y}_{s-}+\alpha\Delta\mathscr{Y}_s)\ge\big((|\mathscr{Y}_{s-}|+\alpha|\Delta\mathscr{Y}_s|)^2+\varepsilon\big)^{\frac p2-1}\ge\big((1+\wp_2)^2|\Delta\mathscr{Y}_s|^2+\varepsilon\big)^{\frac p2-1}\ge(1+\wp_2)^{p-2}\big(|\Delta\mathscr{Y}_s|^2+\varepsilon\big)^{\frac p2-1}$, $\forall\,\alpha\in[0,1]$. So an analogy to (5.11) of [42] and (3.1) show that for $P-$a.s. $\omega\in\Omega$

$$\begin{aligned}&\sum_{s\in(t,T]}e^{A_s}\Big(\varphi^p_\varepsilon\big(\mathscr{Y}_s(\omega)\big)-\varphi^p_\varepsilon\big(\mathscr{Y}_{s-}(\omega)\big)-D\varphi^p_\varepsilon\big(\mathscr{Y}_{s-}(\omega)\big)\Delta\mathscr{Y}_s(\omega)\Big)\\&\quad\ge\frac12(1+\wp_2)^{p-2}p(p-1)\sum_{s\in(t,T]}\mathbf{1}_{\{|\mathscr{Y}_{s-}(\omega)|\le\wp_2|\Delta\mathscr{Y}_s(\omega)|\}}e^{A_s(\omega)}\big|\Delta\mathscr{Y}_s(\omega)\big|^2\big(|\Delta\mathscr{Y}_s(\omega)|^2+\varepsilon\big)^{\frac p2-1}\\&\quad\ge\frac12(1+\wp_2)^{p-2}p(p-1)\sum_{s\in D_{\mathfrak{p}(\omega)}\cap(t,T]}\mathbf{1}_{\{|\mathscr{Y}_{s-}|\le\wp_2|\mathscr{U}_s(x)|\}}e^{A_s}\big|\mathscr{U}\big(s,\omega,\mathfrak{p}_s(\omega)\big)\big|^2\Big(\big|\mathscr{U}\big(s,\omega,\mathfrak{p}_s(\omega)\big)\big|^2+\varepsilon\Big)^{\frac p2-1}\\&\quad=\frac12(1+\wp_2)^{p-2}p(p-1)\bigg(\int_{(t,T]}\int_{\mathcal{X}}\mathbf{1}_{\{|\mathscr{Y}_{s-}|\le\wp_2|\mathscr{U}_s(x)|\}}e^{A_s}\big|\mathscr{U}_s(x)\big|^2\big(|\mathscr{U}_s(x)|^2+\varepsilon\big)^{\frac p2-1}N_{\mathfrak{p}}(ds,dx)\bigg)(\omega).\end{aligned}$$

The function $\psi(x):=x\varphi^{p-2}_\varepsilon(x)=x(x^2+\varepsilon)^{\frac p2-1}$, $x\in\mathbb{R}$ has strictly positive derivative $\frac{d}{dx}\psi(x)=(x^2+\varepsilon)^{\frac p2-2}\big((p-1)x^2+\varepsilon\big)>0$, so it satisfies $\psi(x)\le\psi(x^+)\le(x^+)^{p-1}$, $\forall\,x\in\mathbb{R}$. Then the flat-off condition implies that $P-$a.s.

$$\begin{aligned}\int_t^T e^{A_s}\varphi^{p-2}_\varepsilon(\mathscr{Y}_{s-})\mathscr{Y}_{s-}\big(d\mathcal{K}_s-d\widetilde{\mathcal{K}}_s\big)&=\int_t^T\mathbf{1}_{\{\mathcal{Y}_{s-}=\mathfrak{L}_{s-}\}}e^{A_s}\psi\big(\mathfrak{L}_{s-}-\widetilde{\mathcal{Y}}_{s-}\big)d\mathcal{K}_s+\int_t^T\mathbf{1}_{\{\widetilde{\mathcal{Y}}_{s-}=\mathfrak{L}_{s-}\}}e^{A_s}\psi\big(\mathfrak{L}_{s-}-\mathcal{Y}_{s-}\big)d\widetilde{\mathcal{K}}_s\\&\le\int_t^T\mathbf{1}_{\{\mathcal{Y}_{s-}=\mathfrak{L}_{s-}\}}e^{A_s}\big((\mathfrak{L}_{s-}-\widetilde{\mathcal{Y}}_{s-})^+\big)^{p-1}d\mathcal{K}_s+\int_t^T\mathbf{1}_{\{\widetilde{\mathcal{Y}}_{s-}=\mathfrak{L}_{s-}\}}e^{A_s}\big((\mathfrak{L}_{s-}-\mathcal{Y}_{s-})^+\big)^{p-1}d\widetilde{\mathcal{K}}_s=0.\end{aligned}$$

Set $\eta^\varepsilon_t:=\int_t^T e^{A_s}\big[\varphi^{p-1}_\varepsilon(\mathscr{Y}_s)|\mathfrak{g}_s-\widetilde{\mathfrak{g}}_s|-a_s\varphi^p_\varepsilon(\mathscr{Y}_s)\big]ds$, then (5.26) and (5.25) imply that $P-$a.s.

$$\begin{aligned}&e^{A_t}\varphi^p_\varepsilon(\mathscr{Y}_t)+\frac{p(p-1)}{2}\int_t^T e^{A_s}\varphi^{p-2}_\varepsilon(\mathscr{Y}_s)|\mathscr{Z}_s|^2ds+\frac{p(p-1)}{2}(1+\wp_2)^{p-2}\int_{(t,T]}\int_{\mathcal{X}}\mathbf{1}_{\{|\mathscr{Y}_{s-}|\le\wp_2|\mathscr{U}_s(x)|\}}e^{A_s}\big|\mathscr{U}_s(x)\big|^2\big(|\mathscr{U}_s(x)|^2+\varepsilon\big)^{\frac p2-1}N_{\mathfrak{p}}(ds,dx)\\&\quad\le e^{C_A}\varepsilon^{\frac p2}+p\,\eta^\varepsilon_t-p\big(\mathscr{M}_T-\mathscr{M}_t+\mathfrak{M}_T-\mathfrak{M}_t\big).\end{aligned}\tag{5.27}$$

Since the random variable $\Phi_\varepsilon := \sup_{t\in[0,T]} \big(e^{A_t}\varphi^p_\varepsilon(\mathscr{Y}_t)\big)$ satisfies $E\big[\Phi_\varepsilon\big] \le e^{C_A}E\big[\mathscr{Y}^p_* + \varepsilon^{\frac{p}{2}}\big] = e^{C_A}\big(\|\mathscr{Y}\|^p_{\mathbb{D}^p} + \varepsilon^{\frac{p}{2}}\big) < \infty$ by (1.6), we can deduce from (5.24), Young's inequality and (1.5) that

$$
\begin{aligned}
E\bigg[\sup_{s\in[0,T]}|\mathscr{M}_s| + \sup_{s\in[0,T]}|\mathfrak{M}_s|\bigg] \;&\le\; \kappa E\Bigg[(\Phi_\varepsilon)^{\frac{p-1}{p}}\bigg(\int_0^T e^{\frac{2}{p}A_s}|\mathscr{Z}_s|^2 ds\bigg)^{\frac{1}{2}} + (\Phi_\varepsilon)^{\frac{p-1}{p}}\bigg(\int_{(0,T]}\int_{\mathcal{X}} e^{\frac{2}{p}A_s}|\mathscr{U}_s(x)|^2 N_{\mathfrak{p}}(ds,dx)\bigg)^{\frac{1}{2}}\Bigg] \\
&\le\; \frac{\kappa}{p}E\Bigg[2(p-1)\Phi_\varepsilon + e^{C_A}\bigg(\int_0^T |\mathscr{Z}_s|^2 ds\bigg)^{\frac{p}{2}} + e^{C_A}\int_0^T\int_{\mathcal{X}}|\mathscr{U}_s(x)|^p\nu(dx)ds\Bigg] < \infty.
\end{aligned}
$$

So both $\mathscr{M}$ and $\mathfrak{M}$ are uniformly integrable martingales. Taking $t=0$ and taking expectation in (5.27) yields that

$$
\begin{aligned}
&E\int_0^T e^{A_s}\varphi^{p-2}_\varepsilon(\mathscr{Y}_s)|\mathscr{Z}_s|^2 ds + (1+\wp_2)^{p-2}E\int_0^T\int_{\mathcal{X}} \mathbf{1}_{\{|\mathscr{Y}_{s-}|\le\wp_2|\mathscr{U}_s(x)|\}} e^{A_s}\big|\mathscr{U}_s(x)\big|^2\big(|\mathscr{U}_s(x)|^2+\varepsilon\big)^{\frac{p}{2}-1}\nu(dx)ds \\
&\qquad \le \frac{2}{p(p-1)}\big(e^{C_A}\varepsilon^{\frac{p}{2}} + E[\eta^\varepsilon_0]\big).
\end{aligned}
$$

Using similar arguments to those that lead to (5.94) of [42], we can deduce that $\big\|\big(\mathcal{Y}-\widetilde{\mathcal{Y}}, \mathcal{Z}-\widetilde{\mathcal{Z}}, \mathcal{U}-\widetilde{\mathcal{U}}\big)\big\|^p_\sharp \le \frac{1}{p}\big\|\big(Y-\widetilde{Y}, Z-\widetilde{Z}, U-\widetilde{U}\big)\big\|^p_\sharp$. Hence, Ψ is a contraction mapping on $\mathbb{S}^p$ under the norm $\|\cdot\|_\sharp$, which admits a unique fixed point $(Y,Z,U)\in\mathbb{S}^p$.

Let K be the process in $\mathbb{K}^p$ that is associated to $\Psi(Y,Z,U)=(Y,Z,U)$. Then (Y,Z,U,K) forms a unique solution of RBSDEJ $(\xi,g,\mathfrak{L})$ in $\mathbb{S}^p\times\mathbb{K}^p$. If either (L1) or (L2) holds, Proposition 3.2 shows that K is a continuous process. □

Proof of Proposition 4.1 1): Let $\gamma\in\mathcal{T}$ and $\zeta\in\mathcal{T}_\gamma$. Theorem 2.1 implies that $P-$a.s.

$$
Y^{Y_\zeta,g_\zeta}_t = Y_\zeta + \int_t^\zeta g\big(s, Y^{Y_\zeta,g_\zeta}_s, Z^{Y_\zeta,g_\zeta}_s, U^{Y_\zeta,g_\zeta}_s\big)ds - \int_t^\zeta Z^{Y_\zeta,g_\zeta}_s dB_s - \int_{(t,\zeta]}\int_{\mathcal{X}} U^{Y_\zeta,g_\zeta}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \;\; \forall\, t\in[0,\zeta]. \tag{5.28}
$$

Since it holds $P-$a.s. that

$$
Y_t = Y_\zeta + \int_t^\zeta g(s,Y_s,Z_s,U_s)ds + K_\zeta - K_t - \int_t^\zeta Z_s dB_s - \int_{(t,\zeta]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \qquad \forall\, t\in[0,\zeta],
$$

applying Proposition 3.1 with $(Y^1,Z^1,U^1,V^1) = \big(Y^{Y_\zeta,g_\zeta}, Z^{Y_\zeta,g_\zeta}, U^{Y_\zeta,g_\zeta}, 0\big)$ and $(Y^2,Z^2,U^2,V^2)=(Y,Z,U,K)$ over period $[0,\zeta]$ yields that $P\big\{Y^{Y_\zeta,g_\zeta}_t \le Y_t, \;\; \forall\, t\in[0,\zeta]\big\} = 1$. In particular, $\mathcal{E}^g_{\gamma,\zeta}[Y_\zeta] = Y^{Y_\zeta,g_\zeta}_\gamma \le Y_\gamma$, $P-$a.s., thus Y is a $g-$supermartingale.

