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A WEAK DYNAMIC PROGRAMMING PRINCIPLE FOR ZERO-SUM STOCHASTIC DIFFERENTIAL GAMES WITH UNBOUNDED CONTROLS*

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Abstract. We analyze a zero-sum stochastic differential game between two competing players who can choose unbounded controls. The payoffs of the game are defined through backward stochastic differential equations. We prove that each player's priority value satisfies a weak dynamic programming principle and thus solves the associated fully nonlinear partial differential equation in the viscosity sense.

Key words. zero-sum stochastic differential games, Elliott–Kalton strategies, weak dynamic programming principle, backward stochastic differential equations, viscosity solutions, fully nonlinear PDEs

AMS subject classifications. 49N70, 91A23, 91A60, 49L20, 49L25

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1. Introduction.

In this paper we extend the study of Buckdahn and Li [12] on a zero-sum stochastic differential game (SDG), whose payoffs are generated by backward stochastic differential equations (BSDEs), to the case of super-square-integrable controls (see Remark 2.1).

Since the seminal paper by Fleming and Souganidis [17], the SDG theory has grown rapidly in many aspects (see, e.g., the references in [12, 11]). Among these developments, Hamadène et al. [20, 19, 15] introduced a (decoupled) SDE-BSDE system, with controls only in the drift coefficients, to generate the payoffs in their studies of saddle point problems of SDGs. (For the evolution and applications of the BSDE theory, see Pardoux and Peng [28], El Karoui, Peng, and Quenez [16], and the references therein.) Later on, [12] as well as its sequels [14, 13, 11] generalized the SDE-BSDE framework so that the two competing controllers can also influence the diffusion coefficient of the state dynamics. Unlike [17], [12] used a uniform canonical space $\Omega = \big\{\omega \in \mathbb{C}([0,T];\mathbb{R}^d) : \omega(0) = 0\big\}$ so that admissible control processes can also depend on the information occurring before the start of the game. Such a setting allows the authors of [12] to get around a relatively complicated approximation argument of [17] which was due to a measurability issue (see Remark 2.5), and allows them to adopt the notion of stochastic backward semigroups and a BSDE method, developed in [31, 29], to obtain results similar to [17]: the lower and upper values of the SDG satisfy a dynamic programming principle and solve the associated Hamilton–Jacobi–Bellman–Isaacs equations in the viscosity sense. However, [12, 17] as well as some latest advances to the SDG theory (e.g., Bouchard, Moreau, and Nutz [7] on stochastic target games, Peng and Xu [30] on SDGs in form of a generalized BSDE with random default time) still assume the compactness of control spaces while Pham and Zhang

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[33] on weak formulation of SDGs assumes the boundedness of coefficients in control variables. We are going to address these particular issues.

In the present paper, since two players take super-square-integrable controls over two separable metric spaces $\mathbb{U}$ and $\mathbb{V}$ not necessarily compact, those approximation methods of [17] and [12] in proving the dynamic programming principle are no longer effective. Instead, we derive a weak form of dynamic programming principle in the spirit of Bouchard and Touzi [8] and use it to show that each player's priority value solves the corresponding fully nonlinear PDE in the viscosity sense. Vitoria [34] has tried to extend the SDG for unbounded controls by proving a weak dynamic programming principle. However, it still assumed that the control space of the player with priority is compact; see Theorem 75 of [34].

Square-integrable controls were initially considered by Krylov [25, Chapter 6], however, for cooperative games (i.e., the so-called sup sup case). Browne [10] studied a specific zero-sum investment game between two small investors who control the game via their square-integrable portfolios. Since the PDEs in this case have smooth solutions, the problem can be solved by a verification theorem instead of the dynamic programming principle. Inspired by the "tug-of-war" (a discrete-time random turn game; see, e.g., [32] and [26]), Atar and Budhiraja [1] studied a zero-sum stochastic differential game with $\mathbb{U} = \mathbb{V} = \{x \in \mathbb{R}^n : |x| = 1\} \times [0, \infty)$ played until the state process exits a given domain. As in Chapter 6 of [25], the authors approximated such a game with unbounded controls by a sequence of games with bounded controls which satisfy a dynamic programming principle. They showed the equicontinuity of the approximating sequence and thus proved that the value function of the game is a unique viscosity solution to the inhomogenous infinity Laplace equation. We do not rely on this approximation scheme but directly prove a weak dynamic programming principle for the game with super-square-integrable controls.

Following the probabilistic setting of [12] (see Remark 2.5), our paper takes the canonical space $\Omega = \big\{\omega \in \mathbb{C}([0,T];\mathbb{R}^d) : \omega(0) = 0\big\}$, whose coordinator process B is a Brownian motion under the Wiener measure P. When the game starts from time $t \in [0,T]$, under the super-square-integrable controls $\mu \in \mathcal{U}_t$ and $\nu \in \mathcal{V}_t$ selected by player I and II respectively, the state process $X^{t,\xi,\mu,\nu}$ starting from a random initial state ξ will then evolve according to a stochastic differential equation (SDE):

$$X_s = \xi + \int_t^s b\big(r, X_r, \mu_r, \nu_r\big)\, dr + \int_t^s \sigma\big(r, X_r, \mu_r, \nu_r\big)\, dB_r, \quad s \in [t,T], \tag{1.1}$$

where the drift b and the diffusion σ are Lipschitz continuous in x and have linear growth in (u,v). The payoff player I will receive from player II is determined by the first component of the unique solution $\big(Y^{t,\xi,\mu,\nu}, Z^{t,\xi,\mu,\nu}\big)$ to the following BSDE:

$$Y_s = g\big(X_T^{t,\xi,\mu,\nu}\big) + \int_s^T f\big(r, X_r^{t,\xi,\mu,\nu}, Y_r, Z_r, \mu_r, \nu_r\big)\, dr - \int_s^T Z_r dB_r, \quad s \in [t,T]. \tag{1.2}$$

Here the generator f is Lipschitz continuous in (y,z) and also has linear growth in (u,v). When g and f are $2/p-$Hölder continuous in x for some $p \in (1,2]$, $Y^{t,\xi,\mu,\nu}$ is p-integrable. As we see from (1.1) and (1.2), the controls μ, ν influence the game in two aspects: both affect (1.2) via the state process $X^{t,\xi,\mu,\nu}$ and appear directly in the generator f of (1.2) as parameters. In particular, if f is independent of (y,z), Y is in the form of the conditional linear expectation of the terminal reward $g\big(X_T^{t,\xi,\mu,\nu}\big)$ plus the cumulative reward $\int_s^T f(r, X_r^{t,\xi,\mu,\nu}, \mu_r, \nu_r)\, dr$ (cf. [17]).

When the player (e.g., player I) with the priority chooses first a super-square-integrable control (e.g., $\mu \in \mathcal{U}_t$), its opponent (e.g., player II) will select its reacting control via a nonanticipative mapping $\beta : \mathcal{U}_t \to \mathcal{V}_t$, called *Elliott–Kalton strategy*. In particular, using Elliott–Kalton strategies is essential in proving the dynamic programming principle. This phenomenon already appears in the controller-stopper games, i.e., when one of the players is endowed with the right of stopping the game instead of using a control; see [2], which shows that if the stopper acts second, it is necessary that the stopper uses nonanticipative strategies in order to prove a dynamic programming principle. This type of phenomenon does not appear (or it is implicitly satisfied) if the controllers only control the drift (see, e.g., [3] and the references therein) or when there are two stoppers (the so-called Dynkin games); see, e.g., [4] and the references therein.

By $w_1(t,x) \stackrel{\triangle}{=} \operatorname{essinf}_{\beta \in \mathfrak{B}_t} \operatorname{esssup}_{\mu \in \mathcal{U}_t} Y_t^{t,x,\mu,\beta(\mu)}$ we denote player I's priority value of the game starting from time t and state x, where $\mathfrak{B}_t$ collects all strategies for player II. Switching the priority defines Player II's priority value $w_2(t,x)$.

Although our setting makes the payoffs $Y_t^{t,\xi,\mu,\nu}$ random variables, we can show like [12] that $w_1(t,x)$ and $w_2(t,x)$ are invariant under Girsanov transformation via functions of the Cameron–Martin space and are thus deterministic; see Lemma 2.2. To assure values $w_1(t,x)$ and $w_2(t,x)$ are finite, we assume that each player has some control; neutralizer for coefficients (b,σ,f) (such an assumption holds for additive controls; see Example 2.1), and impose a growth condition on strategies. These two requirements are also crucial in proving our weak dynamic programming principle. When $\mathbb{U}$ and $\mathbb{V}$ are compact, the control neutralizers become futile and the growth condition holds automatically for strategies. Thus our problem degenerates to [12]'s case; see Remark 2.4.

Although value functions $w_1(t,x)$, $w_2(t,x)$ are still $2/p$-Hölder continuous in x (see Proposition 2.3), they may not be continuous in t. Hence we cannot follow [12]'s approach to get a strong form of dynamic programming principle for w_1 and w_2. Instead, we prove a weak dynamic programming principle, say for w_1:

$$\begin{aligned}
&\operatorname*{essinf}_{\beta \in \mathfrak{B}_t} \operatorname*{esssup}_{\mu \in \mathcal{U}_t} Y_t^{t,x,\mu,\beta(\mu)}\Big(\tau_{\beta,\mu}, \phi\big(\tau_{\beta,\mu}, X_{\tau_{\beta,\mu}}^{t,x,\mu,\beta(\mu)}\big)\Big)\\
&\quad \le w_1(t,x) \le \operatorname*{essinf}_{\beta \in \mathfrak{B}_t} \operatorname*{esssup}_{\mu \in \mathcal{U}_t} Y_t^{t,x,\mu,\beta(\mu)}\Big(\tau_{\beta,\mu}, \widetilde{\phi}\big(\tau_{\beta,\mu}, X_{\tau_{\beta,\mu}}^{t,x,\mu,\beta(\mu)}\big)\Big),
\end{aligned}$$

for any two continuous functions $\phi \le w_1 \le \widetilde{\phi}$. Here $\tau_{\beta,\mu}$ denotes the first existing time of state process $X^{t,x,\mu,\beta(\mu)}$ from the given open ball $O_\delta(t,x)$.

To prove the weak dynamic programming principle, we first approximate $w_1(t,x) = \operatorname{essinf}_{\beta \in \mathfrak{B}_t} I(t,x,\beta)$ from above and $I(t,x,\beta) \stackrel{\triangle}{=} \operatorname{esssup}_{\mu \in \mathcal{U}_t} Y_t^{t,x,\mu,\beta(\mu)}$ from below in a probabilistic sense (see Lemma 4.2) so that we can construct approximately optimal controls/strategies by a pasting technique similar to the one used in [8] and [34]. Then we make a series of estimates and eventually obtain the weak dynamic programming principle by using a stochastic backward semigroup property (2.11), the continuous dependence of payoff process on the initial state (see Lemma 2.3) as well as the control-neutralizer assumption and the growth condition on strategies.

Next, one can deduce from the weak dynamic programming principle and the separability of control space $\mathbb{U}$, $\mathbb{V}$ that the value functions w_1 and w_2 are (discontinuous) viscosity solutions of the corresponding fully nonlinear PDEs; see Theorem 3.1. Recently, Krylov [24] and [23] studied the regularity of solutions to related fully nonlinear PDEs: The former obtained $C^{1,1} \cap W^{1,2}_{\infty,loc}$-solutions for the case of bounded measurable coefficients; while the latter showed the existence of L^p-viscosity solutions in $C^{1+\alpha}$ if the fully nonlinear Hamiltonian function is continuous in gradient variable and Lipschitz continuous in Hessian variable.

The rest of the paper is organized as follows: After listing the notations to use, we recall some basic properties of BSDEs in section 1. In section 2, we set up the zero-sum stochastic differential games based on BSDEs and present a weak dynamic programming principle for priority values of both players defined via Elliott–Kalton strategies. With help of the weak dynamic programming principle, we show in section 3 that the priority values are (discontinuous) viscosity solutions of the corresponding fully nonlinear PDEs. The proofs of our results are deferred to section 4.

1.1. Notation and preliminaries. Let $(\mathbb{M}, \rho_{\mathbb{M}})$ be a generic metric space, and let $\mathscr{B}(\mathbb{M})$ be the Borel σ-field of $\mathbb{M}$. For any $x \in \mathbb{M}$ and $\delta > 0$, $O_\delta(x) \stackrel{\triangle}{=} \{x' \in \mathbb{M} : \rho_{\mathbb{M}}(x, x') < \delta\}$ and $\overline{O}_\delta(x) \stackrel{\triangle}{=} \{x' \in \mathbb{M} : \rho_{\mathbb{M}}(x, x') \le \delta\}$, respectively, denote the open and closed ball centered at x with radius δ. For any function $\phi : \mathbb{M} \to \mathbb{R}$, we define the lower/upper semi-continuous envelopes by

$$\varliminf_{x' \to x} \phi(x') \stackrel{\triangle}{=} \lim_{n\to\infty} \uparrow \inf_{x' \in O_{\frac{1}{n}}(x)} \phi(x') \quad \text{ and } \quad \varlimsup_{x' \to x} \phi(x') \stackrel{\triangle}{=} \lim_{n\to\infty} \downarrow \sup_{x' \in O_{\frac{1}{n}}(x)} \phi(x'),$$

where $\lim_{n\to\infty}$ (resp., $\lim_{n\to\infty}$) denotes the limit of a decreasing (resp., increasing) sequence.

Fix $d \in \mathbb{N}$ and a time horizon $T \in (0, \infty)$. We consider the canonical space $\Omega \stackrel{\triangle}{=} \big\{\omega \in \mathbb{C}([0, T]; \mathbb{R}^d) : \omega(0) = 0\big\}$ equipped with Wiener measure P, under which the canonical process B is a d-dimensional Brownian motion. Let $\mathbf{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$ be the filtration generated by B and augmented by all P-null sets. We denote by $\mathscr{P}$ the $\mathbf{F}$-progressively measurable σ-field of $[0, T] \times \Omega$.

Given $t \in [0, T]$, let $\mathcal{S}_{t,T}$ collect all $\mathbf{F}$-stopping times τ with $t \le \tau \le T$, P-almost surely. For any $\tau \in \mathcal{S}_{t,T}$ and $A \in \mathcal{F}_\tau$, we define $[\![t, \tau[\![_A \stackrel{\triangle}{=} \big\{(r, \omega) \in [t, T] \times A : r < \tau(\omega)\big\}$ and $[\![\tau, T]\!]_A \stackrel{\triangle}{=} \big\{(r, \omega) \in [t, T] \times A : r \ge \tau(\omega)\big\}$ for any $A \in \mathcal{F}_\tau$. In particular, $[\![t, \tau[\![\stackrel{\triangle}{=} [\![t, \tau[\![_\Omega$ and $[\![\tau, T]\!] \stackrel{\triangle}{=} [\![\tau, T]\!]_\Omega$ are the stochastic intervals.

Let $\mathbb{E}$ be a generic Euclidian space. For any $p \in [1, \infty)$ and $t \in [0, T]$, we introduce some spaces of functions:

(1) For sub-σ-field $\mathcal{G}$ of $\mathcal{F}_T$, let $\mathbb{L}^p(\mathcal{G}, \mathbb{E})$ be the space of $\mathbb{E}$-valued, $\mathcal{G}$-measurable random variables ξ with $\|\xi\|_{\mathbb{L}^p(\mathcal{G},\mathbb{E})} \stackrel{\triangle}{=} \big\{E\big[|\xi|^p\big]\big\}^{1/p} < \infty$, and let $\mathbb{L}^\infty(\mathcal{G}, \mathbb{E})$ be the space of all $\mathbb{E}$-valued, $\mathcal{G}$-measurable bounded random variables.