As $Y_\zeta \ge \mathbf{1}_{\{\zeta<T\}}\mathfrak{L}_\zeta + \mathbf{1}_{\{\zeta=T\}}\xi = \mathcal{R}_\zeta$, $P-$a.s., the monotonicity of $g-$evaluation further shows that

$$
Y_\gamma \ge \mathcal{E}^g_{\gamma,\zeta}[Y_\zeta] \ge \mathcal{E}^g_{\gamma,\zeta}[\mathcal{R}_\zeta], \quad P-\text{a.s.} \tag{5.29}
$$

Let $\widetilde{\zeta}\in\mathcal{T}_\gamma$ satisfies that $K_{\widetilde{\zeta}} = K_\gamma$, $P-$a.s. Then it follows that $P-$a.s.

$$
Y_t = Y_{\widetilde{\zeta}} + \int_t^{\widetilde{\zeta}} g(s,Y_s,Z_s,U_s)ds - \int_t^{\widetilde{\zeta}} Z_s dB_s - \int_{(t,\widetilde{\zeta}]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \qquad \forall\, t\in\big[\gamma,\widetilde{\zeta}\big].
$$

Similar to (5.28), It holds $P-$a.s. that

$$
Y^{Y_{\widetilde{\zeta}},g_{\widetilde{\zeta}}}_t = Y_{\widetilde{\zeta}} + \int_t^{\widetilde{\zeta}} g\big(s, Y^{Y_{\widetilde{\zeta}},g_{\widetilde{\zeta}}}_s, Z^{Y_{\widetilde{\zeta}},g_{\widetilde{\zeta}}}_s, U^{Y_{\widetilde{\zeta}},g_{\widetilde{\zeta}}}_s\big)ds - \int_t^{\widetilde{\zeta}} Z^{Y_{\widetilde{\zeta}},g_{\widetilde{\zeta}}}_s dB_s - \int_{(t,\widetilde{\zeta}]}\int_{\mathcal{X}} U^{Y_{\widetilde{\zeta}},g_{\widetilde{\zeta}}}_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad \forall\, t\in\big[0,\widetilde{\zeta}\big].
$$

Applying Proposition 3.1 with $(Y^1,Z^1,U^1,V^1) = \big(Y^{Y_{\widetilde{\zeta}},g_{\widetilde{\zeta}}}, Z^{Y_{\widetilde{\zeta}},g_{\widetilde{\zeta}}}, U^{Y_{\widetilde{\zeta}},g_{\widetilde{\zeta}}}, 0\big)$ and $(Y^2,Z^2,U^2,V^2)=(Y,Z,U,0)$ over period $\big[\gamma,\widetilde{\zeta}\big]$ gives that $P\big\{Y_t = Y^{Y_{\widetilde{\zeta}},g_{\widetilde{\zeta}}}_t, \;\; \forall\, t\in\big[\gamma,\widetilde{\zeta}\big]\big\} = 1$. In particular,

$$
\mathcal{E}^g_{\gamma,\widetilde{\zeta}}\big[Y_{\widetilde{\zeta}}\big] = Y^{Y_{\widetilde{\zeta}},g_{\widetilde{\zeta}}}_\gamma = Y_\gamma, \quad P-\text{a.s.} \tag{5.30}
$$

Let $n\in\mathbb{N}$ and define $\tau_n(\gamma):=\{t\in[\gamma,T]: Y_t\le\mathcal{R}_t+1/n\}\in\mathcal{T}_\gamma$. As $Y_s>\mathcal{R}_s+1/n=\mathfrak{L}_s+1/n>\mathfrak{L}_s$ for any $s\in\big[\gamma,\tau_n(\gamma)\big)$, we see that $Y_{s-}\ge\mathfrak{L}_{s-}+1/n>\mathfrak{L}_{s-}$ for any $s\in\big(\gamma,\tau_n(\gamma)\big]$. Then the flat-off condition in RBSDE $(\xi,g,\mathfrak{L})$ and Remark 3.2 (1), (2) imply that $P-$a.s.,

$$K^c_{\tau_n(\gamma)}-K^c_\gamma=\int_\gamma^{\tau_n(\gamma)}\mathbf{1}_{\{Y_s=\mathfrak{L}_s\}}dK^c_s=0 \quad \text{and} \quad 0\le K^d_{\tau_n(\gamma)}-K^d_\gamma\le\sum_{s\in(\gamma,\tau_n(\gamma)]}\mathbf{1}_{\{Y_{s-}=\mathfrak{L}_{s-}\}}(\mathfrak{L}_{s-}-Y_s)^+=0.$$

Putting them together shows that

$$K_{\tau_n(\gamma)}=K_\gamma,\quad P-\text{a.s.} \tag{5.31}$$

For any $\gamma,\zeta\in\mathcal{T}$ with $0\le\gamma\le\zeta\le\tau_n(0)$, as $K_{\tau_n(0)}=K_0=0$, $P-$a.s. by (5.31), The monotonicity of K shows that $K_\zeta=K_\gamma=0$, $P-$a.s. Taking $\widetilde{\zeta}=\zeta$ in (5.30) yields that $\mathcal{E}^g_{\gamma,\zeta}[Y_\zeta]=Y_\gamma$, $P-$a.s. So Y is a $g-$martingale up to $\tau_n(0)$.
2) Let $\gamma\in\mathcal{T}$ and denote $\tau_n(\gamma)$ by $\widetilde{\tau}$. Since $Y_T=\xi=\mathcal{R}_T<\mathcal{R}_T+1/n$, $P-$a.s., the right-continuity of processes Y and $\mathcal{R}$ implies that $Y_{\widetilde{\tau}}\le\mathcal{R}_{\widetilde{\tau}}+1/n$, $P-$a.s. Applying Theorem 2.2 with $(Y^1,Z^1,U^1)=\big(Y^{Y_{\widetilde{\tau}},g_{\widetilde{\tau}}},Z^{Y_{\widetilde{\tau}},g_{\widetilde{\tau}}},U^{Y_{\widetilde{\tau}},g_{\widetilde{\tau}}}\big)$ and $(Y^2,Z^2,U^2)=\big(Y^{\mathcal{R}_{\widetilde{\tau}}+1/n,g_{\widetilde{\tau}}},Z^{\mathcal{R}_{\widetilde{\tau}}+1/n,g_{\widetilde{\tau}}},U^{\mathcal{R}_{\widetilde{\tau}}+1/n,g_{\widetilde{\tau}}}\big)$ over period $[0,\widetilde{\tau}]$ and taking $\widetilde{\zeta}=\widetilde{\tau}$ in (5.30), we see from (5.31) that

$$Y_\gamma=Y_\gamma^{Y_{\widetilde{\tau}},g_{\widetilde{\tau}}}\le Y^{\mathcal{R}_{\widetilde{\tau}}+1/n,g_{\widetilde{\tau}}}=Y_\gamma^{\mathcal{R}_{\widetilde{\tau}},g_{\widetilde{\tau}}}+\eta_n=\mathcal{E}^g_{\gamma,\widetilde{\tau}}[\mathcal{R}_{\widetilde{\tau}}]+\eta_n\le\underset{\zeta\in\mathcal{T}_\gamma}{\operatorname{esssup}}\,\mathcal{E}^g_{\gamma,\zeta}[\mathcal{R}_\zeta]+\eta_n,\quad P-\text{a.s.}, \tag{5.32}$$

where $\eta_n:=Y_\gamma^{\mathcal{R}_{\widetilde{\tau}}+1/n,g_{\widetilde{\tau}}}-Y_\gamma^{\mathcal{R}_{\widetilde{\tau}},g_{\widetilde{\tau}}}$. By (2.1), $E\big[|\eta_n|^p\big]\le\big\|Y^{\mathcal{R}_{\widetilde{\tau}}+1/n,g_{\widetilde{\tau}}}-Y^{\mathcal{R}_{\widetilde{\tau}},g_{\widetilde{\tau}}}\big\|^p_{\mathbb{D}^p}\le\frac{\mathcal{C}}{n^p}$ for some constant $\mathcal{C}$ only depending on T, $\nu(\mathcal{X})$, p and $\widehat{C}$. So there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim\limits_{i\to\infty}\eta_{n_i}=0$, $P-$a.s. Taking $n=n_i$ in (5.32) and letting $i\to\infty$, we can deduce from (5.29) that $Y_\gamma=\underset{\zeta\in\mathcal{T}_\gamma}{\operatorname{esssup}}\,\mathcal{E}^g_{\gamma,\zeta}[\mathcal{R}_\zeta]$, $P-$a.s.

3) Next, assume that the process K is continuous and let $\gamma\in\mathcal{T}$. As $Y_t>\mathcal{R}_t=\mathfrak{L}_t$ for any $t\in\big[\gamma,\tau_*(\gamma)\big)$, the flat-off condition in RBSDE $(\xi,g,\mathfrak{L})$ and (3.2) imply that $P-$a.s., $K_t-K_\gamma=K^c_t-K^c_\gamma=\int_\gamma^t\mathbf{1}_{\{Y_s=\mathfrak{L}_s\}}dK^c_s=0$ for any $t\in\big[\gamma,\tau_*(\gamma)\big]$. In particular, one has $K_{\tau_*(\gamma)}=K_\gamma$, $P-$a.s. Since $Y_T=\xi=\mathcal{R}_T$, $P-$a.s., we can deduce from the right-continuity of processes Y and $\mathcal{R}$ that $Y_{\tau_*(\gamma)}=\mathcal{R}_{\tau_*(\gamma)}$, $P-$a.s., Taking $\widetilde{\zeta}=\tau_*(\gamma)$ in (5.30) and using (4.1) yield

$$\underset{\zeta\in\mathcal{T}_\gamma}{\operatorname{esssup}}\,\mathcal{E}^g_{\gamma,\zeta}\big[\mathcal{R}_\zeta\big]=Y_\gamma=\mathcal{E}^g_{\gamma,\tau_*(\gamma)}\big[Y_{\tau_*(\gamma)}\big]=\mathcal{E}^g_{\gamma,\tau_*(\gamma)}\big[\mathcal{R}_{\tau_*(\gamma)}\big],\quad P-\text{a.s.} \tag{5.33}$$

Set $\mathcal{N}:=\{\omega\in\Omega:$ the path $K_\cdot(\omega)$ is continuous and the path $Y_\cdot(\omega)-\mathcal{R}_\cdot(\omega)$ is right-continuous$\}$ and $A:=\{Y_{\widehat{\tau}(\gamma)}>\mathcal{R}_{\widehat{\tau}(\gamma)}\}\in\mathcal{F}_T$. Clearly, $\mathcal{N}$ is a $P-$null set. Given $\omega\in A\cap\mathcal{N}^c\cap\{\widehat{\tau}(\gamma)<T\}$, there exists a $\delta(\omega)\in\big(0,T-\widehat{\tau}(\gamma)\big]$ such that $\mathfrak{m}(\delta):=\inf\limits_{t\in[\widehat{\tau}(\gamma)(\omega),\widehat{\tau}(\gamma)(\omega)+\delta(\omega)]}\big(Y_t(\omega)-\mathcal{R}_t(\omega)\big)>0$. Since

$$\int_0^T\big(Y_s(\omega)-\mathfrak{L}_s(\omega)\big)dK^c_s(\omega)\ge\int_{\widehat{\tau}(\gamma)(\omega)}^{\widehat{\tau}(\gamma)(\omega)+\delta(\omega)}\big(Y_s(\omega)-\mathfrak{L}_s(\omega)\big)dK^c_s(\omega)\ge\mathfrak{m}(\delta)\big(K\big(\widehat{\tau}(\gamma)(\omega)+\delta(\omega),\omega\big)-K\big(\widehat{\tau}(\gamma)(\omega),\omega\big)\big)>0,$$

we see from (3.2) that $P\big(A\cap\mathcal{N}^c\cap\{\widehat{\tau}(\gamma)<T\}\big)=0$ and thus $P\big(A\cap\{\widehat{\tau}(\gamma)<T\}\big)=0$. It follows that $Y_{\widehat{\tau}(\gamma)}-\mathcal{R}_{\widehat{\tau}(\gamma)}=\mathbf{1}_{\{\widehat{\tau}(\gamma)=T\}}(Y_T-\xi)+\mathbf{1}_{A^c\cap\{\widehat{\tau}(\gamma)<T\}}\big(Y_{\widehat{\tau}(\gamma)}-\mathcal{R}_{\widehat{\tau}(\gamma)}\big)=0$, $P-$a.s. As the continuity of K implies that $K_{\widehat{\tau}(\gamma)}=K_\gamma$, $P-$a.s., taking $\widetilde{\zeta}=\widehat{\tau}(\gamma)$ in (5.30) and using (4.1) yield that

$$\underset{\zeta\in\mathcal{T}_\gamma}{\operatorname{esssup}}\,\mathcal{E}^g_{\gamma,\zeta}\big[\mathcal{R}_\zeta\big]=Y_\gamma=\mathcal{E}^g_{\gamma,\widehat{\tau}(\gamma)}\big[Y_{\widehat{\tau}(\gamma)}\big]=\mathcal{E}^g_{\gamma,\widehat{\tau}(\gamma)}\big[\mathcal{R}_{\widehat{\tau}(\gamma)}\big],\quad P-\text{a.s.},$$

which together with (5.33) shows that both $\tau_*(\gamma)$ and $\widehat{\tau}(\gamma)$ are optimal stopping times for $\underset{\zeta\in\mathcal{T}_\gamma}{\operatorname{esssup}}\,\mathcal{E}^g_{\gamma,\zeta}\big[\mathcal{R}_\zeta\big]$.