(2) $\mathbb{C}^p_{\mathbf{F}}([t, T], \mathbb{E})$ denotes the space of $\mathbb{E}$-valued, $\mathbf{F}$-adapted processes $\{X_s\}_{s\in[t,T]}$ with P-a.s. continuous paths such that $\|X\|_{\mathbb{C}^p_{\mathbf{F}}([t,T],\mathbb{E})} \stackrel{\triangle}{=} \{E[\sup_{s\in[t,T]} |X_s|^p]\}^{1/p} < \infty$.

(3) $\mathbb{H}^{p,loc}_{\mathbf{F}}([t, T], \mathbb{E})$ denotes the space of $\mathbb{E}$-valued, $\mathbf{F}$-progressively measurable processes $\{X_s\}_{s\in[t,T]}$ such that $\int_t^T |X_s|^p \, ds < \infty$, P-a.s. For any $\widehat{p} \in [1, \infty)$, $\mathbb{H}^{p,\widehat{p}}_{\mathbf{F}}([t, T], \mathbb{E})$ denotes the space of $\mathbb{E}$-valued, $\mathbf{F}$-progressively measurable processes $\{X_s\}_{s\in[t,T]}$ with $\|X\|_{\mathbb{H}^{p,\widehat{p}}_{\mathbf{F}}([t,T],\mathbb{E})} \stackrel{\triangle}{=} \{E[(\int_t^T |X_s|^p \, ds)^{\widehat{p}/p}]\}^{1/\widehat{p}} < \infty$.

(4) We also set $\mathbb{G}^p_{\mathbf{F}}([t, T]) \stackrel{\triangle}{=} \mathbb{C}^p_{\mathbf{F}}([t, T], \mathbb{R}) \times \mathbb{H}^{2,p}_{\mathbf{F}}([t, T], \mathbb{R}^d)$.

If $\mathbb{E} = \mathbb{R}$, we will drop it from the above notation. Moreover, we will use the convention $\inf \emptyset = \infty$.

1.2. Backward stochastic differential equations. Given $t \in [0,T]$, a t-parameter set $\big(\eta, f\big)$ consists of a random variable $\eta \in \mathbb{L}^0\big(\mathcal{F}_T\big)$ and a function $f : [t,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ that is $\mathscr{P} \otimes \mathscr{B}(\mathbb{R}) \otimes \mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable. In particular, $\big(\eta, f\big)$ is called a (t,p)-parameter set for some $p \in [1,\infty)$ if $\eta \in \mathbb{L}^p\big(\mathcal{F}_T\big)$.

DEFINITION 1.1. *Given a t-parameter set $\big(\eta, f\big)$ for some $t \in [0,T]$, a pair $(Y,Z) \in \mathbb{C}^0_{\mathbf{F}}([t,T]) \times \mathbb{H}^{2,loc}_{\mathbf{F}}([t,T],\mathbb{R}^d)$ is called a solution of the backward stochastic differential equation on the probability space $(\Omega, \mathcal{F}_T, P)$ over period $[t,T]$ with terminal condition η and generator f $\big(BSDE\big(t,\eta,f\big)$ for short$\big)$ if it holds P-a.s. that*

$$Y_s = \eta + \int_s^T f(r, Y_r, Z_r)\,dr - \int_s^T Z_r dB_r, \quad s \in [t,T]. \tag{1.3}$$

Analogous to Theorem 4.2 of [9], we have the following well-posedness, a priori estimate and comparison results of BSDE (1.3); see [6] for the proofs.

PROPOSITION 1.1. *Given $t \in [0,T]$ and $p \in [1,\infty)$, let $\big(\eta, f\big)$ be a (t,p)-parameter set such that f is Lipschitz continuous in (y,z): i.e., for some $\gamma > 0$, it holds for $ds \times dP$-a.s. $(s,\omega) \in [t,T] \times \Omega$ that*

$$\big|f(s,\omega,y,z) - f(s,\omega,y',z')\big| \le \gamma\big(|y-y'| + |z-z'|\big) \quad \forall\, y,y' \in \mathbb{R},\ \forall\, z,z' \in \mathbb{R}^d.$$

If $E\Big[\big(\int_t^T \big|f(s,0,0)\big|ds\big)^p\Big] < \infty$, BSDE (1.3) admits a unique solution $\big(Y,Z\big) \in \mathbb{G}^p_{\mathbf{F}}\big([t,T]\big)$ that satisfies

$$E\left[\sup_{s\in[t,T]} |Y_s|^p \Big| \mathcal{F}_t\right] \le C(T,p,\gamma) E\left[|\eta|^p + \left(\int_t^T \big|f(s,0,0)\big|ds\right)^p \Bigg| \mathcal{F}_t\right], \quad P\text{-a.s.} \tag{1.4}$$

PROPOSITION 1.2. *Given $t \in [0,T]$ and $p \in [1,\infty)$, let $\big(\eta_i, f_i\big), i = 1,2$ be two (t,p)-parameter sets such that f_1 is Lipschitz continuous in (y,z), and let $\big(Y^i, Z^i\big) \in \mathbb{G}^p_{\mathbf{F}}([t,T])$, $i = 1,2$ be a solution of $BSDE\big(t,\eta_i,f_i\big)$.*

(1) *If $E[(\int_t^T \big|f_1(s,Y^2_s,Z^2_s) - f_2(s,Y^2_s,Z^2_s)\big|ds)^{\widetilde{p}}] < \infty$ for some $\widetilde{p} \in (1,p]$, then it holds P-a.s. that*

$$\begin{aligned} E\left[\sup_{s\in[t,T]} \big|Y^1_s - Y^2_s\big|^{\widetilde{p}} \Big| \mathcal{F}_t\right] \\ \le C(T,\widetilde{p},\gamma) E\left[\big|\eta_1 - \eta_2\big|^{\widetilde{p}} + \left(\int_t^T \big|f_1(s,Y^2_s,Z^2_s) - f_2(s,Y^2_s,Z^2_s)\big|ds\right)^{\widetilde{p}} \Bigg| \mathcal{F}_t\right]. \end{aligned} \tag{1.5}$$

(2) *If $\eta_1 \le$ (resp., $\ge$) η_2, P-a.s. and if $f_1(s,Y^2_s,Z^2_s) \le$ (resp., $\ge$) $f_2(s,Y^2_s,Z^2_s)$, $ds \times dP$-a.s. on $[t,T] \times \Omega$, then it holds P-a.s. that $Y^1_s \le$ (resp., $\ge$) Y^2_s for any $s \in [t,T]$.*

2. Stochastic differential games with unbounded controls. Let $(\mathbb{U}, \rho_{\mathbb{U}})$ and $(\mathbb{V}, \rho_{\mathbb{V}})$ be two separable metric spaces. For some $u_0 \in \mathbb{U}$ and $v_0 \in \mathbb{V}$, we define

$$[u]_{\mathbb{U}} \stackrel{\triangle}{=} \rho_{\mathbb{U}}(u,u_0)\ \ \forall\, u \in \mathbb{U} \quad \text{and} \quad [v]_{\mathbb{V}} \stackrel{\triangle}{=} \rho_{\mathbb{V}}(v,v_0)\ \ \forall\, v \in \mathbb{V}.$$

We shall study a zero-sum stochastic differential game between two players, player I and player II, who choose super-square-integrable $\mathbb{U}$-valued controls and $\mathbb{V}$-valued controls, respectively, to compete.

DEFINITION 2.1. *Given $t \in [0,T]$, an admissible control process $\mu = \{\mu_s\}_{s\in[t,T]}$ for player* I *over period $[t,T]$ is a $\mathbb{U}$-valued, $\mathbf{F}$-progressively measurable process such that $E\int_t^T [\mu_s]_{\mathbb{U}}^q\, ds < \infty$ for some $q > 2$. Admissible control processes for player* II *over period $[t,T]$ are defined similarly. We denote by $\mathcal{U}_t$ (resp., $\mathcal{V}_t$) the set of all admissible controls for player* I *(resp.,* II*) over period $[t,T]$.*

Remark 2.1. The reason why we use super-square-integrable controls lies in the fact that in the proof of Proposition 2.2, the set of $\mathbb{U}$-valued (resp., $\mathbb{V}$-valued) square integrable processes is not closed under Girsanov transformation via functions of the Cameron–Martin space (see, in particular, (4.12)).

Clearly, connecting two $\mathcal{U}_t$-controls along some $\tau \in \mathcal{S}_{t,T}$ results in a new $\mathcal{U}_t$-control.

LEMMA 2.1. *Let $t \in [0,T]$ and $\tau \in \mathcal{S}_{t,T}$. For any $\mu^1, \mu^2 \in \mathcal{U}_t$, $\mu_s \stackrel{\triangle}{=} \mathbf{1}_{\{s<\tau\}}\mu_s^1 + \mathbf{1}_{\{s\ge\tau\}}\mu_s^2$, $s \in [t,T]$ defines a $\mathcal{U}_t$-control. Similarly, for any $\nu^1, \nu^2 \in \mathcal{V}_t$, $\nu_s \stackrel{\triangle}{=} \mathbf{1}_{\{s<\tau\}}\nu_s^1 + \mathbf{1}_{\{s\ge\tau\}}\nu_s^2$, $s \in [t,T]$ defines a $\mathcal{V}_t$-control.*

2.1. Game setting: A controlled SDE-BSDE system. Our zero-sum stochastic differential game is formulated via a (decoupled) SDE-BSDE system with the following parameters: Fix $k \in \mathbb{N}$, $\gamma > 0$ and $p \in (1,2]$.

(1) Let $b : [0,T]\times\mathbb{R}^k\times\mathbb{U}\times\mathbb{V} \to \mathbb{R}^k$ be a $\mathscr{B}([0,T])\otimes\mathscr{B}(\mathbb{R}^k)\otimes\mathscr{B}(\mathbb{U})\otimes\mathscr{B}(\mathbb{V})/\mathscr{B}(\mathbb{R}^k)$-measurable function, and let $\sigma : [0,T]\times\mathbb{R}^k\times\mathbb{U}\times\mathbb{V} \to \mathbb{R}^{k\times d}$ be a $\mathscr{B}([0,T])\otimes\mathscr{B}(\mathbb{R}^k)\otimes\mathscr{B}(\mathbb{U})\otimes\mathscr{B}(\mathbb{V})/\mathscr{B}(\mathbb{R}^{k\times d})$-measurable function such that for any $(t,u,v)\in[0,T]\times\mathbb{U}\times\mathbb{V}$ and $x, x' \in \mathbb{R}^k$,

$$|b(t,0,u,v)| + |\sigma(t,0,u,v)| \le \gamma\big(1 + [u]_{\mathbb{U}} + [v]_{\mathbb{V}}\big) \quad \text{and} \tag{2.1}$$
$$|b(t,x,u,v) - b(t,x',u,v)| + |\sigma(t,x,u,v) - \sigma(t,x',u,v)| \le \gamma|x - x'|. \tag{2.2}$$

(2) Let $g : \mathbb{R}^k \to \mathbb{R}$ be a $2/p$-Hölder continuous function with coefficient γ.

(3) Let $f : [0,T]\times\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{U}\times\mathbb{V} \to \mathbb{R}$ be $\mathscr{B}([0,T])\otimes\mathscr{B}(\mathbb{R}^k)\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)\otimes\mathscr{B}(\mathbb{U})\otimes\mathscr{B}(\mathbb{V})/\mathscr{B}(\mathbb{R})$-measurable function such that for any $(t,u,v) \in [0,T]\times\mathbb{U}\times\mathbb{V}$ and any $(x,y,z), (x',y',z') \in \mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^d$,

$$|f(t,0,0,0,u,v)| \le \gamma\Big(1 + [u]_{\mathbb{U}}^{2/p} + [v]_{\mathbb{V}}^{2/p}\Big) \quad \text{and} \tag{2.3}$$
$$\big|f(t,x,y,z,u,v) - f(t,x',y',z',u,v)\big| \le \gamma\big(|x-x'|^{2/p} + |y-y'| + |z-z'|\big). \tag{2.4}$$

For any $\lambda \ge 0$, we let c_λ denote a generic constant, depending on λ, T, γ, p, and $|g(0)|$, whose form may vary from line to line. (In particular, c_0 stands for a generic constant depending on T, γ, p, and $|g(0)|$.)

Also, we would like to introduce two control neutralizers $\psi, \widetilde{\psi}$ for the coefficients. For some $\kappa > 0$,

(A-u) there exist a function $\psi : [0,T]\times\big(\mathbb{U}\backslash O_\kappa(u_0)\big) \to \mathbb{V}$ that is $\mathscr{B}([0,T])\times\mathscr{B}\big(\mathbb{U}\backslash O_\kappa(u_0)\big)/\mathscr{B}(\mathbb{V})$-measurable and satisfies that for any $(t,x,y,z)\in[0,T]\times\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^d$ and $u, u' \in \mathbb{U}\backslash O_\kappa(u_0)$,

$$b\big(t,x,u,\psi(t,u)\big) = b\big(t,x,u',\psi(t,u')\big), \quad \sigma\big(t,x,u,\psi(t,u)\big) = \sigma\big(t,x,u',\psi(t,u')\big),$$
$$f\big(t,x,y,z,u,\psi(t,u)\big) = f\big(t,x,y,z,u',\psi(t,u')\big), \quad \text{and} \quad [\psi(t,u)]_{\mathbb{V}} \le \kappa(1 + [u]_{\mathbb{U}});$$

(A-v) and there exists a function $\widetilde{\psi} : [0,T]\times\big(\mathbb{V}\backslash O_\kappa(v_0)\big) \to \mathbb{U}$ that is $\mathscr{B}([0,T])\times\mathscr{B}\big(\mathbb{V}\backslash O_\kappa(v_0)\big)/\mathscr{B}(\mathbb{U})$-measurable and satisfies that for any $(t,x,y,z)\in[0,T]\times\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^d$

and $v, v' \in \mathbb{V}\backslash O_\kappa(v_0)$,

$$b\big(t,x,\widetilde{\psi}(t,v),v\big)=b\big(t,x,\widetilde{\psi}(t,v'),v'\big),\quad \sigma\big(t,x,\widetilde{\psi}(t,v),v\big)=\sigma\big(t,x,\widetilde{\psi}(t,v'),v'\big),$$
$$f\big(t,x,y,z,\widetilde{\psi}(t,v),v\big)=f\big(t,x,y,z,\widetilde{\psi}(t,v'),v'\big),\quad \text{and}\quad [\widetilde{\psi}(t,v)]_{\mathbb{U}}\le\kappa(1+[v]_{\mathbb{V}}).$$

A typical example satisfying both (A-u) and (A-v) is the additive-control case.

Example 2.1. Let $\mathbb{U}=\mathbb{V}=\mathbb{R}^\ell$ and consider the following coefficients: $\forall\,(t,x,y,z,u,v)\in[0,T]\times\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{U}\times\mathbb{V}$

$$b(t,x,u,v)=b\big(t,x,u+v\big),\quad \sigma(t,x,u,v)=\sigma\big(t,x,u+v\big),\quad\text{and}$$
$$f(t,x,y,z,u,v)=f\big(t,x,y,z,u+v\big).$$

Then (A-u) and (A-v) hold for functions $\psi(u)=-u$ and $\widetilde{\psi}(v)=-v$, respectively.

Here is another example.

Example 2.2. Given $\gamma>0$, let $b_0,\sigma_0:[0,T]\times\mathbb{R}\to\mathbb{R}$ be two $\mathscr{B}([0,T])\otimes\mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$-measurable functions and let $f_0:[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be $\mathscr{B}([0,T])\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R})/\mathscr{B}(\mathbb{R})$-measurable function such that for any $t\in[0,T]$ and $(x,y,z),(x',y',z')\in\mathbb{R}\times\mathbb{R}\times\mathbb{R}$

$$|b_0(t,x)-b_0(t,x')|+|\sigma_0(t,x)-\sigma_0(t,x')|$$
$$\quad+\big|f_0(t,x,y,z)-f_0(t,x',y',z')\big|\le\gamma\big(|x-x'|+|y-y'|+|z-z'|\big).$$

Also, let $\mathbb{U}=\mathbb{V}=\mathbb{R}$, $\kappa>0$ and $\varphi:[0,T]\times\mathbb{U}\times\mathbb{V}\to\mathbb{R}$ be a jointly continuous function such that φ is Lipschitz continuous in (u,v) with coefficient γ, $\sup_{t\in[0,T]}|\varphi(t,0,0)|\le\gamma$, and for any $(t,u,v)\in[0,T]\times\mathbb{U}\times\mathbb{V}$,

$$\inf_{|v'|\le\kappa|u|}\varphi(t,u,v')\le 0\le\sup_{|v'|\le\kappa|u|}\varphi(t,u,v')\quad\text{and}$$
$$\inf_{|u'|\le\kappa|v|}\varphi(t,u',v)\le 0\le\sup_{|u'|\le\kappa|v|}\varphi(t,u',v).\tag{2.5}$$

Then $b(t,x,u,v)\stackrel{\triangle}{=}b_0(t,x)+\varphi(t,u,v)$, $\sigma(t,x,u,v)\stackrel{\triangle}{=}\sigma_0(t,x)+\varphi(t,u,v)$, and $f(t,x,y,z,u,v)\stackrel{\triangle}{=}f_0(t,x,y,z)+\varphi(t,u,v)$ $\forall\,(t,x,y,z,u,v)\in[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{U}\times\mathbb{V}$ are the measurable functions satisfying (2.1)–(2.4) with $k=d=1$ and $p=2$. We will show in the beginning of section 4.1 that (A-u) and (A-v) hold for these coefficients.

When the game begins at time $t\in[0,T]$, player I and player II select admissible controls $\mu\in\mathcal{U}_t$ and $\nu\in\mathcal{V}_t$ respectively. Then the state process starting from $\xi\in\mathbb{L}^2(\mathcal{F}_t,\mathbb{R}^k)$ will evolve according to SDE (1.1) on the probability space $\big(\Omega,\mathcal{F}_T,P\big)$. The measurability of functions b, σ, μ, and ν implies that $b^{\mu,\nu}(s,\omega,x)\stackrel{\triangle}{=}b\big(s,x,\mu_s(\omega),\nu_s(\omega)\big)$ $\forall\,(s,\omega,x)\in[t,T]\times\Omega\times\mathbb{R}^k$ is $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^k)/\mathscr{B}(\mathbb{R}^k)$-measurable and that $\sigma^{\mu,\nu}(s,\omega,x)\stackrel{\triangle}{=}\sigma\big(s,x,\mu_s(\omega),\nu_s(\omega)\big)$ $\forall\,(s,\omega,x)\in[t,T]\times\Omega\times\mathbb{R}^k$ is $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^k)/\mathscr{B}(\mathbb{R}^{k\times d})$-measurable. Also, (2.2), (2.1), and Hölder's inequality show that $b^{\mu,\nu}$, $\sigma^{\mu,\nu}$ are Lipschitz continuous in x and satisfy $E\,[(\int_t^T|b^{\mu,\nu}(s,0)|\,ds)^2+(\int_t^T|\sigma^{\mu,\nu}(s,0)|\,ds)^2]\le c_0+c_0E\int_t^T\big([\mu_s]^2_{\mathbb{U}}+[\nu_s]^2_{\mathbb{V}}\big)ds<\infty$. Then it is well known that the SDE (1.1) admits a unique solution $\big\{X_s^{t,\xi,\mu,\nu}\big\}_{s\in[t,T]}\in\mathbb{C}^2_{\mathbf{F}}([t,T],\mathbb{R}^k)$ such that

$$E\left[\sup_{s\in[t,T]}\left|X_s^{t,\xi,\mu,\nu}\right|^2\right] \le c_0E\left[|\xi|^2\right]+c_0E\left[\left(\int_t^T\left|b^{\mu,\nu}(s,0)\right|ds\right)^2+\left(\int_t^T\left|\sigma^{\mu,\nu}(s,0)\right|ds\right)^2\right]$$

$$\le c_0\left(1+E\left[|\xi|^2\right]+E\int_t^T\left([\mu_s]_{\mathbb{U}}^2+[\nu_s]_{\mathbb{V}}^2\right)ds\right)<\infty. \tag{2.6}$$

Given $s\in[t,T]$, let $[\mu]^s$ denote the restriction of μ over period $[s,T]$: i.e., $[\mu]_r^s\overset{\triangle}{=}\mu_r$ $\forall\, r\in[s,T]$. Clearly, $[\mu]^s\in\mathcal{U}_s$, similarly, $\left\{[\nu]_r^s\overset{\triangle}{=}\nu_r\right\}_{r\in[s,T]}\in\mathcal{V}_s$. As

$$\begin{aligned}X_r^{t,\xi,\mu,\nu}&=X_s^{t,\xi,\mu,\nu}+\int_s^r b\left(r',X_{r'}^{t,\xi,\mu,\nu},\mu_{r'},\nu_{r'}\right)dr'+\int_s^r\sigma\left(r',X_{r'}^{t,\xi,\mu,\nu},\mu_{r'},\nu_{r'}\right)dB_{r'}\\&=X_s^{t,\xi,\mu,\nu}+\int_s^r b\left(r',X_{r'}^{t,\xi,\mu,\nu},[\mu]_{r'}^s,[\nu]_{r'}^s\right)dr'\\&\quad+\int_s^r\sigma\left(r',X_{r'}^{t,\xi,\mu,\nu},[\mu]_{r'}^s,[\nu]_{r'}^s\right)dB_{r'},\quad r\in[s,T],\end{aligned}$$

we see that $\left\{X_r^{t,\xi,\mu,\nu}\right\}_{r\in[s,T]}\in\mathbb{C}_{\mathbf{F}}^2([s,T],\mathbb{R}^k)$ solves (1.1) with the parameters $\left(s,X_s^{t,\xi,\mu,\nu},[\mu]^s,[\nu]^s\right)$. To wit,

$$P\left(X_r^{t,\xi,\mu,\nu}=X_r^{s,X_s^{t,\xi,\mu,\nu},[\mu]^s,[\nu]^s}\ \ \forall\, r\in[s,T]\right)=1. \tag{2.7}$$

Moreover, the state process depends on controls in the following way.

LEMMA 2.2. *Given $t\in[0,T]$, let $\xi\in\mathbb{L}^2(\mathcal{F}_t,\mathbb{R}^k)$ and $(\mu,\nu),\left(\widetilde{\mu},\widetilde{\nu}\right)\in\mathcal{U}_t\times\mathcal{V}_t$. If $\left(\mu,\nu\right)=\left(\widetilde{\mu},\widetilde{\nu}\right)$, $dr\times dP$-a.s. on $[\![t,\tau[\![\cup[\![\tau,T]\!]_A$ for some $\tau\in\mathcal{S}_{t,T}$ and $A\in\mathcal{F}_\tau$, then it holds P*-a.s. *that*

$$\mathbf{1}_AX_s^{t,\xi,\mu,\nu}+\mathbf{1}_{A^c}X_{\tau\wedge s}^{t,\xi,\mu,\nu}=\mathbf{1}_AX_s^{t,\xi,\widetilde{\mu},\widetilde{\nu}}+\mathbf{1}_{A^c}X_{\tau\wedge s}^{t,\xi,\widetilde{\mu},\widetilde{\nu}}\quad\forall\, s\in[t,T]. \tag{2.8}$$

Now, set $\Theta=(t,\xi,\mu,\nu)$. Given $\tau\in\mathcal{S}_{t,T}$, the measurability of (f,X^Θ,μ,ν) and (2.4) imply that

$$f_\tau^\Theta(s,\omega,y,z)\overset{\triangle}{=}\mathbf{1}_{\{s<\tau(\omega)\}}f\left(s,X_s^\Theta(\omega),y,z,\mu_s(\omega),\nu_s(\omega)\right)$$
$$\forall\,(s,\omega,y,z)\in[t,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d$$

is a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R})\otimes\mathscr{B}(\mathbb{R}^d)/\mathscr{B}(\mathbb{R})$-measurable function that is Lipschitz continuous in (y,z) with coefficient γ. And one can deduce from (2.3), (2.4) and Hölder's inequality that

$$E\left[\left(\int_t^T\left|f_\tau^\Theta(s,0,0)\right|ds\right)^p\right]\le c_0+c_0E\left[\sup_{s\in[t,T]}\left|X_s^\Theta\right|^2+\int_t^T\left([\mu_s]_{\mathbb{U}}^2+[\nu_s]_{\mathbb{V}}^2\right)ds\right]<\infty. \tag{2.9}$$

Thus, for any $\eta\in\mathbb{L}^p(\mathcal{F}_\tau)$, Proposition 1.1 shows that the BSDE(t,η,f_τ^Θ) admits a unique solution $\left(Y^\Theta(\tau,\eta),Z^\Theta(\tau,\eta)\right)\in\mathbb{G}_{\mathbf{F}}^p\left([t,T]\right)$, which has the following estimate as a consequence of (1.5).

COROLLARY 2.1. *Let $t\in[0,T]$, $\xi\in\mathbb{L}^2(\mathcal{F}_t,\mathbb{R}^k)$, $(\mu,\nu)\in\mathcal{U}_t\times\mathcal{V}_t$, and $\tau\in\mathcal{S}_{t,T}$. For any $\eta_1,\eta_2\in\mathbb{L}^p(\mathcal{F}_\tau)$ and $\widetilde{p}\in(1,p]$,*

$$E\bigg[\sup_{s\in[t,T]}\Big|Y_s^{t,\xi,\mu,\nu}(\tau,\eta_1)-Y_s^{t,\xi,\mu,\nu}(\tau,\eta_2)\Big|^{\widetilde{p}}\Big|\mathcal{F}_t\bigg]\le c_{\widetilde{p}}E\big[|\eta_1-\eta_2|^{\widetilde{p}}\big|\mathcal{F}_t\big],\quad P\text{-a.s.}\tag{2.10}$$

Given another stopping time $\zeta\in\mathcal{S}_{t,T}$ with $\zeta\le\tau$, P-a.s., one can easily show that $\{(Y^\Theta_{\zeta\wedge s}(\tau,\eta),\mathbf{1}_{\{s<\zeta\}}Z^\Theta_s(\tau,\eta))\}_{s\in[t,T]}\in\mathbb{G}^p_{\mathbf{F}}\big([t,T]\big)$ solves the BSDE$\big(t,Y^\Theta_\zeta(\tau,\eta),f^\Theta_\zeta\big)$. To wit, we have

$$\begin{aligned}&\Big(Y^\Theta_s\big(\zeta,Y^\Theta_\zeta(\tau,\eta)\big),Z^\Theta_s\big(\zeta,Y^\Theta_\zeta(\tau,\eta)\big)\Big)\\&\quad=\Big(Y^\Theta_{\zeta\wedge s}(\tau,\eta),\mathbf{1}_{\{s<\zeta\}}Z^\Theta_s(\tau,\eta)\Big),\quad s\in[t,T].\end{aligned}\tag{2.11}$$

In particular, when $\zeta=\tau$,

$$\Big(Y^\Theta_s(\tau,\eta),Z^\Theta_s(\tau,\eta)\Big)=\Big(Y^\Theta_{\tau\wedge s}(\tau,\eta),\mathbf{1}_{\{s<\tau\}}Z^\Theta_s(\tau,\eta)\Big),\quad s\in[t,T].\tag{2.12}$$

On the other hand, if $\tau\in\mathcal{S}_{s,T}$ for some $s\in[t,T]$, letting $\Theta^s\overset{\triangle}{=}\big(s,X^\Theta_s,[\mu]^s,[\nu]^s\big)$, we can deduce from (2.7) that $\big\{\big(Y^\Theta_r(\tau,\eta),Z^\Theta_r(\tau,\eta)\big)\big\}_{r\in[s,T]}\in\mathbb{G}^p_{\mathbf{F}}\big([s,T]\big)$ solves the following BSDE$\big(s,\eta,f^{\Theta^s}_\tau\big)$:

$$\begin{aligned}Y_s-\eta+\int_r^T Z_{r'}dB_{r'}&=\int_r^T\mathbf{1}_{\{r'<\tau\}}f(r',X^\Theta_{r'},Y_{r'},Z_{r'},\mu_{r'},\nu_{r'})\,dr'\\&=\int_r^T\mathbf{1}_{\{r'<\tau\}}f(r',X^{\Theta^s}_{r'},Y_{r'},Z_{r'},[\mu]^s_{r'},[\nu]^s_{r'})\,dr',\ r\in[s,T].\end{aligned}$$

Hence,

$$P\Big(Y^\Theta_r(\tau,\eta)=Y^{\Theta^s}_r(\tau,\eta),\ \forall\,r\in[s,T]\Big)=1.\tag{2.13}$$

The $2/p$-Hölder continuity of functions g and (2.6) show that $g\big(X^\Theta_T\big)\in\mathbb{L}^p(\mathcal{F}_T)$. Set $J(\Theta)\overset{\triangle}{=}Y^\Theta_t\big(T,g\big(X^\Theta_T\big)\big)$. From (1.5) and the standard estimate of SDE (1.1), we can deduce the following a priori estimate.

LEMMA 2.3. *Let $t\in[0,T]$ and $(\mu,\nu)\in\mathcal{U}_t\times\mathcal{V}_t$. Given $\xi_1,\xi_2\in\mathbb{L}^2(\mathcal{F}_t,\mathbb{R}^k)$, it holds for any $\widetilde{p}\in(1,p]$ that*

$$\begin{aligned}&E\bigg[\sup_{s\in[t,T]}\Big|Y_s^{t,\xi_1,\mu,\nu}\big(T,g\big(X_T^{t,\xi_1,\mu,\nu}\big)\big)-Y_s^{t,\xi_2,\mu,\nu}\big(T,g\big(X_T^{t,\xi_2,\mu,\nu}\big)\big)\Big|^{\widetilde{p}}\Big|\mathcal{F}_t\bigg]\\&\quad\le c_{\widetilde{p}}|\xi_1-\xi_2|^{\frac{2\widetilde{p}}{p}},\quad P\text{-a.s.}\end{aligned}\tag{2.14}$$

2.2. Definition of the value functions and a weak dynamic programming principle. Now, we are ready to introduce values of the zero-sum stochastic differential games via the following version of Elliott-Kalton strategies (or nonanticipative strategies).

DEFINITION 2.2. *Given $t\in[0,T]$, an admissible strategy α for player* I *over period $[t,T]$ is a mapping $\alpha:\mathcal{V}_t\to\mathcal{U}_t$ satisfying:* (i) *There exists a $C_\alpha>0$ such that for any $\nu\in\mathcal{V}_t$ $\big[\big(\alpha(\nu)\big)_s\big]_{\mathbb{U}}\le\kappa+C_\alpha[\nu_s]_{\mathbb{V}}$, $ds\times dP$-a.s., where κ is the constant that appears in*

(A-u) and (A-v). (ii) *For any $\nu^1, \nu^2 \in \mathcal{V}_t$, $\tau \in \mathcal{S}_{t,T}$, and $A \in \mathcal{F}_\tau$, if $\nu^1 = \nu^2$, $ds \times dP$-a.s. on $[\![t,\tau[\![\cup [\![\tau,T]\!]_A$, then $\alpha(\nu^1) = \alpha(\nu^2)$, $ds \times dP$-a.s. on $[\![t,\tau[\![\cup [\![\tau,T]\!]_A$.*

Admissible strategies $\beta : \mathcal{U}_t \to \mathcal{V}_t$ for player II *over period $[t,T]$ are defined similarly. The collection of all admissible strategies for player* I *(resp.,* II*) over period $[t,T]$ is denoted by $\mathfrak{A}_t$ (resp., $\mathfrak{B}_t$).*

Remark 2.2. The condition (ii) of Definition 2.2 is called the nonanticipativity of strategies. It is said in [12, line 4 of p. 456] that "From the nonanticipativity of β_2 we have $\beta_2(u_2^\varepsilon) = \sum_{j \ge 1} \mathbf{1}_{\Delta_j} \beta_2(u_j^2), \ldots$". What is actually used in this equality is not the nonanticipativity of β_2 as defined in Definition 3.2 therein, but the requirement:

$$\text{For any } u, \widetilde{u} \in \mathcal{U}_{t+\delta,T} \text{ and } A \in \mathcal{F}_{t+\delta}, \text{ if } u = \widetilde{u} \text{ on } [t+\delta,T] \times A, \text{ then } \beta_2(u) = \beta_2(\widetilde{u}) \text{ on } [t+\delta,T] \times A. \tag{2.15}$$

Since β_2 is a restriction of strategy $\beta \in \mathcal{B}_{t,T}$ over period $[t+\delta,T]$, (2.15) entails the following condition on β:

For any $u, \widetilde{u} \in \mathcal{U}_{t,T}$, any $s \in [t,T]$ and any $A \in \mathcal{F}_s$, if $u = \widetilde{u}$ on $\big([t,s) \times \Omega\big) \cup \big([s,T] \times A\big)$, then $\beta(u) = \beta(\widetilde{u})$ on $\big([t,s) \times \Omega\big) \cup \big([s,T] \times A\big)$,

which is exactly a simple version of our nonanticipativity condition on strategies with $\tau = s$.