Let $\tau\in\mathcal{T}_\gamma$ be an arbitrary optimal stopping time for $\underset{\zeta\in\mathcal{T}_\gamma}{\operatorname{esssup}}\,\mathcal{E}^g_{\gamma,\zeta}\big[\mathcal{R}_\zeta\big]$. By the $g-$supermartingality of Y and (4.1),

$$\mathcal{E}^g_{\gamma,\tau}\big[Y_\tau\big]\le Y_\gamma=\underset{\zeta\in\mathcal{T}_\gamma}{\operatorname{esssup}}\,\mathcal{E}^g_{\gamma,\zeta}\big[\mathcal{R}_\zeta\big]=\mathcal{E}^g_{\gamma,\tau}\big[\mathcal{R}_\tau\big],\quad P-\text{a.s.} \tag{5.34}$$

As $Y_\tau\ge\mathcal{R}_\tau$, $P-$a.s., the strict monotonicity (g1) of $g-$evaluations shows that $Y_\tau=\mathcal{R}_\tau$, $P-$a.s. and thus that $\tau_*(\gamma)\le\tau$, $P-$a.s.

Since K is an absolutely continuous process, there exists a positive, $\mathbf{F}-$progressively measurable process Υ such that $P\{K_t=\int_0^t \Upsilon_s ds,\ t\in[0,T]\}=1$. Then it holds $P-$a.s.that

$$Y_t=Y_\tau+\int_t^\tau \big(g(s,Y_s,Z_s,U_s)+\Upsilon_s\big)ds-\int_t^\tau Z_s dB_s-\int_{(t,\tau]}\int_{\mathcal{X}} U_s(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad \forall\, t\in[0,\tau].$$

Also, an analogy to (5.28) shows that $P-$a.s.

$$Y_t^{Y_\tau,g_\tau}=Y_\tau+\int_t^\tau g\big(s,Y_s^{Y_\tau,g_\tau},Z_s^{Y_\tau,g_\tau},U_s^{Y_\tau,g_\tau}\big)ds-\int_t^\tau Z_s^{Y_\tau,g_\tau}dB_s-\int_{(t,\widetilde{\zeta}]}\int_{\mathcal{X}} U_s^{Y_\tau,g_\tau}(x)\widetilde{N}_{\mathfrak{p}}(ds,dx), \quad \forall\, t\in[0,\tau].$$

Applying Theorem 2.2 with $(Y^1,Z^1,U^1,g^1)=\big(Y^{Y_\tau,g_\tau},Z^{Y_\tau,g_\tau},U^{Y_\tau,g_\tau},g\big)$ and $(Y^2,Z^2,U^2,V^2)=(Y,Z,U,g+\Upsilon)$ over period $[\gamma,\tau]$ yields that $Y_\gamma^{Y_\tau,g_\tau}\le Y_\gamma$, $P-$a.s., which together with (5.34) shows that $Y_\gamma^{Y_\tau,g_\tau}\le Y_\gamma=\mathcal{E}^g_{\gamma,\tau}\big[\mathcal{R}_\tau\big]=\mathcal{E}^g_{\gamma,\tau}\big[Y_\tau\big]=Y_\gamma^{Y_\tau,g_\tau}$, $P-$a.s. Then we further see from Theorem 2.2 that $\Upsilon_t=0$ $dt\times dP-$a.s. on $]\!]\gamma,\tau[\![$. It follows that $K_\tau=K_\gamma$, $P-$a.s. and thus that $\tau\le\widehat{\tau}(\gamma)$, $P-$a.s. $\square$

A Appendix

In this appendix, we study the optimal stopping problem for a reward process of class (D) and with unbounded negative values. It is worth pointing out that our results are not simple extension of [13] or Appendix D of [26] since their method heavily depends on the non-negativity of the Snell envelope $\big($see four lines below (2.32.1) of [13] or line 16 on page 357 of [26]$\big)$. Instead, we take a different approach: we first derive a dynamic programming principle (DPP) for the Snell envelope of a reward process. Then we use the DPP as well as a different approximate stopping time (A.15) from [13] to show the martingale property of the Snell envelope.

Assume that $\mathbf{F}=\{\mathcal{F}_t\}_{t\in[0,T]}$ is a general filtration satisfying the *usual hypotheses*, and let $\mathcal{T}$ still denote the collection of all $\mathbf{F}-$stopping times. We consider a real-valued, $\mathbf{F}-$adapted right-continuous process X of class (D) such that $E[X_*^+]<\infty$.

Let us start with a convergence result of uniformly integrable random variables under conditional expectations.

Lemma A.1. *Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of uniformly integrable random variable that converges $P-a.s.$ to a random variable ξ. Then for any sub-sigma-field $\mathcal{G}$ of $\mathcal{F}$, there exists a subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that*

$$\lim_{i\to\infty} E[\xi_{n_i}|\mathcal{G}]=E[\xi|\mathcal{G}], \quad P-a.s. \tag{A.1}$$

Proof: First, we know from e.g. Theorem 16.13 of [3] that ξ is integrable, so the conditional expectation $E[\xi|\mathcal{G}]$ exists. Clearly, the uniform integrability of $\{\xi_n\}_{n\in\mathbb{N}}$ and the integrability of ξ implies the uniform integrability of $\{\xi_n-\xi\}_{n\in\mathbb{N}}$. Applying Theorem 16.13 of [3] again shows that $\lim\limits_{n\to\infty}E[|\xi_n-\xi|]=0$. Since $E\big[\big|E[\xi_n|\mathcal{G}]-E[\xi|\mathcal{G}]\big|\big]\le E\big[E[|\xi_n-\xi|\,|\mathcal{G}]\big]=E[|\xi_n-\xi|]$ for any $n\in\mathbb{N}$, it follows that $\lim\limits_{n\to\infty}E\big[\big|E[\xi_n|\mathcal{G}]-E[\xi|\mathcal{G}]\big|\big]=0$. Thus, we can extract a subsequence $\{n_i\}_{i\in\mathbb{N}}$ from $\mathbb{N}$ such that (A.1) holds. $\square$

Given $\gamma\in\mathcal{T}$, the following lemma shows that $\operatorname*{esssup}\limits_{\rho\in\mathcal{T}_\zeta} E[X_\rho|\mathcal{F}_\gamma]$ with $\gamma\le\zeta$ can be approximated by an increasing sequence, which will play an important role in the arguments of this section.

Lemma A.2. *For any $\gamma\in\mathcal{T}$ and $\zeta\in\mathcal{T}_\gamma$, there exists a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}_\zeta$ such that*

$$\operatorname*{esssup}_{\rho\in\mathcal{T}_\zeta} E[X_\rho|\mathcal{F}_\gamma]=\lim_{n\to\infty}\uparrow E[X_{\rho_n}|\mathcal{F}_\gamma], \quad P-a.s. \tag{A.2}$$

Proof: Let $\rho_1,\rho_2\in\mathcal{T}_\zeta$ and set $A:=\big\{E[X_{\rho_1}|\mathcal{F}_\gamma]\ge E[X_{\rho_2}|\mathcal{F}_\gamma]\big\}$. As the set A is $\mathcal{F}_\gamma-$measurable, $\rho_3:=\mathbf{1}_A\rho_1+\mathbf{1}_{A^c}\rho_3$ is also a stopping time in $\mathcal{T}_\zeta$. It follows that

$$E[X_{\rho_3}|\mathcal{F}_\gamma]=E[\mathbf{1}_A X_{\rho_1}+\mathbf{1}_{A^c}X_{\rho_2}|\mathcal{F}_\gamma]=\mathbf{1}_A E[X_{\rho_1}|\mathcal{F}_\gamma]+\mathbf{1}_{A^c}E[X_{\rho_2}|\mathcal{F}_\gamma]=E[X_{\rho_1}|\mathcal{F}_\gamma]\vee E[X_{\rho_2}|\mathcal{F}_\gamma], \quad P-\text{a.s.}$$

Thus the family $\big\{E[X_\rho|\mathcal{F}_\gamma]\big\}_{\rho\in\mathcal{T}_\zeta}$ is closed under pairwise maximization. In light of [31, Proposition VI-**1**-1], we can find a sequence $\{\rho_n\}_{n\in\mathbb{N}}\subset\mathcal{T}_\zeta$ such that (A.2) holds. $\square$

Now, we define

$$S(\gamma):=\underset{\rho\in\mathcal{T}_\gamma}{\operatorname{esssup}}\, E[X_\rho|\mathcal{F}_\gamma]\in\mathcal{F}_\gamma,\quad \forall\,\gamma\in\mathcal{T}.$$

By Lemma A.2, we have the following basic properties of the family $\{S(\gamma)\}_{\gamma\in\mathcal{T}}$.

Proposition A.1. *(i) $\{S(\gamma)\}_{\gamma\in\mathcal{T}}$ is uniformly integrable; (ii) For any $\gamma,\sigma\in\mathcal{T}$, $S(\gamma)=S(\sigma)$, $P-$a.s. on $\{\gamma=\sigma\}$; (iii) For any $\gamma\in\mathcal{T}$ and $\zeta\in\mathcal{T}_\gamma$, $E[S(\zeta)|\mathcal{F}_\gamma]=\underset{\rho\in\mathcal{T}_\zeta}{\operatorname{esssup}}\, E[X_\rho|\mathcal{F}_\gamma]\le S(\gamma)$, $P-$a.s.*

Proof: 1) Let $\gamma\in\mathcal{T}$, since $E[X_\rho|\mathcal{F}_\gamma]\le E[X^+_*|\mathcal{F}_\gamma]$, $P-$a.s. for any $\rho\in\mathcal{T}_\gamma$, taking supremum of the left-hand-side over $\rho\in\mathcal{T}_\gamma$ yields that $E[X_T|\mathcal{F}_\gamma]\le S(\gamma)\le E[X^+_*|\mathcal{F}_\gamma]$, $P-$a.s., it follows that

$$|S(\gamma)|\le E\big[|X_T|+X^+_*|\mathcal{F}_\gamma\big],\quad P-\text{a.s.} \tag{A.3}$$

As $E[|X_T|+X^+_*]<\infty$, the uniformly integrability of $\big\{E\big[|X_T|+X^+_*|\mathcal{F}_\gamma\big]\big\}_{\gamma\in\mathcal{T}}$ implies that of $\{S(\gamma)\}_{\gamma\in\mathcal{T}}$.
2) For any $\rho\in\mathcal{T}_\sigma$, define $\rho_{_A}:=\rho\mathbf{1}_A+T\mathbf{1}_{A^c}$. Since $A:=\{\gamma=\sigma\}\in\mathcal{F}_{\gamma\wedge\sigma}$ by e.g. Lemma 1.2.16 of [25], one can deduce that $\rho_{_A}$ is a stopping time belonging to $\mathcal{T}_\gamma$. It then holds $P-$a.s. that

$$\mathbf{1}_A E\big[X_\rho|\mathcal{F}_\sigma\big]=\mathbf{1}_A E\big[X_\rho|\mathcal{F}_\gamma\big]=E\big[\mathbf{1}_A X_\rho|\mathcal{F}_\gamma\big]=E\big[\mathbf{1}_A X_{\rho_{_A}}\big|\mathcal{F}_\gamma\big]=\mathbf{1}_A E\big[X_{\rho_{_A}}\big|\mathcal{F}_\gamma\big]\le\mathbf{1}_A S(\gamma),\quad P-\text{a.s.}$$

Taking the essential supremum over $\rho\in\mathcal{T}_\sigma$ on the left-hand-side, we obtain

$$\mathbf{1}_A S(\sigma)=\mathbf{1}_A\underset{\rho\in\mathcal{T}_\sigma}{\operatorname{esssup}}\, E\big[X_\rho\big|\mathcal{F}_\sigma\big]=\underset{\rho\in\mathcal{T}_\sigma}{\operatorname{esssup}}\big(\mathbf{1}_A E\big[X_\rho\big|\mathcal{F}_\sigma\big]\big)\le\mathbf{1}_A S(\gamma),\quad P-\text{a.s.}$$