For any $(t,x) \in [0,T] \times \mathbb{R}^k$, we define

$$\begin{aligned} w_1(t,x) &\stackrel{\triangle}{=} \operatorname*{essinf}_{\beta \in \mathfrak{B}_t} \operatorname*{esssup}_{\mu \in \mathcal{U}_t} J\big(t,x,\mu,\beta(\mu)\big) \\ &= \operatorname*{essinf}_{\beta \in \mathfrak{B}_t} \operatorname*{esssup}_{\mu \in \mathcal{U}_t} Y_t^{t,x,\mu,\beta(\mu)}\Big(T, g\Big(X_T^{t,x,\mu,\beta(\mu)}\Big)\Big) \quad \text{and} \\ w_2(t,x) &\stackrel{\triangle}{=} \operatorname*{esssup}_{\alpha \in \mathfrak{A}_t} \operatorname*{essinf}_{\nu \in \mathcal{V}_t} J\big(t,x,\alpha(\nu),\nu\big) \\ &= \operatorname*{esssup}_{\alpha \in \mathfrak{A}_t} \operatorname*{essinf}_{\nu \in \mathcal{V}_t} Y_t^{t,x,\alpha(\nu),\nu}\Big(T, g\Big(X_T^{t,x,\alpha(\nu),\nu}\Big)\Big) \end{aligned}$$

as player I's and player II's *priority values* of the zero-sum stochastic differential game that starts from time t with initial state x.

Remark 2.3. When f is independent of (y,z), w_1 and w_2 are in form of

$$\begin{aligned} w_1(t,x) &\stackrel{\triangle}{=} \operatorname*{essinf}_{\beta \in \mathfrak{B}_t} \operatorname*{esssup}_{\mu \in \mathcal{U}_t} E\bigg[g\big(X_T^{t,x,\mu,\beta(\mu)}\big) + \int_t^T f\big(s, X_s^{t,x,\mu,\beta(\mu)}, \mu_s, (\beta(\mu))_s\big) ds \bigg| \mathcal{F}_t\bigg] \quad \text{and} \\ w_2(t,x) &\stackrel{\triangle}{=} \operatorname*{esssup}_{\alpha \in \mathfrak{A}_t} \operatorname*{essinf}_{\nu \in \mathcal{V}_t} E\bigg[g\big(X_T^{t,x,\alpha(\nu),\nu}\big) + \int_t^T f\big(s, X_s^{t,x,\alpha(\nu),\nu}, (\alpha(\nu))_s, \nu_s\big) ds \bigg| \mathcal{F}_t\bigg] \\ &\qquad \forall\, (t,x) \in [0,T] \times \mathbb{R}^k. \end{aligned}$$

Remark 2.4. When U and V are compact (say $\mathbb{U} = \overline{O}_\kappa(u_0)$ and $\mathbb{V} = \overline{O}_\kappa(v_0)$), assumptions (A-u), (A-v) are no longer needed, and the integrability condition in Definition 2.1 as well as condition (i) in Definition 2.2 hold automatically. Thus our game problem degenerates to the case of [12].

The values w_1, w_2 are bounded as follows.

Proposition 2.1. *For any $(t,x) \in [0,T] \times \mathbb{R}^k$, it holds P-a.s. that $|w_1(t,x)| + |w_2(t,x)| \le c_\kappa + c_0|x|^{2/p}$.*

Similar to Proposition 3.1 of [12], w_1 and w_2 are actually deterministic functions on $[0,T]\times\mathbb{R}^k$.

PROPOSITION 2.2. *Let $i=1,2$. For any $(t,x)\in[0,T]\times\mathbb{R}^k$, it holds P-a.s. that $w_i(t,x)=E[w_i(t,x)]$.*

Moreover, as a consequence of (2.14), w_1 and w_2 are $2/p$-Hölder continuous in x.

PROPOSITION 2.3. *For any $t\in[0,T]$ and $x_1,x_2\in\mathbb{R}^k$, $\big|w_1(t,x_1)-w_1(t,x_2)\big|+\big|w_2(t,x_1)-w_2(t,x_2)\big|\le c_0|x_1-x_2|^{2/p}$.*

However, the values w_1, w_2 are generally not continuous in t unless $\mathbb{U}$, $\mathbb{V}$ are compact.

Remark 2.5. When trying to directly prove the dynamic programming principle, [17] encountered a measurability issue: The pasted strategies for approximation may not be progressively measurable; see p. 299 of [17]. So they first proved that the value functions are unique viscosity solutions to the associated Bellman–Isaacs equations by a time-discretization approach (assuming that the limiting Isaacs equation has a comparison principle), which relies on the following regularity of the approximating values v_π:

$$|v_\pi(t,x)-v_\pi(t',x')|\le C\big(|t-t'|^{1/2}+|x-x'|\big)\quad\forall\,(t,x),(t',x')\in[0,T]\times\mathbb{R}^k$$

with a uniform coefficient $C>0$ for all partitions π of $[0,T]$. Since our value functions w_1, w_2 may not be $1/2$-Hölder continuous in t, this method is not suitable for our problem. So we adopt Buckdahn and Li's probability setting.

The following weak dynamic programming principle for value functions w_1, w_2 is the main result of the paper.

THEOREM 2.1.

(1) *Given $t\in[0,T)$, let $\phi,\widetilde{\phi}\colon[t,T]\times\mathbb{R}^k\to\mathbb{R}$ be two continuous functions such that $\phi(s,x)\le w_1(s,x)\le\widetilde{\phi}(s,x)$, $(s,x)\in[t,T]\times\mathbb{R}^k$. Then for any $x\in\mathbb{R}^k$ and $\delta\in(0,T-t)$, it holds P-a.s. that*

$$\begin{aligned}&\operatorname*{essinf}_{\beta\in\mathfrak{B}_t}\operatorname*{esssup}_{\mu\in\mathcal{U}_t}Y_t^{t,x,\mu,\beta(\mu)}\Big(\tau_{\beta,\mu},\phi\big(\tau_{\beta,\mu},X^{t,x,\mu,\beta(\mu)}_{\tau_{\beta,\mu}}\big)\Big)\\&\quad\le w_1(t,x)\le\operatorname*{essinf}_{\beta\in\mathfrak{B}_t}\operatorname*{esssup}_{\mu\in\mathcal{U}_t}Y_t^{t,x,\mu,\beta(\mu)}\Big(\tau_{\beta,\mu},\widetilde{\phi}\big(\tau_{\beta,\mu},X^{t,x,\mu,\beta(\mu)}_{\tau_{\beta,\mu}}\big)\Big),\end{aligned}$$

where $\tau_{\beta,\mu}\stackrel{\triangle}{=}\inf\big\{s\in(t,T]\colon\big(s,X_s^{t,x,\mu,\beta(\mu)}\big)\notin O_\delta(t,x)\big\}$.

(2) *Given $t\in[0,T)$, let $\phi,\widetilde{\phi}\colon[t,T]\times\mathbb{R}^k\to\mathbb{R}$ be two continuous functions such that $\phi(s,x)\le w_2(s,x)\le\widetilde{\phi}(s,x)$, $(s,x)\in[t,T]\times\mathbb{R}^k$. Then for any $x\in\mathbb{R}^k$ and $\delta\in(0,T-t)$, it holds P-a.s. that*

$$\begin{aligned}&\operatorname*{esssup}_{\alpha\in\mathfrak{A}_t}\operatorname*{essinf}_{\nu\in\mathcal{V}_t}Y_t^{t,x,\alpha(\nu),\nu}\Big(\tau_{\alpha,\nu},\phi\big(\tau_{\alpha,\nu},X^{t,x,\alpha(\nu),\nu}_{\tau_{\alpha,\nu}}\big)\Big)\le w_2(t,x)\\&\quad\le\operatorname*{esssup}_{\alpha\in\mathfrak{A}_t}\operatorname*{essinf}_{\nu\in\mathcal{V}_t}Y_t^{t,x,\alpha(\nu),\nu}\Big(\tau_{\alpha,\nu},\widetilde{\phi}\big(\tau_{\alpha,\nu},X^{t,x,\alpha(\nu),\nu}_{\tau_{\alpha,\nu}}\big)\Big),\end{aligned}$$

where $\tau_{\alpha,\nu}\stackrel{\triangle}{=}\inf\big\{s\in(t,T]\colon\big(s,X_s^{t,x,\alpha(\nu),\nu}\big)\notin O_\delta(t,x)\big\}$.

The significance of such a weak dynamic programming principle lies in the following fact: Since w_i, $i=1,2$ may not be continuous in t, $w_i\big(\tau_{\beta,\mu},X^{t,x,\mu,\beta(\mu)}_{\tau_{\beta,\mu}}\big)$ may not be $\mathcal{F}_{\tau_{\beta,\mu}}$-measurable. Then $Y_t^{t,x,\mu,\beta(\mu)}\big(\tau_{\beta,\mu},w_i\big(\tau_{\beta,\mu},X^{t,x,\mu,\beta(\mu)}_{\tau_{\beta,\mu}}\big)\big)$ and thus the strong dynamic programming principle may not be well defined.

3. Viscosity solutions of related fully nonlinear PDEs. In this section, we show that the priority values are (discontinuous) viscosity solutions to the following partial differential equation with a fully nonlinear Hamiltonian H:

$$(3.1)\quad -\frac{\partial}{\partial t}w(t,x) - H\big(t,x,w(t,x),D_x w(t,x),D_x^2 w(t,x)\big) = 0 \quad \forall\,(t,x)\in(0,T)\times\mathbb{R}^k.$$

Definition 3.1. *Let us denote by $\mathbb{S}_k$ the set of all $\mathbb{R}^{k\times k}$-valued symmetric matrices, and let $H:[0,T]\times\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^k\times\mathbb{S}_k\to[-\infty,\infty]$. An upper* (*resp., lower*) *semicontinuous function $w:[0,T]\times\mathbb{R}^k\to\mathbb{R}$ is called a viscosity subsolution* (*resp., supersolution*) *of* (3.1) *if for any $(t_0,x_0,\varphi)\in(0,T)\times\mathbb{R}^k\times\mathbb{C}^{1,2}\big([0,T]\times\mathbb{R}^k\big)$ such that $w-\varphi$ attains a strict local maximum* 0 (*resp., strict local minimum* 0) *at (t_0,x_0), we have*

$$-\frac{\partial}{\partial t}\varphi(t_0,x_0) - H\big(t_0,x_0,\varphi(t_0,x_0),D_x\varphi(t_0,x_0),D_x^2\varphi(t_0,x_0)\big) \le (\textit{resp.}, \ge)\ 0.$$

We set $H(t,x,y,z,\Gamma,u,v)\overset{\triangle}{=}\frac{1}{2}\mathrm{trace}\big(\sigma\sigma^T(t,x,u,v)\,\Gamma\big)+z\cdot b(t,x,u,v)+f\big(t,x,y,z\cdot\sigma(t,x,u,v),u,v\big)$ $\forall\,(t,x,y,z,\Gamma,u,v)\in[0,T]\times\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}_k\times\mathbb{U}\times\mathbb{V}$, and consider the following Hamiltonian functions:

$$\underline{H}_1(\Xi)\overset{\triangle}{=}\sup_{u\in\mathbb{U}}\ \varliminf_{\Xi'\to\Xi}\ \inf_{v\in\mathscr{O}_u} H(\Xi',u,v),$$

$$\overline{H}_1(\Xi)\overset{\triangle}{=}\lim_{n\to\infty}\!\downarrow\ \sup_{u\in\mathbb{U}}\ \inf_{v\in\mathscr{O}_u^n}\ \varlimsup_{u'\to u}\ \sup_{\Xi'\in O_{\frac{1}{n}}(\Xi)} H(\Xi',u',v)\ \text{and}$$

$$\overline{H}_2(\Xi)\overset{\triangle}{=}\inf_{v\in\mathbb{V}}\ \varlimsup_{\Xi'\to\Xi}\ \sup_{u\in\mathscr{O}_v} H(\Xi',u,v),$$

$$\underline{H}_2(\Xi)\overset{\triangle}{=}\lim_{n\to\infty}\!\uparrow\ \inf_{v\in\mathbb{V}}\ \sup_{u\in\mathscr{O}_v^n}\ \varliminf_{v'\to v}\ \inf_{\Xi'\in O_{\frac{1}{n}}(\Xi)} H(\Xi',u,v'),$$

where $\Xi=(t,x,y,z,\Gamma)$, $\mathscr{O}_u^n\overset{\triangle}{=}\big\{v\in\mathbb{V}:[v]_{\mathbb{V}}\le\kappa+n[u]_{\mathbb{U}}\big\}$, $\mathscr{O}_v^n\overset{\triangle}{=}\big\{u\in\mathbb{U}:[u]_{\mathbb{U}}\le\kappa+n[v]_{\mathbb{V}}\big\}$, $\mathscr{O}_u\overset{\triangle}{=}\cup_{n\in\mathbb{N}}\mathscr{O}_u^n=\mathbf{1}_{\{u=u_0\}}\overline{O}_\kappa(v_0)+\mathbf{1}_{\{u\ne u_0\}}\mathbb{V}$, and $\mathscr{O}_v\overset{\triangle}{=}\cup_{n\in\mathbb{N}}\mathscr{O}_v^n=\mathbf{1}_{\{v=v_0\}}\overline{O}_\kappa(u_0)+\mathbf{1}_{\{v\ne v_0\}}\mathbb{U}$.

Remark 3.1. When U and V are compact (say $\mathbb{U}=\overline{O}_\kappa(u_0)$ and $\mathbb{V}=\overline{O}_\kappa(v_0)$), it holds for any $(u,v)\in\mathbb{U}\times\mathbb{V}$ and $n\in\mathbb{N}$ that $\big(\mathscr{O}_u^n,\mathscr{O}_v^n\big)=(\mathbb{V},\mathbb{U})$. If further assuming as [12] that for any $(x,y,z)\in\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^d$, $b(\cdot,x,\cdot,\cdot)$, $\sigma(\cdot,x,\cdot,\cdot)$,$f(\cdot,x,y,z,\cdot,\cdot)$ are all continuous in (t,u,v), one can deduce from (2.1)−(2.4) that the continuity of $H(\Xi,u,v)$ in Ξ is uniform in (u,v). It follows that

$$\begin{aligned}
\underline{H}_1(\Xi) &= \sup_{u\in\mathbb{U}}\ \varliminf_{\Xi'\to\Xi}\ \inf_{v\in\mathbb{V}} H(\Xi',u,v)\\
&= \sup_{u\in\mathbb{U}}\ \inf_{v\in\mathbb{V}}\ \varliminf_{\Xi'\to\Xi} H(\Xi',u,v) = \sup_{u\in\mathbb{U}}\ \inf_{v\in\mathbb{V}} H(\Xi,u,v)\quad\text{and}\\
\overline{H}_1(\Xi) &= \lim_{n\to\infty}\!\downarrow\ \sup_{u\in\mathbb{U}}\ \inf_{v\in\mathbb{V}}\ \varlimsup_{u'\to u}\ \sup_{\Xi'\in O_{\frac{1}{n}}(\Xi)} H(\Xi',u',v)\\
&= \sup_{u\in\mathbb{U}}\ \inf_{v\in\mathbb{V}}\ \varlimsup_{u'\to u}\ \lim_{n\to\infty}\!\downarrow\ \sup_{\Xi'\in O_{\frac{1}{n}}(\Xi)} H(\Xi',u',v)\\
&= \sup_{u\in\mathbb{U}}\ \inf_{v\in\mathbb{V}}\ \varlimsup_{u'\to u} H(\Xi,u',v) = \sup_{u\in\mathbb{U}}\ \inf_{v\in\mathbb{V}} H(\Xi,u,v) = \underline{H}_1(\Xi).
\end{aligned}$$

Similarly, $\underline{H}_2(\Xi) = \overline{H}_2(\Xi) = \inf_{v\in\mathbb{V}} \sup_{u\in\mathbb{U}} H(\Xi, u, v)$.

For $i = 1, 2$, Proposition 2.3 implies that

$$\underline{w}_i(t,x) \stackrel{\triangle}{=} \varliminf_{t'\to t} w_i(t',x) = \varliminf_{(t',x')\to(t,x)} w_i(t',x') \quad \text{and}$$
$$\overline{w}_i(t,x) \stackrel{\triangle}{=} \varlimsup_{t'\to t} w_i(t',x) = \varlimsup_{(t',x')\to(t,x)} w_i(t',x') \;\; \forall\,(t,x)\in[0,T]\times\mathbb{R}^k.$$

In fact, $\underline{w}_i$ is the largest lower semicontinuous function below w_i (known as the lower semicontinuous envelope of w_i) while $\overline{w}_i$ is the smallest upper semicontinuous function above w_i (known as the upper semicontinuous envelope of w_i).

THEOREM 3.1. *For $i = 1, 2$, $\underline{w}_i$ (resp., $\overline{w}_i$) is a viscosity supersolution (resp., subsolution) of* (3.1) *with the fully nonlinear Hamiltonian $\underline{H}_i$ (resp., $\overline{H}_i$).*

Since there is no regularity, even semicontinuity, in the fully nonlinear Hamiltonian functions $\underline{H}_i$ and $\overline{H}_i$, this existence result of viscosity solutions to the fully nonlinear PDEs (3.1) is quite general. In general, a comparison result for the PDEs we analyze may not hold since it is not clear whether $\underline{H}_i = \overline{H}_i$ unless the control spaces are compact.

Remark 3.2. Given $i = 1, 2$ and $x \in \mathbb{R}^k$, although $w_i(T, x) = g(x)$, it is possible that neither $\underline{w}_i(T, x)$ nor $\overline{w}_i(T, x)$ equals to $g(x)$ since w_i may not be continuous in t. This phenomenon already appears in stochastic control problems with unbounded control; see, e.g., [5].

4. Proofs.

4.1. Proofs of the results in section 2.

Proof of Example 2.2. For any $(t, u) \in [0, T] \times \mathbb{U}$, the continuity of φ and (2.5) show that $\{v \in [-\kappa|u|, \kappa|u|] : \varphi(t, u, v) = 0\}$ is a nonempty closed set. So we can define $\mathscr{V}(t, u) \stackrel{\triangle}{=} \min\{v \in [-\kappa|u|, \kappa|u|] : \varphi(t, u, v) = 0\}$.

Given $n \in \mathbb{N}$, for any $i = 0, \dots, 2^n - 1$ and $j \in \mathbb{Z}$, we set $t^n_i = i2^{-n}T$, $u^n_j = j2^{-n}$, and $\psi^n_{i,j} \stackrel{\triangle}{=} \inf_{(t,u)\in\mathcal{D}^n_{i,j}} \mathscr{V}(t, u) \in [-\kappa - \kappa|u|, \kappa + \kappa|u|]$ with

$$\mathcal{D}^n_{i,j} \stackrel{\triangle}{=} \begin{cases} [t^n_i, t^n_{i+1}) \times [u^n_j, u^n_{j+1}) & \text{if } i < 2^n - 1, \\ [t^n_i, T] \times [u^n_j, u^n_{j+1}) & \text{if } i = 2^n - 1. \end{cases}$$

Clearly, $\psi_n(t,u) \stackrel{\triangle}{=} \sum_{i=0}^{2^n-1} \sum_{j\in\mathbb{Z}} \psi^n_{i,j} \mathbf{1}_{\{(t,u)\in\mathcal{D}^n_{i,j}\}} \in [-\kappa-\kappa|u|, \kappa+\kappa|u|] \;\; \forall\,(t,u)\in[0,T]\times\mathbb{U}$ defines a $\mathscr{B}([0, T]) \otimes \mathscr{B}(\mathbb{U})/\mathscr{B}(\mathbb{R})$-measurable function. As $\psi_n \le \psi_{n+1}$, the function $\psi(t,u) \stackrel{\triangle}{=} \lim_{n\to\infty} \psi_n(t,u) \in [-\kappa-\kappa|u|, \kappa+\kappa|u|] \;\; \forall\,(t,u)\in[0,T]\times\mathbb{U}$ is also $\mathscr{B}([0,T])\otimes \mathscr{B}(\mathbb{U})/\mathscr{B}(\mathbb{R})$-measurable.

Now, let $(t, u) \in [0, T] \times \mathbb{U}$ and $\varepsilon > 0$. By the continuity of φ in t, there exists a $\delta \in (0, \varepsilon/3\gamma)$ such that

$$\big|\varphi\big(s, u, \psi(t, u)\big) - \varphi\big(t, u, \psi(t, u)\big)\big| < \varepsilon/3 \quad \forall\, s \in [t - \delta, t + \delta] \cap [0, T]. \tag{4.1}$$

For any $n > \log_2(1 \vee T) - \log_2(\delta)$, $(t, u) \in \mathcal{D}^n_{i,j}$ for some $(i, j) \in \{0, \dots, 2^n - 1\} \times \mathbb{Z}$, and we can find $(t', u') \in \mathcal{D}^n_{i,j}$ such that $\mathscr{V}(t', u') \le \psi^n_{i,j} + \delta$. Then (4.1) and the Lipschitz continuity of φ in (u, v) show that

$$\begin{aligned}\big|\varphi(t,u,\psi(t,u))\big| &= \big|\varphi(t,u,\psi(t,u)) - \varphi(t',u',\mathscr{V}(t',u'))\big| \\ &\le \big|\varphi(t,u,\psi(t,u)) - \varphi(t',u,\psi(t,u))\big| + \big|\varphi(t',u,\psi(t,u)) \\ &\quad - \varphi(t',u,\psi_n(t,u))\big| + \big|\varphi(t',u,\psi^n_{i,j}) - \varphi(t',u',\mathscr{V}(t',u'))\big| \\ &\le \varepsilon/3 + \gamma\big|\psi(t,u) - \psi_n(t,u)\big| + \gamma\big(|u-u'| + |\psi^n_{i,j} - \mathscr{V}(t',u')|\big) \\ &\le \varepsilon + \gamma\big|\psi(t,u) - \psi_n(t,u)\big| + 2\gamma\delta \le \varepsilon + \gamma\big|\psi(t,u) - \psi_n(t,u)\big|.\end{aligned}$$

Letting $n \to \infty$ yields that $\big|\varphi(t,u,\psi(t,u))\big| \le \varepsilon$. Then as $\varepsilon \to 0$, we obtain that $\varphi(t,u,\psi(t,u)) = 0$.

Similarly, we can construct a measurable function $\widetilde{\psi}$ on $[0,T] \times \mathbb{V}$ such that

$$\varphi\big(t,\widetilde{\psi}(t,v),v\big) = 0 \quad \text{and} \quad |\widetilde{\psi}(t,v)| \le \kappa(1+|v|) \quad \forall\,(t,v) \in [0,T] \times \mathbb{V}.$$

Hence (A-u) and (A-v) are satisfied. □

Proof of Lemma 2.2. Both $\big\{X^{t,\xi,\mu,\nu}_{\tau\wedge s}\big\}_{s\in[t,T]}$ and $\big\{X^{t,\xi,\widetilde{\mu},\widetilde{\nu}}_{\tau\wedge s}\big\}_{s\in[t,T]}$ satisfy the same SDE:

$$X_s = \xi + \int_t^s b^{\mu,\nu}_\tau(r,X_r)\,dr + \int_t^s \sigma^{\mu,\nu}_\tau(r,X_r)\,dB_r, \quad s \in [t,T], \tag{4.2}$$

where $b^{\mu,\nu}_\tau(r,\omega,x) \stackrel{\triangle}{=} \mathbf{1}_{\{r<\tau(\omega)\}} b^{\mu,\nu}(r,\omega,x)$ and $\sigma^{\mu,\nu}_\tau(r,\omega,x) \stackrel{\triangle}{=} \mathbf{1}_{\{r<\tau(\omega)\}}\sigma^{\mu,\nu}(r,\omega,x)$ $\forall\,(r,\omega,x) \in [t,T]\times\Omega\times\mathbb{R}^k$. Like $b^{\mu,\nu}$ and $\sigma^{\mu,\nu}$, $b^{\mu,\nu}_\tau$ is a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^k)/\mathscr{B}(\mathbb{R}^k)$-measurable function and $\sigma^{\mu,\nu}_\tau$ is a $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^k)/\mathscr{B}(\mathbb{R}^{k\times d})$-measurable function that is Lipschitz continuous in (y,z) with coefficient γ and satisfies $E[(\int_t^T \big|b^{\mu,\nu}_\tau(s,0)\big|ds)^2 + (\int_t^T \big|\sigma^{\mu,\nu}_\tau(s,0)\big|ds)^2] < \infty$. Thus (4.2) has a unique solution, i.e.,

$$P\Big(X^{t,\xi,\mu,\nu}_{\tau\wedge s} = X^{t,\xi,\widetilde{\mu},\widetilde{\nu}}_{\tau\wedge s} \;\; \forall\, s \in [t,T]\Big) = 1. \tag{4.3}$$

One can deduce that

$$\begin{aligned}&X^{t,\xi,\mu,\nu}_s - X^{t,\xi,\mu,\nu}_{\tau\wedge s} = X^{t,\xi,\mu,\nu}_{\tau\vee s} - X^{t,\xi,\mu,\nu}_\tau \\ &\quad = \int_\tau^{\tau\vee s} b\big(r, X^{t,\xi,\mu,\nu}_r, \mu_r, \nu_r\big)dr + \int_\tau^{\tau\vee s} \sigma\big(r, X^{t,\xi,\mu,\nu}_r, \mu_r, \nu_r\big)dB_r, \quad s \in [t,T].\end{aligned}$$

Multiplying $\mathbf{1}_A$ on both sides yields that

$$\begin{aligned}\mathcal{X}_s &\stackrel{\triangle}{=} \mathbf{1}_A\big(X^{t,\xi,\mu,\nu}_s - X^{t,\xi,\mu,\nu}_{\tau\wedge s}\big) \\ &= \int_\tau^{\tau\vee s} \mathbf{1}_A b\big(r, X^{t,\xi,\mu,\nu}_r, \mu_r, \nu_r\big)dr + \int_\tau^{\tau\vee s} \mathbf{1}_A \sigma\big(r, X^{t,\xi,\mu,\nu}_r, \mu_r, \nu_r\big)dB_r \\ &= \int_t^s \mathbf{1}_{\{r\ge\tau\}}\mathbf{1}_A b\big(r, \mathcal{X}_r + X^{t,\xi,\mu,\nu}_{\tau\wedge r}, \mu_r, \nu_r\big)dr \\ &\quad + \int_t^s \mathbf{1}_{\{r\ge\tau\}}\mathbf{1}_A \sigma\big(r, \mathcal{X}_r + X^{t,\xi,\mu,\nu}_{\tau\wedge r}, \mu_r, \nu_r\big)dB_r, \quad s \in [t,T].\end{aligned}$$

Similarly, we see from (4.3) that

$$\widetilde{\mathcal{X}}_s \triangleq \mathbf{1}_A\big(X^{t,\xi,\widetilde{\mu},\widetilde{\nu}}_s - X^{t,\xi,\widetilde{\mu},\widetilde{\nu}}_{\tau\wedge s}\big) = \int_t^s \mathbf{1}_{\{r\ge\tau\}}\mathbf{1}_A b\big(r, \widetilde{\mathcal{X}}_r + X^{t,\xi,\widetilde{\mu},\widetilde{\nu}}_{\tau\wedge r}, \widetilde{\mu}_r, \widetilde{\nu}_r\big)dr$$
$$+\int_t^s \mathbf{1}_{\{r\ge\tau\}}\mathbf{1}_A \sigma\big(r, \widetilde{\mathcal{X}}_r + X^{t,\xi,\widetilde{\mu},\widetilde{\nu}}_{\tau\wedge r}, \widetilde{\mu}_r, \widetilde{\nu}_r\big)dB_r$$
$$= \int_t^s \mathbf{1}_{\{r\ge\tau\}}\mathbf{1}_A b\big(r, \widetilde{\mathcal{X}}_r + X^{t,\xi,\mu,\nu}_{\tau\wedge r}, \mu_r, \nu_r\big)dr$$
$$+\int_t^s \mathbf{1}_{\{r\ge\tau\}}\mathbf{1}_A \sigma\big(r, \widetilde{\mathcal{X}}_r + X^{t,\xi,\mu,\nu}_{\tau\wedge r}, \mu_r, \nu_r\big)dB_r, \quad s\in[t,T].$$

To wit, $\mathcal{X}, \widetilde{\mathcal{X}} \in \mathbb{C}^2_{\mathbf{F}}([t,T],\mathbb{R}^k)$ satisfy the same SDE:

$$X_s = \int_t^s \widehat{b}(r, X_r)\,dr + \int_t^s \widehat{\sigma}(r, X_r)\,dB_r, \quad s\in[t,T], \tag{4.4}$$

where $\widehat{b}(r,\omega,x) \triangleq \mathbf{1}_{\{r\ge\tau(\omega)\}}\mathbf{1}_{\{\omega\in A\}} b\big(r, x + X^{t,\xi,\mu,\nu}_{\tau\wedge r}(\omega), \mu_r(\omega), \nu_r(\omega)\big)$ and $\widehat{\sigma}(r,\omega,x) \triangleq \mathbf{1}_{\{r\ge\tau(\omega)\}}\mathbf{1}_{\{\omega\in A\}}\sigma^{\mu,\nu}\big(r, x + X^{t,\xi,\mu,\nu}_{\tau\wedge r}(\omega), \mu_r(\omega), \nu_r(\omega)\big)$ $\forall\,(r,\omega,x)\in[t,T]\times\Omega\times\mathbb{R}^k$. The measurability of functions b, $X^{t,\xi,\mu,\nu}$, μ and ν implies that the mapping $(r,\omega,x)\to b\big(r,\omega, x + X^{t,\xi,\mu,\nu}_{\tau\wedge r}(\omega), \mu_r(\omega), \nu_r(\omega)\big)$ is $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^k)/\mathscr{B}(\mathbb{R}^k)$-measurable. Clearly, $\{\mathbf{1}_{\{r\ge\tau\}\cap A}\}_{r\in[t,T]}$ is a right-continuous $\mathbf{F}$-adapted process. Thus $\widehat{b}$ is also $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^k)/\mathscr{B}(\mathbb{R}^k)$-measurable. Similarly, $\widehat{\sigma}$ is $\mathscr{P}\otimes\mathscr{B}(\mathbb{R}^k)/\mathscr{B}(\mathbb{R}^{k\times d})$-measurable. By (2.2), both $\widehat{b}$ and $\widehat{\sigma}$ are Lipschitz continuous in x. Since

$$E\left[\left(\int_t^T \big|\widehat{b}(r,0)\big|\,dr\right)^2 + \left(\int_t^T \big|\widehat{\sigma}(r,0)\big|\,dr\right)^2\right]$$
$$\le c_0 + c_0 E\Big[\big|X^{t,\xi,\mu,\nu}_\tau\big|^2\Big] + c_0 E\int_t^T \big([\mu_r]^2_{\mathbb{U}} + [\nu_r]^2_{\mathbb{V}}\big)dr < \infty$$

by (2.1), (2.2), and Hölder's inequality, the SDE (4.4) admits a unique solution. Hence, $P\big(\mathcal{X}_s = \widetilde{\mathcal{X}}_s\ \forall\, s\in[t,T]\big) = 1$, which together with (4.3) proves (2.8). $\square$