Reversing the roles of γ and σ yields that $\mathbf{1}_A S(\sigma)=\mathbf{1}_A S(\gamma)$, $P-$a.s.
3) Let $\zeta\in\mathcal{T}_\gamma$. As $\mathcal{T}_\zeta\subset\mathcal{T}_\gamma$, one clearly has $\underset{\rho\in\mathcal{T}_\zeta}{\operatorname{esssup}}\, E\big[X_\rho\big|\mathcal{F}_\gamma\big]\le\underset{\rho\in\mathcal{T}_\gamma}{\operatorname{esssup}}\, E\big[X_\rho\big|\mathcal{F}_\gamma\big]=S(\gamma)$, $P-$a.s. By Lemma A.2, there exists a sequence $\big\{\widetilde{\rho}_n\big\}_{n\in\mathbb{N}}\subset\mathcal{T}_\zeta$ such that $S(\zeta)=\lim\limits_{n\to\infty}\uparrow E\big[X_{\widetilde{\rho}_n}\big|\mathcal{F}_\zeta\big]$, $P-$a.s. A conditional-expectation version of the monotone convergence theorem implies that

$$\begin{aligned}E[S(\zeta)|\mathcal{F}_\gamma]\;&=\;E\big[S(\zeta)-E[X_{\widetilde{\rho}_1}|\mathcal{F}_\zeta]\big|\mathcal{F}_\gamma\big]+E[X_{\widetilde{\rho}_1}|\mathcal{F}_\gamma]=\lim_{n\to\infty}\uparrow E\big[E[X_{\widetilde{\rho}_n}-X_{\widetilde{\rho}_1}|\mathcal{F}_\zeta]\big|\mathcal{F}_\gamma\big]+E[X_{\widetilde{\rho}_1}|\mathcal{F}_\gamma]\\&=\;\lim_{n\to\infty}\uparrow E[X_{\widetilde{\rho}_n}|\mathcal{F}_\gamma]\le\underset{\rho\in\mathcal{T}_\zeta}{\operatorname{esssup}}\, E\big[X_\rho\big|\mathcal{F}_\gamma\big],\quad P-\text{a.s.}\end{aligned} \tag{A.4}$$

On the other hand, it holds for any $\rho\in\mathcal{T}_\zeta$ that $E\big[X_\rho\big|\mathcal{F}_\gamma\big]=E\big[E\big[X_\rho\big|\mathcal{F}_\zeta\big]\big|\mathcal{F}_\gamma\big]\le E\big[S(\zeta)\big|\mathcal{F}_\gamma\big]$, $P-$a.s. Taking the essential supremum over $\rho\in\mathcal{T}_\zeta$ on the left-hand-side, we see from (A.4) that $\underset{\rho\in\mathcal{T}_\zeta}{\operatorname{esssup}}\, E\big[X_\rho\big|\mathcal{F}_\gamma\big]=E\big[S(\zeta)\big|\mathcal{F}_\gamma\big]$, $P-$a.s. □

Proposition A.2. *The supermartingale $\big\{S(t)\big\}_{t\in[0,T]}$ admits an càdlàg modification $\mathfrak{S}$ such that for any $\gamma\in\mathcal{T}$*

$$S(\gamma)=\mathfrak{S}_\gamma,\quad P-a.s. \tag{A.5}$$

Moreover, $\mathfrak{S}$ is the smallest càdlàg supermartingale that dominates X $\big($i.e. for any càdlàg supermartingale M with $P\{M_t\ge X_t,\ \forall\,t\in[0,T]\}=1$, it holds $P-$a.s. that $M_t\ge\mathfrak{S}_t,\ \forall\,t\in[0,T]\big)$.

We call $\mathfrak{S}$ the “Snell envelope” of process X. Proposition A.1 and (A.5) imply that

$$\{\mathfrak{S}_\gamma\}_{\gamma\in\mathcal{T}}\text{ is uniformly integrable} \tag{A.6}$$

and that for any $\gamma\in\mathcal{T}$ and $\zeta\in\mathcal{T}_\gamma$

$$E[\mathfrak{S}_\zeta|\mathcal{F}_\gamma]\le\mathfrak{S}_\gamma,\quad P-\text{a.s.} \tag{A.7}$$

Proof: 1) For any $0\le t<t'\le T$, we see from Proposition A.1 (iii) that $E\big[S(t')\big|\mathcal{F}_t\big]\le S(t)$, $P-$a.s. So $\{S(t)\}_{t\in[0,T]}$ is a supermartingale.

For any $t\in[0,T]$, define $\mathfrak{S}_t:=\varliminf_{n\to\infty} S\big(q_n^+(t)\big)$, where $q_n^+(t):=\frac{[2^n t]}{2^n}\wedge T$. By Proposition 1.3.14 of [25] and the right-continuity of filtration $\mathbf{F}$, the process $\mathfrak{S}$ is a real-valued càdlàg supermartingale such that $P\big\{\mathfrak{S}_t=\lim_{n\to\infty} S\big(q_n^+(t)\big)\in\mathbb{R},\ \forall\, t\in[0,T]\big\}=1$ and that $P\{\mathfrak{S}_t\le S(t)\}=1$ for any $t\in[0,T]$. So to see that $\mathfrak{S}$ is a modification of process $\{S(t)\}_{t\in[0,T]}$, one only needs to show that $P\{\mathfrak{S}_t\ge S(t)\}=1$ for any $t\in[0,T]$.

Fix $t\in[0,T]$ and $\rho\in\mathcal{T}_t$. For any $n\in\mathbb{N}$, we set $t_n:=q_n^+(t)$ and define $\rho_n:=(\rho+2^{-n})\wedge T\in\mathcal{T}_t$. Let $m\ge n$, since $t_m\le t_n\le(t+2^{-n})\wedge T\le\rho_n$, $P-$a.s. (i.e. $\rho_n\in\mathcal{T}_{t_m}$), one has $E\big[X_{\rho_n}\big|\mathcal{F}_{t_m}\big]\le S(t_m)$, $P-$a.s. As $m\to\infty$, the right-continuity of the processes $E\big[X_{\rho_n}\big|\mathcal{F}_\cdot\big]$ shows that

$$E\big[X_{\rho_n}\big|\mathcal{F}_t\big]=\lim_{m\to\infty}E\big[X_{\rho_n}\big|\mathcal{F}_{t_m}\big]\le\varliminf_{m\to\infty}S(t_m)=\mathfrak{S}_t,\quad P-\text{a.s.}\tag{A.8}$$

Since $\lim_{n\to\infty}\downarrow\rho_n=\rho$, $P-$a.s., the right continuity of process X implies that $\lim_{n\to\infty}X_{\rho_n}=X_\rho$, $P-$a.s. Then the uniform integrability of $\big\{X_{\rho_n}\big\}_{n\in\mathbb{N}}$, Lemma A.1 and (A.8) yield that for some subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$, $E\big[X_\rho\big|\mathcal{F}_t\big]=\lim_{i\to\infty}E[X_{\rho_{n_i}}|\mathcal{F}_t]\le\mathfrak{S}_t$, $P-$a.s. Letting ρ run throughout $\mathcal{T}_t$ yields that

$$S(t)=\operatorname*{esssup}_{\rho\in\mathcal{T}_t}E\big[X_\rho\big|\mathcal{F}_t\big]=\mathfrak{S}_t,\quad P-\text{a.s.}\tag{A.9}$$

Namely, $\mathfrak{S}$ is a modification of $\{S(t)\}_{t\in[0,T]}$ and thus a supermartingale. For any $t\in[0,T]$, taking $\rho=t$ in the definition of $S(t)$, we see from (A.9) that $X_t\le S(t)=\mathfrak{S}_t$, $P-$a.s. The right continuity of X and $\mathfrak{S}$ then implies that

$$P\{\mathfrak{S}_t\ge X_t,\ t\in[0,T]\}=1.\tag{A.10}$$

2) Let $\gamma\in\mathcal{T}$ takes values in a finite subset $\{t_1<\cdots<t_n\}$ of $[0,T]$. For any $i\in\{1,\cdots,n\}$, Proposition A.1 (ii) and (A.9) imply that $\mathbf{1}_{\{\gamma=t_i\}}S(\gamma)=\mathbf{1}_{\{\gamma=t_i\}}S(t_i)=\mathbf{1}_{\{\gamma=t_i\}}\mathfrak{S}_{t_i}=\mathbf{1}_{\{\gamma=t_i\}}\mathfrak{S}_\gamma$, $P-$a.s. Summing up over i leads to (A.5).

Next, let γ be a general stopping time in $\mathcal{T}$. Given $n\in\mathbb{N}$, we set $\gamma_n:=\sum_{i=1}^{2^n}\mathbf{1}_{\{(i-1)2^{-n}T<\gamma\le i2^{-n}T\}}i2^{-n}T\in\mathcal{T}$. Since $\mathfrak{S}$ is a supermartingale, the optional sampling theorem imply that $E\big[\mathfrak{S}_{\gamma_n}\big|\mathcal{F}_{\gamma_{n+1}}\big]\le\mathfrak{S}_{\gamma_{n+1}}$, $P-$a.s. and thus

$$E\big[\mathfrak{S}_{\gamma_n}\big|\mathcal{F}_\gamma\big]=E\big[E\big[\mathfrak{S}_{\gamma_n}\big|\mathcal{F}_{\gamma_{n+1}}\big]\big|\mathcal{F}_\gamma\big]\le E\big[\mathfrak{S}_{\gamma_{n+1}}\big|\mathcal{F}_\gamma\big]\le\mathfrak{S}_\gamma,\quad P-\text{a.s.}\tag{A.11}$$

This means $\lim_{n\to\infty}\uparrow E\big[\mathfrak{S}_{\gamma_n}\big|\mathcal{F}_\gamma\big]\le\mathfrak{S}_\gamma$, $P-$a.s. Clearly, $\lim_{n\to\infty}\downarrow\gamma_n=\gamma$. The right-continuity of $\mathfrak{S}$ shows that $\lim_{n\to\infty}\mathfrak{S}_{\gamma_n}=\mathfrak{S}_\gamma$, $P-$a.s. Since $\big\{\mathfrak{S}_{\gamma_n}=S(\gamma_n)\big\}_{n\in\mathbb{N}}$ is uniformly integrable by (A.6), Lemma A.1 and (A.11) imply that for some subsequence $\{n_i\}_{i\in\mathbb{N}}$ of $\mathbb{N}$, $\mathfrak{S}_\gamma=E\big[\mathfrak{S}_\gamma\big|\mathcal{F}_\gamma\big]=\lim_{i\to\infty}\uparrow E\big[\mathfrak{S}_{\gamma_{n_i}}\big|\mathcal{F}_\gamma\big]\le\mathfrak{S}_\gamma$, $P-$a.s. Then by Proposition A.1 (iii),

$$\mathfrak{S}_\gamma=\lim_{i\to\infty}\uparrow E\big[\mathfrak{S}_{\gamma_{n_i}}\big|\mathcal{F}_\gamma\big]=\lim_{i\to\infty}\uparrow E\big[S(\gamma_{n_i})\big|\mathcal{F}_\gamma\big]\le S(\gamma),\quad P-\text{a.s.}\tag{A.12}$$

On the other hand, for any $\rho\in\mathcal{T}_\gamma$ and $n\in\mathbb{N}$, we define $\rho_n:=\rho\vee\gamma_n\in\mathcal{T}_{\gamma_n}$. As $E\big[X_{\rho_n}\big|\mathcal{F}_{\gamma_n}\big]\le S(\gamma_n)$, $P-$a.s., taking the conditional expectation $E\big[\cdot\big|\mathcal{F}_\gamma\big]$ on both sides gives that

$$E[X_{\rho_n}|\mathcal{F}_\gamma]=E\big[E[X_{\rho_n}|\mathcal{F}_{\gamma_n}]\big|\mathcal{F}_\gamma\big]\le E\big[S(\gamma_n)\big|\mathcal{F}_\gamma\big],\quad P-\text{a.s.}$$

As $\lim_{n\to\infty}\downarrow\rho_n=\rho\vee\gamma=\rho$, the uniform integrability of $\big\{X_{\rho_{n_i}}\big\}_{i\in\mathbb{N}}$, Lemma A.1 and (A.12) imply that for some subsequence $\{\widetilde{n}_i\}_{i\in\mathbb{N}}$ of $\{n_i\}_{i\in\mathbb{N}}$, $E\big[X_\rho\big|\mathcal{F}_\gamma\big]=\lim_{i\to\infty}E\big[X_{\rho_{\widetilde{n}_i}}\big|\mathcal{F}_\gamma\big]\le\lim_{i\to\infty}\uparrow E\big[S(\gamma_{\widetilde{n}_i})\big|\mathcal{F}_\gamma\big]=\mathfrak{S}_\gamma$, $P-$a.s. Taking the essential supremum over $\rho\in\mathcal{T}_\gamma$, we see from (A.12) that $S(\gamma)=\mathfrak{S}_\gamma$, $P-$a.s.