Proof of Proposition 2.1. Given $\beta\in\mathfrak{B}_t$, (1.4) and Hölder's inequality imply that

$$\big|J\big(t,x,u_0,\beta(u_0)\big)\big|^p$$
$$\le c_0 E\left[\big|g\big(X^{t,x,u_0,\beta(u_0)}_T\big)\big|^p + \int_t^T \big|f^{t,x,u_0,\beta(u_0)}_T(s,0,0)\big|^p ds\Big|\mathcal{F}_t\right], \quad P\text{-a.s.} \tag{4.5}$$

Since $\big[\big(\beta(u_0)\big)_s\big]_{\mathbb{V}} \le \kappa$, $ds\times dP$-a.s., the $2/p$-Hölder continuity of g, (2.3), (2.4) as well as a conditional-expectation version of (2.6) show that P-a.s.

$$\big|J\big(t,x,u_0,\beta(u_0)\big)\big|^p$$
$$\le c_0 + c_0 E\left[\big|X^{t,x,u_0,\beta(u_0)}_T\big|^2 + \int_t^T \Big(\big|X^{t,x,u_0,\beta(u_0)}_s\big|^2 + \big[\big(\beta(u_0)\big)_s\big]^2_{\mathbb{V}}\Big)ds\Big|\mathcal{F}_t\right]$$
$$\le c_\kappa + c_0 E\left[\sup_{s\in[t,T]}\big|X^{t,x,u_0,\beta(u_0)}_s\big|^2\Big|\mathcal{F}_t\right]$$
$$\le c_\kappa + c_0|x|^2 + c_0 E\left[\int_t^T \big[\big(\beta(u_0)\big)_s\big]^2_{\mathbb{V}} ds\Big|\mathcal{F}_t\right] \le c_\kappa + c_0|x|^2. \tag{4.6}$$

So it follows that $w_1(t,x) \ge \operatorname{essinf}_{\beta\in\mathfrak{B}_t} J\big(t,x,u_0,\beta(u_0)\big) \ge -c_\kappa - c_0|x|^{2/p}$, P-a.s.

We extensively set $\psi(t,u) \overset{\triangle}{=} v_0 \ \ \forall\,(t,u)\in[0,T]\times O_\kappa(u_0)$, then it is $\mathscr{B}([0,T])\times\mathscr{B}(\mathbb{U})/\mathscr{B}(\mathbb{V})$-measurable. For any $\mu\in\mathcal{U}_t$, the measurability of function ψ and process μ implies that

$$(4.7)\qquad \big(\beta_\psi(\mu)\big)_s \overset{\triangle}{=} \psi(s,\mu_s), \quad s\in[t,T]$$

defines a $\mathbb{V}$-valued, $\mathbf{F}$-progressively measurable process. By (A-u), $[\big(\beta_\psi(\mu)\big)_s]_{\mathbb{V}} \le \kappa+\kappa[\mu_s]_{\mathbb{U}} \ \ \forall\, s\in[t,T]$, so $\beta_\psi(\mu)\in\mathcal{V}_t$. Let $\mu^1,\mu^2\in\mathcal{U}_t$ such that $\mu^1=\mu^2$, $ds\times dP$-a.s. on $[\![t,\tau[\![\cup[\![\tau,T]\!]_A$ for some $\tau\in\mathcal{S}_{t,T}$ and $A\in\mathcal{F}_\tau$. It clearly holds that $ds\times dP$-a.s. on $[\![t,\tau[\![\cup[\![\tau,T]\!]_A$ that $\big(\beta_\psi(\mu^1)\big)_s=\psi(s,\mu^1_s)=\psi(s,\mu^2_s)=\big(\beta_\psi(\mu^2)\big)_s$. Hence, $\beta_\psi\in\mathfrak{B}_t$.

Fix a $u_\sharp\in\partial O_\kappa(u_0)$. For any $\mu\in\mathcal{U}_t$, similar to (4.5) and (4.6), we can deduce that P-a.s.,

$$\begin{aligned}
&\big|J\big(t,x,\mu,\beta_\psi(\mu)\big)\big|^p\\
&\quad\le c_0E\bigg[\big|g\big(X_T^{t,x,\mu,\beta_\psi(\mu)}\big)\big|^p+\int_t^T\Big(\mathbf{1}_{\{\mu_s\in O_\kappa(u_0)\}}\big|f(s,X_s^{t,x,\mu,\beta_\psi(\mu)},0,0,\mu_s,v_0)\big|^p\\
&\qquad\quad+\mathbf{1}_{\{\mu_s\notin O_\kappa(u_0)\}}\big|f\big(s,X_s^{t,x,\mu,\beta_\psi(\mu)},0,0,u_\sharp,\psi(s,u_\sharp)\big)\big|^p\Big)ds\Big|\mathcal{F}_t\bigg]\\
&\quad\le c_\kappa+c_0E\bigg[\sup_{s\in[t,T]}\big|X_s^{t,x,\mu,\beta_\psi(\mu)}\big|^2\Big|\mathcal{F}_t\bigg]\\
&\quad\le c_\kappa+c_0|x|^2+c_0E\left[\left(\int_t^T\big|b\big(s,0,\mu_s,(\beta_\psi(\mu))_s\big)\big|\,ds\right)^2\right.\\
&\qquad\left.+\left(\int_t^T\big|\sigma\big(s,0,\mu_s,(\beta_\psi(\mu))_s\big)\big|\,ds\right)^2\Big|\mathcal{F}_t\right],
\end{aligned}\tag{4.8}$$

where we used a conditional-expectation version of (2.6) in the last inequality. Then an analogous decomposition and estimation to (4.8) leads to $\big|J\big(t,x,\mu,\beta_\psi(\mu)\big)\big|^p\le c_\kappa+c_0|x|^2$, P-a.s. It follows that

$$w_1(t,x)\le \operatorname*{esssup}_{\mu\in\mathcal{U}_t} J\big(t,x,\mu,\beta_\psi(\mu)\big)\le c_\kappa+c_0|x|^{2/p},\quad P\text{-a.s.}$$

Similarly, one has $|w_2(t,x)|\le c_\kappa+c_0|x|^{2/p}$, P-a.s. ∎

Proof of Proposition 2.2. Let $\mathcal{H}$ denote the Cameron–Martin space of all absolutely continuous functions $h\in\Omega$ whose derivative $\dot h$ belongs to $\mathbb{L}^2([0,T],\mathbb{R}^d)$. For any $h\in\mathcal{H}$, we define $\mathcal{T}_h(\omega)\overset{\triangle}{=}\omega+h \ \ \forall\,\omega\in\Omega$. Clearly, $\mathcal{T}_h:\Omega\to\Omega$ is a bijection and its law is given by $P_h\overset{\triangle}{=}P\circ\mathcal{T}_h^{-1}=\exp\big\{\int_0^T\dot h_s dB_s-\frac12\int_0^T|\dot h_s|^2ds\big\}P$. Fix $(t,x)\in[0,T]\times\mathbb{R}^k$ and set $\mathcal{H}_t\overset{\triangle}{=}\{h\in\mathcal{H}:h(s)=h(s\wedge t)\ \forall\,s\in[0,T]\}$.

(a) *Let* $h\in\mathcal{H}_t$. *We first show that*

$$(4.9)\qquad \big(\mu(\mathcal{T}_h),\nu(\mathcal{T}_h)\big)\in\mathcal{U}_t\times\mathcal{V}_t\quad\forall\,(\mu,\nu)\in\mathcal{U}_t\times\mathcal{V}_t.$$

Fix $\mu\in\mathcal{U}_t$. Given $s\in[t,T]$, we set $\Upsilon_s^h(\mathcal{D})\overset{\triangle}{=}\big\{(r,\omega)\in[t,s]\times\Omega:\big(r,\mathcal{T}_h(\omega)\big)\in\mathcal{D}\big\}$ for any $\mathcal{D}\subset[t,s]\times\Omega$. As the mapping

$$\mathcal{T}_h = B+h \text{ is } \mathcal{F}_s/\mathcal{F}_s\text{-measurable,} \tag{4.10}$$

it holds for any $\mathcal{E}\in\mathscr{B}\big([t,s]\big)$ and $A\in\mathcal{F}_s$ that $\Upsilon^h_s(\mathcal{E}\times A)=\big\{(r,\omega)\in[t,s]\times\Omega\colon \big(r,\mathcal{T}_h(\omega)\big)\in \mathcal{E}\times A\big\} = \big(\mathcal{E}\cap[t,s]\big)\times\mathcal{T}_h^{-1}(A)\in\mathscr{B}\big([t,s]\big)\otimes\mathcal{F}_s$. So $\mathcal{E}\times A\in\Lambda^h_s \stackrel{\triangle}{=} \big\{\mathcal{D}\subset[t,s]\times\Omega: \Upsilon^h_s(\mathcal{D})\in\mathscr{B}\big([t,s]\big)\otimes\mathcal{F}_s\big\}$. In particular, $\emptyset\times\emptyset\in\Lambda^h_s$ and $[t,s]\times\Omega\in\Lambda^h_s$. For any $\mathcal{D}\in\Lambda^h_s$ and $\{\mathcal{D}_n\}_{n\in\mathbb{N}}\subset\Lambda^h_s$, one can deduce that

$$\Upsilon^h_s\big(([t,s]\times\Omega)\backslash\mathcal{D}\big)=([t,s]\times\Omega)\backslash\Upsilon^h_s(\mathcal{D})\in\mathscr{B}\big([t,s]\big)\otimes\mathcal{F}_s \quad\text{and}$$
$$\Upsilon^h_s\Big(\underset{n\in\mathbb{N}}{\cup}\mathcal{D}_n\Big)=\underset{n\in\mathbb{N}}{\cup}\Upsilon^h_s(\mathcal{D}_n)\in\mathscr{B}\big([t,s]\big)\otimes\mathcal{F}_s,$$

i.e., $([t,s]\times\Omega)\backslash\mathcal{D}$, $\cup_{n\in\mathbb{N}}\mathcal{D}_n\in\Lambda^h_s$. Thus Λ^h_s is a σ-field of $[t,s]\times\Omega$. It follows that $\mathscr{B}([t,s])\otimes\mathcal{F}_s=\sigma\big\{\mathcal{E}\times A:\ \mathcal{E}\in\mathscr{B}\big([t,s]\big), A\in\mathcal{F}_s\big\}\subset\Lambda^h_s$. Then for $U\in\mathscr{B}(\mathbb{U})$, the **F**-progressive measurability of μ implies that $\mathcal{D}_U\stackrel{\triangle}{=}\big\{(r,\omega)\in[t,s]\times\Omega:\ \mu_r(\omega)\in U\big\}\in\mathscr{B}([t,s])\otimes\mathcal{F}_s\subset\Lambda^h_s$. That is

$$\begin{aligned}&\big\{(r,\omega)\in[t,s]\times\Omega\colon \mu_r\big(\mathcal{T}_h(\omega)\big)\in U\big\}\\ &\quad=\big\{(r,\omega)\in[t,s]\times\Omega\colon \big(r,\mathcal{T}_h(\omega)\big)\in\mathcal{D}_U\big\}=\Upsilon^h_s\big(\mathcal{D}_U\big)\in\mathscr{B}([t,s])\otimes\mathcal{F}_s,\end{aligned}\tag{4.11}$$

which shows the **F**-progressive measurability of process $\mu(\mathcal{T}_h)$.

Suppose that $E\int_t^T[\mu_s]^q_{\mathbb{U}}ds<\infty$ for some $q>2$. Then one can deduce from Hölder's inequality that for any $\widetilde{q}\in(2,q)$,

$$\begin{aligned}&E\int_t^T\big[\mu_s(\mathcal{T}_h)\big]^{\widetilde{q}}_{\mathbb{U}}ds\\ &\quad=E_{P_h}\int_t^T[\mu_s]^{\widetilde{q}}_{\mathbb{U}}ds=E\bigg[\exp\bigg\{\int_0^T\dot h_s dB_s-\frac12\int_0^T|\dot h_s|^2ds\bigg\}\int_t^T[\mu_s]^{\widetilde{q}}_{\mathbb{U}}ds\bigg]\\ &\quad\le T^{\frac{q-\widetilde{q}}{q}}\exp\bigg\{\frac{\widetilde{q}}{2(q-\widetilde{q})}\int_0^T|\dot h_s|^2ds\bigg\}\\ &\qquad\times\bigg(E\bigg[\exp\bigg\{\frac{q}{q-\widetilde{q}}\int_0^T\dot h_s dB_s-\frac{q^2}{2(q-\widetilde{q})^2}\int_0^T|\dot h_s|^2ds\bigg\}\bigg]\bigg)^{\frac{q-\widetilde{q}}{q}}\bigg(E\int_t^T[\mu_s]^q_{\mathbb{U}}ds\bigg)^{\frac{\widetilde{q}}{q}}\\ &\quad=T^{\frac{q-\widetilde{q}}{q}}\exp\bigg\{\frac{\widetilde{q}}{2(q-\widetilde{q})}\int_0^T|\dot h_s|^2ds\bigg\}\bigg(E\int_t^T[\mu_s]^q_{\mathbb{U}}ds\bigg)^{\frac{\widetilde{q}}{q}}<\infty.\end{aligned}\tag{4.12}$$

Hence, $\mu(\mathcal{T}_h)\in\mathcal{U}_t$. Similarly, $\nu(\mathcal{T}_h)\in\mathcal{V}_t$ for any $\nu\in\mathcal{V}_t$.