3) Let M be a càdlàg supermartingale with $P\{M_t\ge X_t,\ t\in[0,T]\}=1$. Given $t\in[0,T]$, since the optional sampling theorem shows that $E\big[X_\gamma\big|\mathcal{F}_t\big]\le E\big[M_\gamma\big|\mathcal{F}_t\big]\le M_t$, $P-$a.s. for any $\gamma\in\mathcal{T}_t$, taking the essential supremum of the left-hand-side over $\gamma\in\mathcal{T}_t$ yields that $\mathfrak{S}_t=S(t)=\operatorname*{esssup}_{\gamma\in\mathcal{T}_t}E\big[X_\gamma\big|\mathcal{F}_t\big]\le M_t$, $P-$a.s. Then the right-continuity of $\mathfrak{S}$ and M implies that $P\{\mathfrak{S}_t\le M_t,\ t\in[0,T]\}=1$. $\square$

We have the following dynamic programming principle of Snell envelope $\mathfrak{S}$.

Proposition A.3. *For any $\gamma\in\mathcal{T}$ and $\zeta\in\mathcal{T}_\gamma$, $\mathfrak{S}_\gamma=\operatorname*{esssup}_{\rho\in\mathcal{T}_\gamma}E\big[\mathbf{1}_{\{\rho<\zeta\}}X_\rho+\mathbf{1}_{\{\rho\ge\zeta\}}\mathfrak{S}_\zeta\big|\mathcal{F}_\gamma\big]$, $P-$a.s.*

Proof: Let $\gamma\in\mathcal{T}$ and $\zeta\in\mathcal{T}_\gamma$. For any $\rho\in\mathcal{T}_\gamma$, one can deduce from (A.5) that $P-$a.s.

$$E[X_\rho|\mathcal{F}_\gamma]=E\big[\mathbf{1}_{\{\rho<\zeta\}}X_{\rho\wedge\zeta}+\mathbf{1}_{\{\rho\ge\zeta\}}X_{\rho\vee\zeta}|\mathcal{F}_\gamma\big]=E\Big[E\big[\mathbf{1}_{\{\rho<\zeta\}}X_{\rho\wedge\zeta}+\mathbf{1}_{\{\rho\ge\zeta\}}X_{\rho\vee\zeta}|\mathcal{F}_\zeta\big]\big|\mathcal{F}_\gamma\Big]$$
$$=E\Big[\mathbf{1}_{\{\rho<\zeta\}}X_{\rho\wedge\zeta}+\mathbf{1}_{\{\rho\ge\zeta\}}E\big[X_{\rho\vee\zeta}|\mathcal{F}_\zeta\big]\big|\mathcal{F}_\gamma\Big]\le E\Big[\mathbf{1}_{\{\rho<\zeta\}}X_\rho+\mathbf{1}_{\{\rho\ge\zeta\}}S(\zeta)\big|\mathcal{F}_\gamma\Big]=E\Big[\mathbf{1}_{\{\rho<\zeta\}}X_\rho+\mathbf{1}_{\{\rho\ge\zeta\}}\mathfrak{S}_\zeta\big|\mathcal{F}_\gamma\Big].$$

Taking supremum over $\rho\in\mathcal{T}_\gamma$ on both sides yields that $\mathfrak{S}_\gamma=S(\gamma)\le\underset{\rho\in\mathcal{T}_\gamma}{\operatorname{esssup}}E\big[\mathbf{1}_{\{\rho<\zeta\}}X_\rho+\mathbf{1}_{\{\rho\ge\zeta\}}\mathfrak{S}_\zeta|\mathcal{F}_\gamma\big]$, $P-$a.s.

On the other hand, Lemma A.2 and (A.5) show that for some sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}_\zeta$,

$$\mathfrak{S}_\zeta=S(\zeta)=\underset{\rho\in\mathcal{T}_\zeta}{\operatorname{esssup}}E[X_\rho|\mathcal{F}_\zeta]=\lim_{n\to\infty}\uparrow E[X_{\rho_n}|\mathcal{F}_\zeta],\quad P-\text{a.s.}\tag{A.13}$$

For any $\rho\in\mathcal{T}_\gamma$ and $n\in\mathbb{N}$, we set $\widetilde{\rho}_n:=\mathbf{1}_{\{\rho<\zeta\}}\rho+\mathbf{1}_{\{\rho\ge\zeta\}}\rho_n\ge\gamma$. Since $\{\rho<\zeta\}\in\mathcal{F}_{\rho\wedge\zeta}\subset\mathcal{F}_{\rho_n}$, it holds for any $t\in[0,T]$ that $\{\widetilde{\rho}_n\le t\}=\big(\{\rho<\zeta\}\cap\{\rho\wedge\zeta\le t\}\big)\cup\big(\{\rho\ge\zeta\}\cap\{\rho_n\le t\}\big)\in\mathcal{F}_t$, which shows that $\widetilde{\rho}_n\in\mathcal{T}_\gamma$. It follows from (A.5) that

$$E\Big[\mathbf{1}_{\{\rho<\zeta\}}X_\rho+\mathbf{1}_{\{\rho\ge\zeta\}}E\big[X_{\rho_n}|\mathcal{F}_\zeta\big]\big|\mathcal{F}_\gamma\Big]=E\Big[E\big[\mathbf{1}_{\{\rho<\zeta\}}X_\rho+\mathbf{1}_{\{\rho\ge\zeta\}}X_{\rho_n}|\mathcal{F}_\zeta\big]\big|\mathcal{F}_\gamma\Big]=E\big[X_{\widetilde{\rho}_n}|\mathcal{F}_\gamma\big]\le S(\gamma)=\mathfrak{S}_\gamma,\ P-\text{a.s.}\tag{A.14}$$

Since a conditional-expectation version of the monotone convergence theorem and (A.13) imply that

$$E\big[\mathbf{1}_{\{\rho\ge\zeta\}}\big(\mathfrak{S}_\zeta-E[X_{\rho_1}|\mathcal{F}_\zeta]\big)\big|\mathcal{F}_\gamma\big]=\lim_{n\to\infty}\uparrow E\big[\mathbf{1}_{\{\rho\ge\zeta\}}\big(E[X_{\rho_n}-X_{\rho_1}|\mathcal{F}_\zeta]\big)\big|\mathcal{F}_\gamma\big],\quad P-\text{a.s.}$$

we see from (A.14) that $E\big[\mathbf{1}_{\{\rho<\zeta\}}X_\rho+\mathbf{1}_{\{\rho\ge\zeta\}}\mathfrak{S}_\zeta|\mathcal{F}_\gamma\big]=\lim_{n\to\infty}\uparrow E\Big[\mathbf{1}_{\{\rho<\zeta\}}X_\rho+\mathbf{1}_{\{\rho\ge\zeta\}}E\big[X_{\rho_n}|\mathcal{F}_\zeta\big]\big|\mathcal{F}_\gamma\Big]\le\mathfrak{S}_\gamma$, $P-$a.s. Taking supremum over $\rho\in\mathcal{T}_\gamma$ yields that $\underset{\rho\in\mathcal{T}_\gamma}{\operatorname{esssup}}E\big[\mathbf{1}_{\{\rho<\zeta\}}X_\rho+\mathbf{1}_{\{\rho\ge\zeta\}}\mathfrak{S}_\zeta|\mathcal{F}_\gamma\big]\le\mathfrak{S}_\gamma$, $P-$a.s. □

To solve the optimal stopping problem, let us introduce *approximately* optimal stopping times: Given $k\in\mathbb{N}$ and $\gamma\in\mathcal{T}$, we define

$$\tau_k(\gamma):=\inf\big\{t\in[\gamma,T]:\ \mathfrak{S}_t\le X_t+1/k\big\}\in\mathcal{T}_\gamma.\tag{A.15}$$

The next result shows that the Snell envelope $\mathfrak{S}$ is a martingale over each period $\big[\gamma,\tau_k(\gamma)\big]$.

Proposition A.4. *Let $k\in\mathbb{N}$ and $\gamma\in\mathcal{T}$. It holds for any $\zeta\in\mathcal{T}_\gamma$ that $\mathfrak{S}_{\zeta\wedge\tau_k(\gamma)}=E\big[\mathfrak{S}_{\tau_k(\gamma)}\big|\mathcal{F}_\zeta\big]$, $P-$a.s.*

Proof: Let $k\in\mathbb{N}$, $\gamma\in\mathcal{T}$ and $\zeta\in\mathcal{T}_\gamma$. We set $\zeta_k:=\zeta\wedge\tau_k(\gamma)$. An analogy to Proposition A.3 as well as Lemma A.2 imply that for some sequence $\{\rho^k_n\}_{n\in\mathbb{N}}\subset\mathcal{T}_{\zeta_k}$,

$$\mathfrak{S}_{\zeta_k}=\underset{\rho\in\mathcal{T}_{\zeta_k}}{\operatorname{esssup}}E\big[\mathbf{1}_{\{\rho<\tau_k(\gamma)\}}X_\rho+\mathbf{1}_{\{\rho\ge\tau_k(\gamma)\}}\mathfrak{S}_{\tau_k(\gamma)}|\mathcal{F}_{\zeta_k}\big]=\lim_{n\to\infty}\uparrow\xi^k_n,\quad P-\text{a.s.},\tag{A.16}$$

where $\xi^k_n:=E\big[\mathbf{1}_{\{\rho^k_n<\tau_k(\gamma)\}}X_{\rho^k_n}+\mathbf{1}_{\{\rho^k_n\ge\tau_k(\gamma)\}}\mathfrak{S}_{\tau_k(\gamma)}\big|\mathcal{F}_{\zeta_k}\big]$.

Given $n\in\mathbb{N}$, since it holds for any $\omega\in\{\rho^k_n<\tau_k(\gamma)\}$ that $\mathfrak{S}\big(\rho^k_n(\omega),\omega\big)>X\big(\rho^k_n(\omega),\omega\big)+1/k$, (A.7) shows that

$$\xi^k_n+\frac{1}{k}E\big[\mathbf{1}_{\{\rho^k_n<\tau_k(\gamma)\}}|\mathcal{F}_{\zeta_k}\big]\le E\big[\mathbf{1}_{\{\rho^k_n<\tau_k(\gamma)\}}\mathfrak{S}_{\rho^k_n}+\mathbf{1}_{\{\rho^k_n\ge\tau_k(\gamma)\}}\mathfrak{S}_{\tau_k(\gamma)}|\mathcal{F}_{\zeta_k}\big]=E\big[\mathfrak{S}_{\rho^k_n\wedge\tau_k(\gamma)}|\mathcal{F}_{\zeta_k}\big]\le\mathfrak{S}_{\zeta_k},\quad P-\text{a.s.}$$

It follows that $E\big[\mathbf{1}_{\{\rho^k_n<\tau_k(\gamma)\}}\big]\le kE\big[\mathfrak{S}_{\zeta_k}-\xi^k_n\big]$. The uniform integrability of $\big\{X_\gamma\big\}_{\gamma\in\mathcal{T}}$ and (A.6) show that

$$E\big[\mathfrak{S}_{\zeta_k}-\xi^k_1\big]\le E\big[|\mathfrak{S}_{\zeta_k}|+|\xi^k_1|\big]\le E\big[|\mathfrak{S}_{\zeta_k}|+E[|X_{\rho^k_1}|+|\mathfrak{S}_{\tau_k(\gamma)}||\mathcal{F}_{\zeta_k}]\big]\le\sup_{\rho\in\mathcal{T}}E\big[|X_\rho|\big]+2\sup_{\rho\in\mathcal{T}}E\big[|\mathfrak{S}_\rho|\big]<\infty.$$

Then (A.16) and the dominated convergence theorem imply that $\lim_{n\to\infty}E\big[\mathbf{1}_{\{\rho^k_n<\tau_k(\gamma)\}}\big]=\lim_{n\to\infty}kE\big[\mathfrak{S}_{\zeta_k}-\xi^k_n\big]=0$. So there exists a subsequence $\{n_i=n_i(k)\}_{i\in\mathbb{N}}$ of $\mathbb{N}$ such that $\lim_{i\to\infty}\mathbf{1}_{\{\rho^k_{n_i}<\tau_k(\gamma)\}}=0$. Applying a conditional-expectation version of the dominated convergence theorem and using (A.16), we can deduce from (A.7) that

$$\mathfrak{S}_{\zeta_k}=\lim_{i\to\infty}\uparrow\xi^k_{n_i}\le\lim_{i\to\infty}E\big[\mathbf{1}_{\{\rho^k_{n_i}<\tau_k(\gamma)\}}X^+_*+\mathbf{1}_{\{\rho^k_{n_i}\ge\tau_k(\gamma)\}}\mathfrak{S}_{\tau_k(\gamma)}\big|\mathcal{F}_{\zeta_k}\big]=E\big[\mathfrak{S}_{\tau_k(\gamma)}\big|\mathcal{F}_{\zeta_k}\big]\le\mathfrak{S}_{\zeta_k},\quad P-\text{a.s.},$$

which leads to that $\mathfrak{S}_{\zeta\wedge\tau_k(\gamma)}=E\big[\mathfrak{S}_{\tau_k(\gamma)}\big|\mathcal{F}_{\zeta_k}\big]=E\big[E[\mathfrak{S}_{\tau_k(\gamma)}|\mathcal{F}_{\tau_k(\gamma)}]\big|\mathcal{F}_\zeta\big]=E\big[\mathfrak{S}_{\tau_k(\gamma)}\big|\mathcal{F}_\zeta\big]$, $P-$a.s. □

Moreover, the Snell envelope $\mathfrak{S}$ admits a Doob-Meyer decomposition.