(b) *We next show that*

$$J(t,x,\mu,\nu)(\mathcal{T}_h)=J\big(t,x,\mu(\mathcal{T}_h),\nu(\mathcal{T}_h)\big),\quad P\text{-a.s.}\tag{4.13}$$

Let $\{\Phi_s\}_{s\in[t,T]}$ be an $\mathbb{R}^{k\times d}$-valued, **F**-progressively measurable process and set $M_s\stackrel{\triangle}{=}\int_t^s\Phi_r dB_r$, $s\in[t,T]$. We know that (see, e.g., Problem 3.2.27 of [22], which is proved on page 228 therein) there exists a sequence of $\mathbb{R}^{k\times d}$-valued, **F**-simple processes $\{\Phi^n_s=\sum_{i=1}^{\ell_n}\xi^n_i\,\mathbf{1}_{\{s\in(t^n_i,t^n_{i+1}]\}},\ s\in[t,T]\}_{n\in\mathbb{N}}$ $\big($where $t=t^n_1<\cdots<t^n_{\ell_n+1}=T$ and $\xi^n_i\in\mathcal{F}_{t^n_i}$ for $i=1,\ldots,\ell_n\big)$ such that

$$P-\lim_{n\to\infty}\int_t^T \text{trace}\Big\{\big(\Phi^n_r-\Phi_r\big)\big(\Phi^n_r-\Phi_r\big)^T\Big\}\,ds=0 \text{ and } P-\lim_{n\to\infty}\sup_{s\in[t,T]}|M^n_s-M_s|=0,$$

where $M^n_s \stackrel{\triangle}{=} \int_t^s \Phi^n_r dB_s = \sum_{i=1}^{\ell_n}\xi^n_i\big(B_{s\wedge t^n_{i+1}}-B_{s\wedge t^n_i}\big)$. By the equivalence of P_h to P, one has

$$\begin{aligned}
&P_h-\lim_{n\to\infty}\int_t^T \text{trace}\Big\{\big(\Phi^n_r-\Phi_r\big)\big(\Phi^n_r-\Phi_r\big)^T\Big\}\,ds\\
&\quad= P_h-\lim_{n\to\infty}\sup_{s\in[t,T]}|M^n_s-M_s|=0 \quad\text{or}\\
&P-\lim_{n\to\infty}\int_t^T \text{trace}\Big\{\big(\Phi^n_r(\mathcal{T}_h)-\Phi_r(\mathcal{T}_h)\big)\big(\Phi^n_r(\mathcal{T}_h)-\Phi_r(\mathcal{T}_h)\big)^T\Big\}\,ds\\
&\quad= P-\lim_{n\to\infty}\sup_{s\in[t,T]}\big|M^n_s(\mathcal{T}_h)-M_s(\mathcal{T}_h)\big|=0.
\end{aligned}\tag{4.14}$$

Applying Proposition 3.2.26 of [22] yields that

$$0=P-\lim_{n\to\infty}\sup_{s\in[t,T]}\Big|\int_t^s\Phi^n_r(\mathcal{T}_h)dB_r-\int_t^s\Phi_r(\mathcal{T}_h)dB_r\Big|.\tag{4.15}$$

As $h\in\mathcal{H}_t$, one can deduce that

$$\begin{aligned}
M^n_s(\mathcal{T}_h)&=\left(\sum_{i=1}^{\ell_n}\xi^{\,n}_i\big(B_{s\wedge t^{\,n}_{i+1}}-B_{s\wedge t^{\,n}_i}\big)\right)(\mathcal{T}_h)=\sum_{i=1}^{\ell_n}\xi^{\,n}_i(\mathcal{T}_h)\big(B_{s\wedge t^{\,n}_{i+1}}(\mathcal{T}_h)-B_{s\wedge t^{\,n}_i}(\mathcal{T}_h)\big)\\
&=\sum_{i=1}^{\ell_n}\xi^{\,n}_i(\mathcal{T}_h)\big(B_{s\wedge t^{\,n}_{i+1}}-h(s\wedge t^{\,n}_{i+1})-B_{s\wedge t^{\,n}_i}+h(s\wedge t^{\,n}_i)\big)\\
&=\int_t^s\Phi^n_r(\mathcal{T}_h)dB_r \quad \forall\, s\in[t,T],
\end{aligned}$$

which together with (4.14) and (4.15) leads to P-a.s.

$$\int_t^s\Phi_r(\mathcal{T}_h)dB_r=M_s(\mathcal{T}_h)=\left(\int_t^s\Phi_r dB_r\right)(\mathcal{T}_h),\quad s\in[t,T].\tag{4.16}$$

Let $(\mu,\nu)\in\mathcal{U}_t\times\mathcal{V}_t$ and set $\Theta=(t,x,\mu,\nu)$. By (4.10), the process $X^\Theta(\mathcal{T}_h)$ is $\mathbf{F}$-adapted, and the equivalence of P_h to P implies that $X^\Theta(\mathcal{T}_h)$ has P-a.s. continuous paths. Suppose that $E\int_t^T[\mu_s]^q_{\mathbb{U}}ds+E\int_t^T[\nu_s]^q_{\mathbb{V}}ds<\infty$ for some $q>2$. A standard estimate of SDEs (see, e.g., [21, pp. 166–168] and [22, pp. 289–290]) shows that

$$\begin{aligned}
E\left[\sup_{s\in[t,T]}\big|X^\Theta_s\big|^q\right]&\le c_q|x|^q+c_qE\left[\left(\int_t^T\big|b^{\mu,\nu}(s,0)\big|\,ds\right)^q+\left(\int_t^T\big|\sigma^{\mu,\nu}(s,0)\big|\,ds\right)^q\right]\\
&\le c_q\left(1+|x|^q+E\int_t^T\big([\mu_s]^q_{\mathbb{U}}+[\nu_s]^q_{\mathbb{V}}\big)ds\right)<\infty.
\end{aligned}\tag{4.17}$$

Similar to (4.12), one can deduce that $E[\sup_{s\in[t,T]}\big|X^\Theta_s(\mathcal{T}_h)\big|^{\widetilde{q}}]<\infty$ for any $\widetilde{q}\in[2,q)$. In particular, $X^\Theta(\mathcal{T}_h)\in\mathbb{C}^2_{\mathbf{F}}([t,T],\mathbb{R}^k)$. It follows from (4.16) that

$$X_s^{\Theta}(\mathcal{T}_h) = x + \int_t^s b\big(r, X_r^{\Theta}(\mathcal{T}_h), \mu_r(\mathcal{T}_h), \nu_r(\mathcal{T}_h)\big)\, dr + \bigg(\int_t^s \sigma(r, X_r^{\Theta}, \mu_r, \nu_r)\, dB_r\bigg)(\mathcal{T}_h)$$
$$= x + \int_t^s b\big(r, X_r^{\Theta}(\mathcal{T}_h), \mu_r(\mathcal{T}_h), \nu_r(\mathcal{T}_h)\big)\, dr + \int_t^s \sigma\big(r, X_r^{\Theta}(\mathcal{T}_h), \mu_r(\mathcal{T}_h), \nu_r(\mathcal{T}_h)\big)\, dB_r,$$
$$s \in [t,T].$$

Thus the uniqueness of SDE (1.1) with parameters $\Theta_h = \big(t, x, \mu(\mathcal{T}_h), \nu(\mathcal{T}_h)\big)$ shows that

$$X_s^{\Theta_h} = X_s^{\Theta}(\mathcal{T}_h) \quad \forall\, s \in [t,T]. \tag{4.18}$$

Let $\big(\widehat{Y}, \widehat{Z}\big) = \big(Y^{\Theta}\big(T, g(X_T^{\Theta})\big), Z^{\Theta}\big(T, g(X_T^{\Theta})\big)\big)$. Analogous to $X^{\Theta}(\mathcal{T}_h)$, $\widehat{Y}(\mathcal{T}_h)$ is an $\mathbf{F}$-adapted continuous process. And using the similar arguments that lead to (4.11), we see that the process $\widehat{Z}(\mathcal{T}_h)$ is $\mathbf{F}$-progressively measurable. By (4.17), $g\big(X_T^{\Theta}\big) \in \mathbb{L}^{\frac{pq}{2}}(\mathcal{F}_T)$ and a similar argument to (2.9) yields $E[(\int_t^T |f_T^{\Theta}(s,0,0)|ds)^{\frac{pq}{2}}] \le c_q + c_q E[\sup_{s\in[t,T]} |X_s^{\Theta}|^q + \int_t^T ([\mu_s]_{\mathbb{U}}^q + [\nu_s]_{\mathbb{V}}^q) ds] < \infty$. Then Proposition 1.1 shows that the unique solution $\big(\widehat{Y}, \widehat{Z}\big)$ of $\mathrm{BSDE}\big(t, g(X_T^{\Theta}), f_T^{\Theta}\big)$ in $\mathbb{G}_{\mathbf{F}}^p([t,T])$ also belongs to $\mathbb{G}_{\mathbf{F}}^{\frac{pq}{2}}([t,T])$. Similar to (4.12), one can deduce that $E[\sup_{s\in[t,T]} |\widehat{Y}_s(\mathcal{T}_h)|^{\widetilde{q}} + (\int_t^T |\widehat{Z}_s(\mathcal{T}_h)|^2 ds)^{\widetilde{q}/2}] < \infty$ for any $\widetilde{q} \in [p, \frac{pq}{2})$. In particular, $\big(\widehat{Y}(\mathcal{T}_h), \widehat{Z}(\mathcal{T}_h)\big) \in \mathbb{G}_{\mathbf{F}}^p([t,T])$.

Applying (4.16) again, we can deduce from (4.18) that

$$\widehat{Y}_s(\mathcal{T}_h) = g\big(X_T^{\Theta}(\mathcal{T}_h)\big) + \int_s^T f\big(r, X_r^{\Theta}(\mathcal{T}_h), \widehat{Y}_r(\mathcal{T}_h), \widehat{Z}_r(\mathcal{T}_h), \mu_r(\mathcal{T}_h), \nu_r(\mathcal{T}_h)\big)\, dr$$
$$- \bigg(\int_s^T \widehat{Z}_r\, dB_r\bigg)(\mathcal{T}_h)$$
$$= g\big(X_T^{\Theta_h}\big) + \int_s^T f\big(r, X_r^{\Theta_h}, \widehat{Y}_r(\mathcal{T}_h), \widehat{Z}_r(\mathcal{T}_h), \mu_r(\mathcal{T}_h), \nu_r(\mathcal{T}_h)\big)\, dr$$
$$- \int_t^s \widehat{Z}_r(\mathcal{T}_h)\, dB_r, \quad s \in [t,T].$$

Thus the uniqueness of $\mathrm{BSDE}(t, g(X_T^{\Theta_h}), f_T^{\Theta_h})$ yields $P(Y_s^{\Theta_h}(T, g(X_T^{\Theta_h})) = \widehat{Y}_s(\mathcal{T}_h)$ $\forall\, s \in [t,T]) = 1$. In particular,

$$J(t,x,\mu,\nu)(\mathcal{T}_h) = \widehat{Y}_t(\mathcal{T}_h) = Y_t^{\Theta_h}\big(T, g\big(X_T^{\Theta_h}\big)\big) = J\big(t, x, \mu(\mathcal{T}_h), \nu(\mathcal{T}_h)\big), \quad P\text{-a.s.}$$

(c) *Now, we show that* $w_1(t,x)(\mathcal{T}_h) = w_1(t,x)$, P-a.s.

Let $\beta \in \mathfrak{B}_t$ and define $\beta_h(\mu) \overset{\triangle}{=} \beta\big(\mu(\mathcal{T}_{-h})\big)(\mathcal{T}_h)$ $\forall\, \mu \in \mathcal{U}_t$. Similar to (4.9), $\mu(\mathcal{T}_{-h}) \in \mathcal{U}_t$ as $-h \in \mathcal{H}$. Then $\beta\big(\mu(\mathcal{T}_{-h})\big) \in \mathcal{V}_t$. Using (4.9) again shows that $\beta_h(\mu) = \beta\big(\mu(\mathcal{T}_{-h})\big)(\mathcal{T}_h) \in \mathcal{V}_t$. Since $\big[\big(\beta\big(\mu(\mathcal{T}_{-h})\big)\big)_s\big]_{\mathbb{V}} \le \kappa + C_\beta [\mu_s(\mathcal{T}_{-h})]_{\mathbb{U}}$, $ds \times dP$-a.s., the equivalence of P_h to P shows that $\big[\big(\beta\big(\mu(\mathcal{T}_{-h})\big)\big)_s\big]_{\mathbb{V}} \le \kappa + C_\beta [\mu_s(\mathcal{T}_{-h})]_{\mathbb{U}}$, $ds \times dP_h$-a.s., or $\big[\big(\beta_h(\mu)\big)_s\big]_{\mathbb{V}} = \big[\big(\beta\big(\mu(\mathcal{T}_{-h})\big)(\mathcal{T}_h)\big)_s\big]_{\mathbb{V}} \le \kappa + C_\beta [\mu_s]_{\mathbb{U}}$, $ds \times dP$-a.s.

Let $\mu^1, \mu^2 \in \mathcal{U}_t$ such that $\mu^1 = \mu^2$, $ds \times dP$-a.s. on $[\![t,\tau[\![\cup [\![\tau,T]\!]_A$ for some $\tau \in \mathcal{S}_{t,T}$ and $A \in \mathcal{F}_\tau$. By the equivalence of P_{-h} to P, $\mu^1 = \mu^2$, $ds \times dP_{-h}$-a.s. on $[\![t,\tau[\![\cup [\![\tau,T]\!]_A$, or $\mu^1(\mathcal{T}_{-h}) = \mu^2(\mathcal{T}_{-h})$, $ds \times dP$-a.s. on $[\![t,\tau(\mathcal{T}_{-h})[\![\cup [\![\tau(\mathcal{T}_{-h}),T]\!]_{\mathcal{T}_h(A)}$. Given $s \in [t,T]$, similar to (4.10), $\mathcal{T}_{-h}$ is also $\mathcal{F}_s/\mathcal{F}_s$-measurable. It follows that

$$\{\tau(\mathcal{T}_{-h}) \le s\} = \big\{\omega : \mathcal{T}_{-h}(\omega) \in \{\tau \le s\}\big\} = \mathcal{T}_{-h}^{-1}\big(\{\tau \le s\}\big) \in \mathcal{F}_s$$
$$\text{and } \mathcal{T}_h(A) \cap \{\tau(\mathcal{T}_{-h}) \le s\} = \mathcal{T}_{-h}^{-1}(A) \cap \mathcal{T}_{-h}^{-1}\big(\{\tau \le s\}\big) = \mathcal{T}_{-h}^{-1}\big(A \cap \{\tau \le s\}\big) \in \mathcal{F}_s,$$

which shows that $\tau(\mathcal{T}_{-h})$ is an $\mathbf{F}$-stopping time and $\mathcal{T}_h(A) \in \mathcal{F}_{\tau(\mathcal{T}_{-h})}$. As $t \le \tau \le T$, P-a.s., the equivalence of P_{-h} to P shows that $t \le \tau \le T$, P_{-h}-a.s., or $t \le \tau(\mathcal{T}_{-h}) \le T$, P-a.s. So $\tau(\mathcal{T}_{-h}) \in \mathcal{S}_{t,T}$, and we see from Definition 2.2 that $\beta\big(\mu^1(\mathcal{T}_{-h})\big) = \beta\big(\mu^2(\mathcal{T}_{-h})\big)$, $ds \times dP$-a.s. on $[\![t, \tau(\mathcal{T}_{-h})[\![\cup [\![\tau(\mathcal{T}_{-h}), T]\!]_{\mathcal{T}_h(A)}$. The equivalence of P_h to P then shows that $\beta\big(\mu^1(\mathcal{T}_{-h})\big) = \beta\big(\mu^2(\mathcal{T}_{-h})\big)$, $ds \times dP_h$-a.s. on $[\![t, \tau(\mathcal{T}_{-h})[\![\cup [\![\tau(\mathcal{T}_{-h}), T]\!]_{\mathcal{T}_h(A)}$, or $\beta_h(\mu^1) = \beta\big(\mu^1(\mathcal{T}_{-h})\big)(\mathcal{T}_h) = \beta\big(\mu^2(\mathcal{T}_{-h})\big)(\mathcal{T}_h) = \beta_h(\mu^2)$, $ds \times dP$-a.s. on $[\![t, \tau[\![\cup [\![\tau, T]\!]_A$. Hence, $\beta_h \in \mathfrak{B}_t$.