Proposition A.5. *There exist a uniformly integrable càdlàg martingale M and a* $\mathbf{F}-$*predictable càdlàg increasing process K with $K(0)=0$ such that $P-$a.s.*

$$\mathfrak{S}_t = M_t - K_t, \quad t\in[0,T]. \tag{A.17}$$

For any $\mathbf{F}-$predictable stopping time $\gamma\in\mathcal{T}$, $\{\Delta K_\gamma>0\}\cap\{\mathfrak{S}_{\gamma-}>X_{\gamma-}\}$ is a $P-$null set. Moreover, If $E\big[|X_T|^p+(X_*^+)^p\big]<\infty$ for some $p\in[1,\infty)$, then $E\big[M_*^p+K_T^p\big]<\infty$.

Proof: 1) We have seen from Proposition A.2 and (A.6) that the Snell envelope $\mathfrak{S}$ is a càdlàg supermartingale of class (D). In light of Theorem VII.12 of [11] (or Theorem III.3.8 of [36]), there exist a uniformly integrable càdlàg martingale M and a $\mathbf{F}-$predictable càdlàg increasing process K with $K(0)=0$ such that (A.17) holds.

Let $\gamma\in\mathcal{T}$ be a $\mathbf{F}-$predictable stopping time. In virtue of Meyer's PFA Theorem (see e.g. Theorem VI.12.6 of [39] or Theorem IV.77 of [10]), γ is announceable, i.e. there exists an increasing sequence $\{\rho_n\}_{n\in\mathbb{N}}$ in $\mathcal{T}$ such that $P-$a.s.

$$\rho_n \le \rho_{n+1} < \gamma, \quad \forall\, n\in\mathbb{N} \quad \text{and} \quad \lim_{n\to\infty}\uparrow \rho_n=\gamma.$$

Let $k\in\mathbb{N}$ and $n\in\mathbb{N}$. applying Proposition A.4 with $\gamma=\zeta=\rho_n$, we can deduce from the optional sampling theorem

$$K_{\rho_n}=M_{\rho_n}-\mathfrak{S}_{\rho_n}=E[M_{\tau_k(\rho_n)}-\mathfrak{S}_{\tau_k(\rho_n)}|\mathcal{F}_{\rho_n}]=E[K_{\tau_k(\rho_n)}|\mathcal{F}_{\rho_n}], \quad P-\text{a.s.}$$

So the monotonicity of K shows that $K_{\tau_k(\rho_n)}=K_{\rho_n}$ holds except on a $P-$null set $\mathcal{N}_n^k$. As

$$\mathfrak{S}_T=S(T)=X_T, \quad P-\text{a.s.}, \tag{A.18}$$

by (A.5), the set $\big\{t\in[\rho_n,T]:\ \mathfrak{S}_t\le X_t+1/k\big\}$ is not empty $P-$a.s. Then (A.10) and the right-continuity of processes X, $\mathfrak{S}$ imply that

$$X_{\tau_k(\rho_n)}\le\mathfrak{S}_{\tau_k(\rho_n)}\le X_{\tau_k(\rho_n)}+1/k \tag{A.19}$$

holds except on a $P-$null set $\widetilde{\mathcal{N}}_n^k$.

Let $\omega\in\{\Delta K_\gamma>0\}\cap\Big(\underset{k,n\in\mathbb{N}}{\cap}\big(\mathcal{N}_n^k\cup\widetilde{\mathcal{N}}_n^k\big)^c\Big)$ and let $k\in\mathbb{N}$. For any $n\in\mathbb{N}$, one can deduce that $K\big((\tau_k(\rho_n))(\omega),\omega\big)=K\big(\rho_n(\omega),\omega\big)\le K\big(\gamma(\omega)-,\omega\big)<K\big(\gamma(\omega),\omega\big)$. It follows that

$$(\tau_k(\rho_n))(\omega)<\gamma(\omega) \quad \text{and thus} \quad \gamma(\omega)=\lim_{i\to\infty}\uparrow\rho_n(\omega)\le\lim_{i\to\infty}\uparrow(\tau_k(\rho_n))(\omega)\le\gamma(\omega).$$

Then letting $n\to\infty$ in (A.19) yields that $X\big(\gamma(\omega)-,\omega\big)\le\mathfrak{S}\big(\gamma(\omega)-,\omega\big)\le X\big(\gamma(\omega)-,\omega\big)+1/k$. As $k\to\infty$, we obtain $\mathfrak{S}\big(\gamma(\omega)-,\omega\big)=X\big(\gamma(\omega)-,\omega\big)$, which implies that $\{\Delta K_\gamma>0\}\cap\{\mathfrak{S}_{\gamma-}>X_{\gamma-}\}\subset\mathcal{N}\cup\Big(\underset{k,n\in\mathbb{N}}{\cup}\big(\mathcal{N}_n^k\cup\widetilde{\mathcal{N}}_n^k\big)\Big)$.

2) Assume further that $E\big[|X_T|^p+(X_*^+)^p\big]<\infty$ for some $p\in[1,\infty)$. For any $t\in[0,T]$, (A.18) and the supermartingality of $\mathfrak{S}$ show that $\widetilde{\mathfrak{S}}_t:=\mathfrak{S}_t-E[X_T|\mathcal{F}_t]=\mathfrak{S}_t-E[\mathfrak{S}_T|\mathcal{F}_t]\ge 0$, $P-$a.s. As $\{E[X_T|\mathcal{F}_t]\}_{t\in[0,T]}$ is an càdlàg martingale, we see from Proposition A.2 that $\widetilde{\mathfrak{S}}$ is an non-negative càdlàg supermartingale.

By (A.5) and (A.3), $|\mathfrak{S}_t|\vee\big|\widetilde{\mathfrak{S}}_t\big|\le E[2|X_T|+X_*^+|\mathcal{F}_t]$, $P-$a.s. for any $t\in[0,T]$. The right-continuity of processes $\mathfrak{S}$, $\big\{E\big[|X_T|\big|\mathcal{F}_t\big]\big\}_{t\in[0,T]}$ and $\big\{E[X_*^+|\mathcal{F}_t]\big\}_{t\in[0,T]}$ then implies that $P\big\{|\mathfrak{S}_t|\vee\big|\widetilde{\mathfrak{S}}_t\big|\le E\big[2|X_T|+X_*^+|\mathcal{F}_t\big],\ t\in[0,T]\big\}=1$. It follows that $\mathfrak{S}_*\vee\widetilde{\mathfrak{S}}_*\le\sup_{t\in[0,T]}E\big[2|X_T|+X_*^+|\mathcal{F}_t\big]$, $P-$a.s. Thus Doob's martingale inequality yields that

$$E\Big[\mathfrak{S}_*^p\vee\widetilde{\mathfrak{S}}_*^p\Big]\le\big(\tfrac{p}{p-1}\big)^pE\Big[\big(2|X_T|+X_*^+\big)^p\Big]<\infty.$$

As the Doob-Meyer decomposition of $\widetilde{\mathfrak{S}}$ is $\widetilde{\mathfrak{S}}=\widetilde{M}-K$ with $\widetilde{M}_t:=M_t-E[X_T|\mathcal{F}_t]$, $t\in[0,T]$, we can deduce from the estimate (VII.15.1) of [11] that $E[K_T^p]\le p^pE\Big[\widetilde{\mathfrak{S}}_*^p\Big]<\infty$. It follows that $E\big[M_*^p\big]\le E\big[(\mathfrak{S}_*+K_T)^p\big]<\infty$. □

Let $\gamma\in\mathcal{T}$, as $\tau_k(\gamma)\le\tau_{k+1}(\gamma)$, one can define a limiting stopping time

$$\overline{\tau}(\gamma):=\lim_{k\to\infty}\uparrow\tau_k(\gamma)\in\mathcal{T}_\gamma.$$

When X is further quasi left-continuous, the next theorem demonstrates that the Snell envelope $\mathfrak{S}$ of X is a martingale over period $\big[\gamma,\overline{\tau}(\gamma)\big]$. Consequently, $\overline{\tau}(\gamma)$ is not only an optimal stopping time after γ but also the first time when X meets $\mathfrak{S}$ after γ.

Theorem A.1. *Assume that X is additionally quasi left-continuous and let $\gamma \in \mathcal{T}$.*
(1) For any $\zeta \in \mathcal{T}_\gamma$, $\mathfrak{S}_{\zeta\wedge\overline{\tau}(\gamma)} = E[\mathfrak{S}_{\overline{\tau}(\gamma)}|\mathcal{F}_\zeta]$, $P-$a.s.
(2) It holds $P-$a.s. that $\underset{\rho\in\mathcal{T}_\gamma}{\mathrm{esssup}}\, E[X_\rho|\mathcal{F}_\gamma] = E[X_{\overline{\tau}(\gamma)}|\mathcal{F}_\gamma]$ and that $\overline{\tau}(\gamma) = \tau_*(\gamma) := \inf\big\{t\in[\gamma,T]:\ \mathfrak{S}_t = X_t\big\}$.

Proof: 1) Let $\zeta\in\mathcal{T}_\gamma$. For any $k\in\mathbb{N}$, an analogy to (A.19) shows that

$$X_{\tau_k(\gamma)} \le \mathfrak{S}_{\tau_k(\gamma)} \le X_{\tau_k(\gamma)} + \frac{1}{k} \tag{A.20}$$

holds except on a $P-$null set $\mathcal{N}_k$. So we see from Proposition A.4 that

$$\mathfrak{S}_{\zeta\wedge\tau_k(\gamma)} = E\big[\mathfrak{S}_{\tau_k(\gamma)}\big|\mathcal{F}_\zeta\big] \le E\big[X_{\tau_k(\gamma)}\big|\mathcal{F}_\zeta\big] + 1/k, \quad P-\text{a.s.} \tag{A.21}$$

Since X is quasi left-continuous, (A.10) and (A.7) imply that $P-$a.s.

$$\begin{aligned}\mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}\mathfrak{S}_{\zeta\wedge\overline{\tau}(\gamma)} &= \mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}\mathfrak{S}_{\zeta} = \lim_{k\to\infty}\mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}\mathfrak{S}_{\zeta\wedge\tau_k(\gamma)} = \mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}\varliminf_{k\to\infty}\mathfrak{S}_{\zeta\wedge\tau_k(\gamma)} = \mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}\varliminf_{k\to\infty}E\big[X_{\tau_k(\gamma)}\big|\mathcal{F}_\zeta\big]\\ &\le \mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}E\big[X_{\overline{\tau}(\gamma)}\big|\mathcal{F}_\zeta\big] \le \mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}E\big[\mathfrak{S}_{\overline{\tau}(\gamma)}\big|\mathcal{F}_\zeta\big] = \mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}E\big[\mathfrak{S}_{\overline{\tau}(\gamma)}\big|\mathcal{F}_{\zeta\wedge\overline{\tau}(\gamma)}\big] \le \mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}\mathfrak{S}_{\zeta\wedge\overline{\tau}(\gamma)}\,.\end{aligned} \tag{A.22}$$

It follows that $\mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}\mathfrak{S}_{\zeta\wedge\overline{\tau}(\gamma)} = \mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}E\big[\mathfrak{S}_{\overline{\tau}(\gamma)}\big|\mathcal{F}_\zeta\big] = E\big[\mathbf{1}_{\{\zeta<\overline{\tau}(\gamma)\}}\mathfrak{S}_{\overline{\tau}(\gamma)}\big|\mathcal{F}_\zeta\big]$, $P-$a.s. As we also have

$$E[\mathbf{1}_{\{\zeta\ge\overline{\tau}(\gamma)\}}\mathfrak{S}_{\overline{\tau}(\gamma)}|\mathcal{F}_\zeta] = E[\mathbf{1}_{\{\zeta\ge\overline{\tau}(\gamma)\}}\mathfrak{S}_{\zeta\wedge\overline{\tau}(\gamma)}|\mathcal{F}_\zeta] = \mathbf{1}_{\{\zeta\ge\overline{\tau}(\gamma)\}}\mathfrak{S}_{\zeta\wedge\overline{\tau}(\gamma)}, \quad P-\text{a.s.},$$

it then holds $P-$a.s. that $\mathfrak{S}_{\zeta\wedge\overline{\tau}(\gamma)} = E[\mathfrak{S}_{\overline{\tau}(\gamma)}|\mathcal{F}_\zeta]$.
2) For any $\omega\in\{\gamma=\overline{\tau}(\gamma)\}\cap\big(\underset{k\in\mathbb{N}}{\cup}\mathcal{N}_k\big)^c$ and $k\in\mathbb{N}$, since $\big(\tau_k(\gamma)\big)(\omega) = \gamma(\omega)$, one has $X\big(\gamma(\omega),\omega\big) \le \mathfrak{S}\big(\gamma(\omega),\omega\big) \le X\big(\gamma(\omega),\omega\big) + 1/k$. Letting $k\to\infty$, we obtain

$$\mathbf{1}_{\{\gamma=\overline{\tau}(\gamma)\}}\mathfrak{S}_\gamma = \mathbf{1}_{\{\gamma=\overline{\tau}(\gamma)\}}X_\gamma = E[\mathbf{1}_{\{\gamma=\overline{\tau}(\gamma)\}}X_\gamma|\mathcal{F}_\gamma] = \mathbf{1}_{\{\gamma=\overline{\tau}(\gamma)\}}E\big[X_{\overline{\tau}(\gamma)}|\mathcal{F}_\gamma\big], \quad P-\text{a.s.} \tag{A.23}$$

On the other hand, taking $\zeta=\gamma$ in (A.22) gives that $\mathbf{1}_{\{\gamma<\overline{\tau}(\gamma)\}}\mathfrak{S}_\gamma = \mathbf{1}_{\{\gamma<\overline{\tau}(\gamma)\}}E\big[X_{\overline{\tau}(\gamma)}\big|\mathcal{F}_\gamma\big]$, $P-$a.s., which together with (A.23) and (A.5) shows that $\underset{\rho\in\mathcal{T}_\gamma}{\mathrm{esssup}}\, E[X_\rho|\mathcal{F}_\gamma] = S(\gamma) = \mathfrak{S}_\gamma = E[X_{\overline{\tau}(\gamma)}|\mathcal{F}_\gamma]$, $P-$a.s.

By (A.10) and (A.7) again, $\mathfrak{S}_\gamma = E\big[X_{\overline{\tau}(\gamma)}\big|\mathcal{F}_\gamma\big] \le E\big[\mathfrak{S}_{\overline{\tau}(\gamma)}\big|\mathcal{F}_\gamma\big] \le \mathfrak{S}_\gamma$, $P-$a.s., which implies that $\mathfrak{S}_{\overline{\tau}(\gamma)} = X_{\overline{\tau}(\gamma)}$, $P-$a.s. For any $k\in\mathbb{N}$, it follows that $\overline{\tau}(\gamma)\ge\tau_*(\gamma) = \inf\big\{t\in[\gamma,T]:\ \mathfrak{S}_t = X_t\big\} \ge \inf\big\{t\in[\gamma,T]:\ \mathfrak{S}_t\le X_t+1/k\big\} = \tau_k(\gamma)$, $P-$a.s. Letting $k\to\infty$ yields that $\overline{\tau}(\gamma) = \tau_*(\gamma)$, $P-$a.s. □

The optimal stopping time also exists in the following situation:

Theorem A.2. *Assume that the process K in decomposition (A.17) is continuous. Let $\gamma\in\mathcal{T}$ and set $\widehat{\tau}(\gamma) := \inf\{t\in(\gamma,T]:\ K_t > K_\gamma\}\wedge T\in\mathcal{T}$.*
(1) For any $\zeta\in\mathcal{T}_\gamma$, $\mathfrak{S}_{\zeta\wedge\widehat{\tau}(\gamma)} = E[\mathfrak{S}_{\widehat{\tau}(\gamma)}|\mathcal{F}_\zeta]$, $P-$a.s.
(2) It holds $P-$a.s. that $\mathfrak{S}_{\widehat{\tau}(\gamma)} = X_{\widehat{\tau}(\gamma)}$. Consequently, $\underset{\rho\in\mathcal{T}_\gamma}{\mathrm{esssup}}\, E[X_\rho|\mathcal{F}_\gamma] = E[X_{\widehat{\tau}(\gamma)}|\mathcal{F}_\gamma] = E[X_{\tau_*(\gamma)}|\mathcal{F}_\gamma]$, $P-$a.s.
(3) For any $\tau\in\mathcal{T}_\gamma$ satisfying $\underset{\rho\in\mathcal{T}_\gamma}{\mathrm{esssup}}\, E[X_\rho|\mathcal{F}_\gamma] = E[X_\tau|\mathcal{F}_\gamma]$, $P-$a.s. one has $\tau_*(\gamma)\le\tau\le\widehat{\tau}(\gamma)$, $P-$a.s. To wit, $\tau_*(\gamma)$ $\big($resp. $\widehat{\tau}(\gamma)\big)$ is the minimal (resp. maximal) optimal stopping time for $S(\gamma)$.

Proof: 1) Let $\zeta\in\mathcal{T}_\gamma$. Since the continuity of K shows that $K_{\widehat{\tau}(\gamma)} = K_\gamma$, $P-$a.s., we see from the monotonicity of K that $K_{\widehat{\tau}(\gamma)} = K_{\zeta\wedge\widehat{\tau}(\gamma)}$, $P-$a.s. The optional sampling theorem then implies that

$$E[\mathfrak{S}_{\widehat{\tau}(\gamma)}|\mathcal{F}_{\zeta\wedge\widehat{\tau}(\gamma)}] = E[M_{\widehat{\tau}(\gamma)} - K_{\zeta\wedge\widehat{\tau}(\gamma)}|\mathcal{F}_{\zeta\wedge\widehat{\tau}(\gamma)}] = M_{\zeta\wedge\widehat{\tau}(\gamma)} - K_{\zeta\wedge\widehat{\tau}(\gamma)} = \mathfrak{S}_{\zeta\wedge\widehat{\tau}(\gamma)}, \quad P-\text{a.s.}$$

It follows that $\mathbf{1}_{\{\zeta\le\widehat{\tau}(\gamma)\}}E[\mathfrak{S}_{\widehat{\tau}(\gamma)}|\mathcal{F}_\zeta] = \mathbf{1}_{\{\zeta\le\widehat{\tau}(\gamma)\}}\mathfrak{S}_{\zeta\wedge\widehat{\tau}(\gamma)}$, $P-$a.s. On the other hand, one clearly has

$$\mathbf{1}_{\{\zeta>\widehat{\tau}(\gamma)\}}E[\mathfrak{S}_{\widehat{\tau}(\gamma)}|\mathcal{F}_\zeta] = E[\mathbf{1}_{\{\zeta>\widehat{\tau}(\gamma)\}}\mathfrak{S}_{\widehat{\tau}(\gamma)}|\mathcal{F}_\zeta] = E[\mathbf{1}_{\{\zeta>\widehat{\tau}(\gamma)\}}\mathfrak{S}_{\zeta\wedge\widehat{\tau}(\gamma)}|\mathcal{F}_\zeta] = \mathbf{1}_{\{\zeta>\widehat{\tau}(\gamma)\}}\mathfrak{S}_{\zeta\wedge\widehat{\tau}(\gamma)}, \quad P-\text{a.s.}$$

Thus, $\mathfrak{S}_{\zeta\wedge\widehat{\tau}(\gamma)} = E[\mathfrak{S}_{\widehat{\tau}(\gamma)}|\mathcal{F}_\zeta]$, $P-$a.s.
2) Let $k\in\mathbb{N}$. Since $\rho_n := (\widehat{\tau}(\gamma)+\frac{1}{n})\wedge T$, $n\in\mathbb{N}$ is a decreasing sequence in $\mathcal{T}_{\widehat{\tau}(\gamma)}$ with $\lim_{n\to\infty}\downarrow\rho_n = \widehat{\tau}(\gamma)$, $\{\tau_k(\rho_n)\}_{n\in\mathbb{N}}$ is also a decreasing sequence in $\mathcal{T}_{\widehat{\tau}(\gamma)}$ with limit $\zeta_k := \lim_{n\to\infty}\downarrow\tau_k(\rho_n)\in\mathcal{T}_{\widehat{\tau}(\gamma)}$.

Given $n\in\mathbb{N}$, applying Proposition A.4 with $\gamma=\zeta=\rho_n$ and using the optional sampling theorem yield that

$$K_{\rho_n} = M_{\rho_n} - \mathfrak{S}_{\rho_n} = E[M_{\tau_k(\rho_n)} - \mathfrak{S}_{\tau_k(\rho_n)}|\mathcal{F}_{\rho_n}] = E[K_{\tau_k(\rho_n)}|\mathcal{F}_{\rho_n}], \quad P-\text{a.s.}$$

The monotonicity of K and an analogy to (A.20) then show that $P-$a.s.

$$K_{\tau_k(\rho_n)}=K_{\rho_n} \quad \text{and} \quad X_{\tau_k(\rho_n)}\le\mathfrak{S}_{\tau_k(\rho_n)}\le X_{\tau_k(\rho_n)}+1/k$$

Letting $n\to\infty$, we can deduce from the continuity of K and the right-continuity of processes X, $\mathfrak{S}$ that $P-$a.s.

$$K_{\zeta_k}=K_{\widehat{\tau}(\gamma)}=K_\gamma \quad \text{and} \quad X_{\zeta_k}\le\mathfrak{S}_{\zeta_k}\le X_{\zeta_k}+1/k.$$

The former and the definition of $\widehat{\tau}(\gamma)$ imply that $\zeta_k=\widehat{\tau}(\gamma)$, $P-$a.s., which together with the latter shows that $X_{\widehat{\tau}(\gamma)}\le\mathfrak{S}_{\widehat{\tau}(\gamma)}\le X_{\widehat{\tau}(\gamma)}+1/k$, $P-$a.s. As $k\to\infty$, we obtain

$$\mathfrak{S}_{\widehat{\tau}(\gamma)}=X_{\widehat{\tau}(\gamma)}, \quad P-\text{a.s.}, \tag{A.24}$$

it follows that $\tau_*(\gamma)=\inf\big\{t\in[\gamma,T]:\ \mathfrak{S}_t=X_t\big\}\le\widehat{\tau}(\gamma)$, $P-$a.s.

Now, applying part (1) with $\zeta=\gamma$, we see from (A.5) and (A.24) that $S(\gamma)=\mathfrak{S}_\gamma=E\big[\mathfrak{S}_{\widehat{\tau}(\gamma)}|\mathcal{F}_\gamma\big]=E\big[X_{\widehat{\tau}(\gamma)}|\mathcal{F}_\gamma\big]$, $P-$a.s. Since (A.18) shows that the set $\big\{t\in[\gamma,T]:\ \mathfrak{S}_t=X_t\big\}$ is not empty $P-$a.s., the right-continuity of processes X and $\mathfrak{S}$ implies that $\mathfrak{S}_{\tau_*(\gamma)}=X_{\tau_*(\gamma)}$, $P-$a.s. Using part (1) with $\zeta=\tau_*(\gamma)$ and taking the conditional expectation $E[\cdot|\mathcal{F}_\gamma]$ yield that

$$E\big[X_{\tau_*(\gamma)}|\mathcal{F}_\gamma\big]=E\big[\mathfrak{S}_{\tau_*(\gamma)}|\mathcal{F}_\gamma\big]=E\big[E[\mathfrak{S}_{\widehat{\tau}(\gamma)}|\mathcal{F}_{\tau_*(\gamma)}]\big|\mathcal{F}_\gamma\big] = E\big[\mathfrak{S}_{\widehat{\tau}(\gamma)}|\mathcal{F}_\gamma\big] = S(\gamma), \quad P-\text{a.s.}$$

3) Let $\tau\in\mathcal{T}_\gamma$ satisfy $S(\gamma)=E[X_\tau|\mathcal{F}_\gamma]$, $P-$a.s. Since (A.5), (A.10) and (A.7) show that $\mathfrak{S}_\gamma=S(\gamma)=E[X_\tau|\mathcal{F}_\gamma]\le E[\mathfrak{S}_\tau|\mathcal{F}_\gamma]\le\mathfrak{S}_\gamma$, $P-$a.s., we have

$$E[X_\tau|\mathcal{F}_\gamma]=E[\mathfrak{S}_\tau|\mathcal{F}_\gamma]=\mathfrak{S}_\gamma, \quad P-\text{a.s.} \tag{A.25}$$

The first equality and (A.10) imply that $\mathfrak{S}_\tau=X_\tau$, $P-$a.s. and thus that $\tau_*(\gamma)\le\tau$, $P-$a.s. On the other hand, one can deduce from the second equality of (A.25) and the optional sampling theorem that

$$K_\gamma = M_\gamma - \mathfrak{S}_\gamma = E[M_\tau - \mathfrak{S}_\tau|\mathcal{F}_\gamma] = E[K_\tau|\mathcal{F}_\gamma], \quad P-\text{a.s.}$$

It follows from the monotonicity of K that $K_\tau=K_\gamma$, $P-$a.s. and thus that $\tau\le\widehat{\tau}(\gamma)$, $P-$a.s. □

Example A.1. *When X is l.u.s.c.e., the process K in decomposition* (A.17) *is continuous.*

Proof: Let $\gamma\in\mathcal{T}$. For any $k\in\mathbb{N}$, taking $\zeta=\gamma$ in (A.21) and then taking expectation yield that $E\big[\mathfrak{S}_\gamma\big]\le E\big[X_{\tau_k(\gamma)}\big]+1/k$. As $k\to\infty$, the l.u.s.c.e. of X, (A.10) and (A.7) imply that

$$E\big[\mathfrak{S}_\gamma\big]\le\varliminf_{k\to\infty}E\big[X_{\tau_k(\gamma)}\big]\le E\big[X_{\overline{\tau}(\gamma)}\big]\le E\big[\mathfrak{S}_{\overline{\tau}(\gamma)}\big]=E\big[E\big[\mathfrak{S}_{\overline{\tau}(\gamma)}|\mathcal{F}_\gamma\big]\big]\le E\big[\mathfrak{S}_\gamma\big]. \tag{A.26}$$

Next, fix $\{\gamma_n\}_{n\in\mathbb{N}}$ be an increasing sequence in $\mathcal{T}$ and set $\overline{\gamma}:=\lim_{n\to\infty}\uparrow\gamma_n\in\mathcal{T}$. Given $n\in\mathbb{N}$, we see from (A.7) that

$$\mathfrak{S}_{\gamma_n}\ge E\big[\mathfrak{S}_{\gamma_{n+1}}|\mathcal{F}_{\gamma_n}\big]\ge E\Big[E\big[\mathfrak{S}_{\overline{\gamma}}|\mathcal{F}_{\gamma_{n+1}}\big]\big|\mathcal{F}_{\gamma_n}\Big]\ge E\big[\mathfrak{S}_{\overline{\gamma}}|\mathcal{F}_{\gamma_n}\big], \quad P-\text{a.s.}$$

Taking expectation gives that $E\big[\mathfrak{S}_{\gamma_n}\big]\ge E\big[\mathfrak{S}_{\gamma_{n+1}}\big]\ge E\big[\mathfrak{S}_{\overline{\gamma}}\big]$, it follows that $\lim_{n\to\infty}\downarrow E\big[\mathfrak{S}_{\gamma_n}\big]\ge E\big[\mathfrak{S}_{\overline{\gamma}}\big]$.

For any $n\in\mathbb{N}$, taking $\gamma=\gamma_n$ in (A.26) shows that $E\big[\mathfrak{S}_{\gamma_n}\big]=E\big[X_{\overline{\tau}(\gamma_n)}\big]$. Clearly, $\big\{\overline{\tau}(\gamma_n)\big\}_{n\in\mathbb{N}}$ is also an increasing sequence in $\mathcal{T}$. As $\zeta:=\lim_{n\to\infty}\uparrow\overline{\tau}(\gamma_n)\in\mathcal{T}$ satisfies that $\zeta\ge\lim_{n\to\infty}\uparrow\gamma_n=\overline{\gamma}$, we can deduce from the l.u.s.c.e. of X, (A.10) and (A.7) that

$$E\big[\mathfrak{S}_{\overline{\gamma}}\big]\le\lim_{n\to\infty}\downarrow E\big[\mathfrak{S}_{\gamma_n}\big]=\lim_{n\to\infty}\downarrow E\big[X_{\overline{\tau}(\gamma_n)}\big]\le E\big[X_\zeta\big]\le E\big[\mathfrak{S}_\zeta\big]=E\big[E[\mathfrak{S}_\zeta|\mathcal{F}_{\overline{\gamma}}]\big]\le E\big[\mathfrak{S}_{\overline{\gamma}}\big].$$

So $\lim_{n\to\infty}\downarrow E\big[\mathfrak{S}_{\gamma_n}\big]=E\big[\mathfrak{S}_{\overline{\gamma}}\big]$, which further implies that the $\mathbf{F}-$predictable projection of $\mathfrak{S}$ is $\{\mathfrak{S}_{t-}\}_{t\in[0,T]}$ $\big($see e.g. Remark VI.50 of [11]$\big)$. Then we know from e.g. Theorem VII.10 of [11] that K is a continuous process. □

Proposition A.6. *Let K be the $\mathbf{F}-$predictable càdlàg increasing process in the decomposition* (A.17)*. If process X is also càdlàg, then*

$$\int_0^T (\mathfrak{S}_{t-} - X_{t-}) dK_t = 0, \quad P-a.s. \tag{A.27}$$

Proof: By Remark 3.2 (1), the jumps of process K are exhausted by a sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ of $\mathbf{F}-$predictable stopping times. Let us denote by K^c (resp. K^d) the continuous part (resp. purely discontinuous part) of K. Similar to (5.2) and (5.3), the demonstration of (A.27) is equivalent to show that

$$\int_0^T (\mathfrak{S}_t - X_t) dK_t^c = 0, \quad P-\text{a.s.}, \tag{A.28}$$

and that

$$0 = \sum_{t\in[0,T]} \big(\mathfrak{S}_{s-} - X_{s-}\big) \Delta K_s^d = \sum_{n\in\mathbb{N}} \big(\mathfrak{S}_{\gamma_n -} - X_{\gamma_n -}\big) \Delta K_{\gamma_n}^d, \quad P-\text{a.s.} \tag{A.29}$$

By Proposition A.2, $\mathfrak{S}$ is the smallest càdlàg supermartingale that dominates X. Correspondingly, $\widehat{\mathfrak{S}} := \mathfrak{S} + K^d$ the smallest càdlàg supermartingale that dominates $\widehat{X} := X + K^d$. To wit, $\widehat{\mathfrak{S}}$ is the Snell envelope of $\widehat{X}$ with Doob-Meyer decomposition $\widehat{\mathfrak{S}} = M - K^c$.

Given $t \in [0,T)$, we define $\tau_t := \inf\{s \in (t,T] : K_s^c > K_t^c\} \wedge T \in \mathcal{T}$. Theorem A.2 (2) show that $\widehat{\mathfrak{S}}_{\tau_t} = \widehat{X}_{\tau_t}$, $P-$a.s. or

$$\mathfrak{S}_{\tau_t} = X_{\tau_t} \text{ holds except on a } P-\text{null set } \mathcal{N}_t. \tag{A.30}$$

By (A.10) and (A.18), there exists another $P-$null set $\widehat{\mathcal{N}}$ such that for any $\omega \in \widehat{\mathcal{N}}^c$, the path $\mathfrak{S}_\cdot(\omega) - X_\cdot(\omega) \ge 0$ is càdlàg, $\mathfrak{S}_T(\omega) = X_T(\omega)$ and the path $K_\cdot^c(\omega)$ is continuous.

Let $\omega \in \widehat{\mathcal{N}}^c \cap \Big(\underset{r\in[0,T)\cap\mathbb{Q}}{\cap} \mathcal{N}_r^c\Big)$ and set $I(\omega) := \big\{t \in (0,T) : K_{t-\varepsilon}^c(\omega) = K_{t+\varepsilon}^c(\omega) \text{ for some } \varepsilon \in \big(0, t \wedge (T-t]\big)\big\}$. As an open set, $I(\omega)$ can be written as a countable union of disjoint open intervals: $I(\omega) = \underset{i\in\mathbb{N}}{\cup} \big(a_i(\omega), b_i(\omega)\big)$. The continuity of $K_\cdot^c(\omega)$ implies that

$$K^c\big(b_i(\omega), \omega\big) = K^c\big(a_i(\omega), \omega\big) \quad \forall\, i \in \mathbb{N} \quad \text{and} \tag{A.31}$$

$$\widetilde{I}(\omega) := \underset{i\in\mathbb{N}}{\cup} \big[a_i(\omega), b_i(\omega)\big) = \big\{t \in [0,T) : K_t^c(\omega) = K_{t+\varepsilon}^c(\omega) \text{ for some } \varepsilon \in (0, T-t]\big\}. \tag{A.32}$$

Let $t \in [0,T) \backslash \widetilde{I}(\omega)$. By the continuity of $K_\cdot^c(\omega)$ again, there exists a strictly decreasing sequence $\{t_n(\omega)\}_{n\in\mathbb{N}}$ with $\lim\limits_{n\to\infty} \downarrow t_n(\omega) = t$ such that $K^c\big(t_n(\omega), \omega\big)$ is also a strictly decreasing sequence with $\lim\limits_{n\to\infty} \downarrow K^c\big(t_n(\omega), \omega\big) = K^c(t, \omega)$. Given $n \in \mathbb{N}$, let $r_n \in (t, t_{n+1}) \cap \mathbb{Q}$. Since $K^c(r_n, \omega) \le K^c\big(t_{n+1}(\omega), \omega\big) < K^c\big(t_n(\omega), \omega\big)$, we see from the definition of τ_{r_n} that $t < r_n \le \tau_{r_n}(\omega) < t_n(\omega)$, it follows that $\lim\limits_{n\to\infty} \tau_{r_n}(\omega) = t$. Then the continuity of $\mathfrak{S}_\cdot(\omega) - X_\cdot(\omega)$ and (A.30) imply that $\mathfrak{S}_t(\omega) - X_t(\omega) = \lim\limits_{n\to\infty} \big(\mathfrak{S}\big(\tau_{r_n}(\omega), \omega\big) - X\big(\tau_{r_n}(\omega), \omega\big)\big) = 0$, which shows that $[0,T] \backslash \widetilde{I}(\omega) \subset \{t \in [0,T] : \mathfrak{S}_t(\omega) = X_t(\omega)\}$, or equivalently, $\{t \in [0,T] : \mathfrak{S}_t(\omega) > X_t(\omega)\} \subset \widetilde{I}(\omega)$. Consequently, one can deduce from (A.31) and (A.32) that

$$0 \le \int_0^T \mathbf{1}_{\{\mathfrak{S}_t(\omega) > X_t(\omega)\}} dK_t^c(\omega) \le \int_0^T \mathbf{1}_{\{t\in\widetilde{I}(\omega)\}} dK_t^c(\omega) \le \sum_{i\in\mathbb{N}} \Big(K^c\big(b_i(\omega), \omega\big) - K^c\big(a_i(\omega), \omega\big)\Big) = 0,$$

which leads to (A.28).

Moreover, for any $n \in \mathbb{N}$, Proposition A.5 implies that $\mathbf{1}_{\{\mathfrak{S}_{\gamma_n -} > X_{\gamma_n -}\}} \Delta K_{\gamma_n}^d = \mathbf{1}_{\{\mathfrak{S}_{\gamma_n -} > X_{\gamma_n -}\}} \Delta K_{\gamma_n} = 0$, $P-$a.s. or equivalently $(\mathfrak{S}_{\gamma_n -} - X_{\gamma_n -}) \Delta K_{\gamma_n}^d = 0$, $P-$a.s. Summing them up over $n \in \mathbb{N}$ leads to (A.29). □

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