Set $I(t,x,\beta) \stackrel{\triangle}{=} \operatorname{esssup}_{\mu \in \mathcal{U}_t} J\big(t,x,\mu,\beta(\mu)\big)$. For any $\mu \in \mathcal{U}_t$, as $I(t,x,\beta) \ge J\big(t,x,\mu,\beta(\mu)\big)$, P-a.s., the equivalence of P_h to P shows that $I(t,x,\beta) \ge J\big(t,x,\mu,\beta(\mu)\big)$, P_h-a.s., or

$$I(t,x,\beta)(\mathcal{T}_h) \ge J\big(t,x,\mu,\beta(\mu)\big)(\mathcal{T}_h), \quad P\text{-a.s.} \tag{4.19}$$

Let ξ be another random variable such that $\xi \ge J\big(t,x,\mu,\beta(\mu)\big)(\mathcal{T}_h)$, P-a.s., or $\xi(\mathcal{T}_{-h}) \ge J\big(t,x,\mu,\beta(\mu)\big)$, P_h-a.s. for any $\mu \in \mathcal{U}_t$. By the equivalence of P_h to P, it holds for any $\mu \in \mathcal{U}_t$ that $\xi(\mathcal{T}_{-h}) \ge J\big(t,x,\mu,\beta(\mu)\big)$, P-a.s. Taking essential supremum over $\mu \in \mathcal{U}_t$ yields that $\xi(\mathcal{T}_{-h}) \ge I(t,x,\beta)$, P-a.s. or $\xi \ge I(t,x,\beta)(\mathcal{T}_h)$, P_{-h}-a.s. Then it follows from the equivalence of P_{-h} to P that $\xi \ge I(t,x,\beta)(\mathcal{T}_h)$, P-a.s., which together with (4.19) implies that

$$\begin{aligned}\operatorname*{esssup}_{\mu \in \mathcal{U}_t} \Big(J\big(t,x,\mu,\beta(\mu)\big)(\mathcal{T}_h)\Big) \\ = I(t,x,\beta)(\mathcal{T}_h) = \Big(\operatorname*{esssup}_{\mu \in \mathcal{U}_t} J\big(t,x,\mu,\beta(\mu)\big)\Big)(\mathcal{T}_h), \quad P\text{-a.s.}\end{aligned} \tag{4.20}$$

Similarly, $w_1(t,x)(\mathcal{T}_h) = \operatorname*{essinf}_{\beta \in \mathfrak{B}_t} \big(I(t,x,\beta)(\mathcal{T}_h)\big)$, P-a.s., which (4.13) and (4.20) together yields that

$$\begin{aligned} w_1(t,x)(\mathcal{T}_h) &= \operatorname*{essinf}_{\beta \in \mathfrak{B}_t} \operatorname*{esssup}_{\mu \in \mathcal{U}_t} \Big(J\big(t,x,\mu,\beta(\mu)\big)(\mathcal{T}_h)\Big) \\ &= \operatorname*{essinf}_{\beta \in \mathfrak{B}_t} \operatorname*{esssup}_{\mu \in \mathcal{U}_t} J\big(t,x,\mu(\mathcal{T}_h),\beta_h(\mu(\mathcal{T}_h))\big) = \operatorname*{essinf}_{\beta \in \mathfrak{B}_t} \operatorname*{esssup}_{\mu \in \mathcal{U}_t} J\big(t,x,\mu,\beta_h(\mu)\big) \\ &= \operatorname*{essinf}_{\beta \in \mathfrak{B}_t} \operatorname*{esssup}_{\mu \in \mathcal{U}_t} J\big(t,x,\mu,\beta(\mu)\big) = w_1(t,x), \quad P\text{-a.s.,} \end{aligned} \tag{4.21}$$

where we used the facts that $\big\{\mu(\mathcal{T}_h) : \mu \in \mathcal{U}_t\big\} = \mathcal{U}_t$ and $\big\{\beta_h : \beta \in \mathfrak{B}_t\big\} = \mathfrak{B}_t$.

(d) As an $\mathcal{F}_t$-measurable random variable, $w_1(t,x)$ only depends on the restriction of $\omega \in \Omega$ to the time interval $[0,t]$. So (4.21) holds even for any $h \in \mathcal{H}$. Then an application of Lemma 3.4 of [12] yields that $w_1(t,x) = E[w_1(t,x)]$, P-a.s. Similarly, one can deduce that $w_2(t,x) = E[w_2(t,x)]$, P-a.s. ∎

4.2. Proof of the weak dynamic programming principle. To prove the weak dynamic programming principle (Theorem 2.1), we begin with two auxiliary results. The first one shows that the pasting of state processes (resp., payoff processes) is exactly the state process (resp., payoff process) with the pasted controls.

LEMMA 4.1. *Given $t \in [0,T]$, let $\{A_i\}_{i=1}^n \subset \mathcal{F}_t$ be a partition of Ω. For any $\big\{(\xi_i, \mu^i, \nu^i)\big\}_{i=0}^n \subset \mathbb{L}^2(\mathcal{F}_t, \mathbb{R}^k) \times \mathcal{U}_t \times \mathcal{V}_t$, if $\xi_0 = \sum_{i=1}^n \mathbf{1}_{A_i}\xi_i$, P-a.s., and if $(\mu^0, \nu^0) = \big(\sum_{i=1}^n \mathbf{1}_{A_i}\mu^i, \sum_{i=1}^n \mathbf{1}_{A_i}\nu^i\big)$, $ds \times dP$-a.s., then it holds P-a.s. that*

$$X_s^{t,\xi_0,\mu^0,\nu^0} = \sum_{i=1}^n \mathbf{1}_{A_i} X_s^{t,\xi_i,\mu^i,\nu^i} \quad \forall\, s \in [t,T]. \tag{4.22}$$

Moreover, for any $\{(\tau_i,\eta_i)\}_{i=0}^n \subset \mathcal{S}_{t,T}\times\mathbb{L}^p(\mathcal{F}_T)$ *such that each* η_i *is* $\mathcal{F}_{\tau_i}$*-measurable, if* $\tau_0=\sum_{i=1}^n \mathbf{1}_{A_i}\tau_i$, P-a.s., *and if* $\eta_0=\sum_{i=1}^n \mathbf{1}_{A_i}\eta_i$, P-a.s., *then it holds* P-a.s. *that*

$$Y_s^{t,\xi_0,\mu^0,\nu^0}\big(\tau_0,\eta_0\big) = \sum_{i=1}^n \mathbf{1}_{A_i} Y_s^{t,\xi_i,\mu^i,\nu^i}\big(\tau_i,\eta_i\big) \quad \forall\, s\in[t,T]. \tag{4.23}$$

In particular,

$$J(t,\xi_0,\mu^0,\nu^0) = \sum_{i=1}^n \mathbf{1}_{A_i} J(t,\xi_i,\mu^i,\nu^i), \quad P\text{-a.s.} \tag{4.24}$$

Proof. Let $\big(X^i,Y^i,Z^i\big)=\big(X^{t,\xi_i,\mu^i,\nu^i}, Y^{t,\xi_i,\mu^i,\nu^i}(\tau_i,\eta_i), Z^{t,\xi_i,\mu^i,\nu^i}(\tau_i,\eta_i)\big)$ for $i=0,\dots,n$. We define $\big(\mathcal{X},\mathcal{Y},\mathcal{Z}\big)\stackrel{\triangle}{=}\sum_{i=1}^n \mathbf{1}_{A_i}\big(X^i,Y^i,Z^i\big)\in\mathbb{C}^2_{\mathbf{F}}([t,T],\mathbb{R}^k)\times\mathbb{G}^p_{\mathbf{F}}([t,T])$. For any $s\in[t,T]$ and $i=1,\dots,n$, multiplying $\mathbf{1}_{A_i}$ to SDE (1.1) with parameters (t,ξ_i,μ^i,ν^i), we can deduce that

$$\begin{aligned}\mathbf{1}_{A_i}X_s^i-\mathbf{1}_{A_i}\xi_i &= \mathbf{1}_{A_i}\int_t^s b(r,X_r^i,\mu_r^i,\nu_r^i)dr+\mathbf{1}_{A_i}\int_t^s \sigma(r,X_r^i,\mu_r^i,\nu_r^i)dB_r\\ &= \int_t^s \mathbf{1}_{A_i} b(r,\mathcal{X}_r,\mu_r^0,\nu_r^0)dr+\int_t^s \mathbf{1}_{A_i}\sigma(r,\mathcal{X}_r,\mu_r^0,\nu_r^0)dB_r\\ &= \mathbf{1}_{A_i}\int_t^s b(r,\mathcal{X}_r,\mu_r^0,\nu_r^0)\,dr+\mathbf{1}_{A_i}\int_t^s \sigma(r,\mathcal{X}_r,\mu_r^0,\nu_r^0)\,dB_r, \quad P\text{-a.s.}\end{aligned} \tag{4.25}$$

Adding them up over $i\in\{1,\dots,n\}$ and using the continuity of $\mathcal{X}$ show that P-a.s., $\mathcal{X}_s=\xi_0+\int_t^s b(r,\mathcal{X}_r,\mu_r^0,\nu_r^0)dr+\int_t^s\sigma(r,\mathcal{X}_r,\mu_r^0,\nu_r^0)dB_r$ $\forall\, s\in[t,T]$. So $\mathcal{X}=X^{t,\xi_0,\mu^0,\nu^0}$, i.e., (4.22).

Next, for any $s\in[t,T]$ and $i=1,\dots,n$, similar to (4.25), multiplying $\mathbf{1}_{A_i}$ to BSDE$(t,\eta_i,f_{\tau_i}^{t,\xi_i,\mu^i,\nu^i})$ yields that

$$\begin{aligned}\mathbf{1}_{A_i}Y_s^i &= \mathbf{1}_{A_i}\eta_i+\mathbf{1}_{A_i}\int_s^T \mathbf{1}_{\{r<\tau_i\}} f(r,X_r^i,Y_r^i,Z_r^i,\mu_r^i,\nu_r^i)\,dr-\mathbf{1}_{A_i}\int_s^T Z_r^i dB_r\\ &= \mathbf{1}_{A_i}\eta_i+\mathbf{1}_{A_i}\int_s^T \mathbf{1}_{\{r<\tau_0\}} f(r,\mathcal{X}_r,\mathcal{Y}_r,\mathcal{Z}_r,\mu_r^0,\nu_r^0)\,dr-\mathbf{1}_{A_i}\int_s^T \mathcal{Z}_r dB_r, \quad P\text{-a.s.}\end{aligned}$$

Adding them up and using the continuity of $\mathcal{Y}$ yield that P-a.s., $\mathcal{Y}_s=\eta_0+\int_s^T\mathbf{1}_{\{r<\tau_0\}}f(r,X_r^{t,\xi_0,\mu^0,\nu^0},\mathcal{Y}_r,\mathcal{Z}_r,\mu_r^0,\nu_r^0)dr-\int_s^T\mathcal{Z}_r dB_r$ $\forall\, s\in[t,T]$. Thus $\big(\mathcal{Y},\mathcal{Z}\big)=\big(Y^{t,\xi_0,\mu^0,\nu^0}(\tau_0,\eta_0), Z^{t,\xi_0,\mu^0,\nu^0}(\tau_0,\eta_0)\big)$, proving (4.23).

Taking $\tau_i=T$ and $\eta_i=g\big(X_T^{t,\xi_i,\mu^i,\nu^i}\big)\in\mathbb{L}^p(\mathcal{F}_T)$ for $i=0,\dots,n$, we can deduce from (4.22) that $\sum_{i=1}^n\mathbf{1}_{A_i}\eta_i=\sum_{i=1}^n\mathbf{1}_{A_i}g(X_T^{t,\xi_i,\mu^i,\nu^i})=\sum_{i=1}^n\mathbf{1}_{A_i}g\big(X_T^{t,\xi_0,\mu^0,\nu^0}\big)=g\big(X_T^{t,\xi_0,\mu^0,\nu^0}\big)=\eta_0$, P-a.s. Then (4.23) shows that $J(t,\xi_0,\mu^0,\nu^0)=Y_t^{t,\xi_0,\mu^0,\nu^0}\big(T,\eta_0\big)=\sum_{i=1}^n\mathbf{1}_{A_i}Y_t^{t,\xi_i,\mu^i,\nu^i}\big(T,\eta_i\big)=\sum_{i=1}^n\mathbf{1}_{A_i}J(t,\xi_i,\mu^i,\nu^i)$, P-a.s. ∎

In the next lemma, we approach $I(t,x,\beta)\stackrel{\triangle}{=}\operatorname{esssup}_{\mu\in\mathcal{U}_t} Y_t^{t,x,\mu,\beta(\mu)}$ from above and $w_1(t,x)=\operatorname{essinf}_{\beta\in\mathfrak{B}_t} I(t,x,\beta)$ from below in a probabilistic sense.

LEMMA 4.2. *Let* $(t,x)\in[0,T]\times\mathbb{R}^k$ *and* $\varepsilon>0$. *For any* $\beta\in\mathfrak{B}_t$, *there exist* $\big\{(A_n,\mu^n)\big\}_{n\in\mathbb{N}}\subset\mathcal{F}_t\times\mathcal{U}_t$ *with* $\lim_{n\to\infty}\mathbf{1}_{A_n}=1$, P-a.s. *such that for any* $n\in\mathbb{N}$,

$$J\big(t,x,\mu^n,\beta(\mu^n)\big)\ge\big(I(t,x,\beta)-\varepsilon\big)\wedge\varepsilon^{-1}, \quad P\text{-a.s. } on\ A_n. \tag{4.26}$$

Similarly, there exist $\{(\mathcal{A}_n,\beta_n)\}_{n\in\mathbb{N}}\subset\mathcal{F}_t\times\mathfrak{B}_t$ *with* $\lim_{n\to\infty}\mathbf{1}_{\mathcal{A}_n}=1$, P-a.s. *such that for any* $n\in\mathbb{N}$,

$$w_1(t,x)\ge I\big(t,x,\beta_n\big)-\varepsilon,\quad P\text{-a.s. on }\mathcal{A}_n. \tag{4.27}$$

Proof. (i) Let $\beta\in\mathfrak{B}_t$. Given $\mu^1,\mu^2\in\mathcal{U}_t$, we set $A\stackrel{\triangle}{=}\{J\big(t,x,\mu^1,\beta(\mu^1)\big)\ge J\big(t,x,\mu^2,\beta(\mu^2)\big)\}\in\mathcal{F}_t$ and define $\widehat{\mu}_s\stackrel{\triangle}{=}\mathbf{1}_A\mu^1_s+\mathbf{1}_{A^c}\mu^2_s$, $s\in[t,T]$. Clearly, process $\widehat{\mu}$ is $\mathbf{F}$-progressively measurable. For $i=1,2$, suppose that $E\int_t^T\big[\mu^i_s\big]_{\mathbb{U}}^{q_i}ds<\infty$ for some $q_i>2$. It follows that $E\int_t^T[\widehat{\mu}_s]_{\mathbb{U}}^{q_1\wedge q_2}ds\le E\int_t^T\big[\mu^1_s\big]_{\mathbb{U}}^{q_1\wedge q_2}ds+E\int_t^T\big[\mu^2_s\big]_{\mathbb{U}}^{q_1\wedge q_2}ds<\infty$. Thus, $\widehat{\mu}\in\mathcal{U}_t$. As $\widehat{\mu}=\mu^1$ on $[t,T]\times A$, taking $(\tau,A)=(t,A)$ in Definition 2.2 yields that $\beta(\widehat{\mu})=\beta(\mu^1)$, $ds\times dP$-a.s. on $[t,T]\times A$. Similarly, $\beta(\widehat{\mu})=\beta(\mu^2)$, $ds\times dP$-a.s. on $[t,T]\times A^c$. So $\beta(\widehat{\mu})=\mathbf{1}_A\beta(\mu^1)+\mathbf{1}_{A^c}\beta(\mu^2)$, $ds\times dP$-a.s. Then (4.24) shows that

$$\begin{aligned}J\big(t,x,\widehat{\mu},\beta(\widehat{\mu})\big)&=\mathbf{1}_AJ\big(t,x,\mu^1,\beta(\mu^1)\big)+\mathbf{1}_{A^c}J\big(t,x,\mu^2,\beta(\mu^2)\big)\\&=J\big(t,x,\mu^1,\beta(\mu^1)\big)\vee J\big(t,x,\mu^2,\beta(\mu^2)\big),\quad P\text{-a.s.}\end{aligned}$$

So $\big\{J\big(t,x,\mu,\beta(\mu)\big)\big\}_{\mu\in\mathcal{U}_t}$ is directed upwards (see Theorem A.32 of [18]). By Proposition VI-**1**-1 of [27] or Theorem A.32 of [18], there exists $\big\{\widetilde{\mu}^i\big\}_{i\in\mathbb{N}}\subset\mathcal{U}_t$ such that

$$I(t,x,\beta)=\operatorname*{esssup}_{\mu\in\mathcal{U}_t}J\big(t,x,\mu,\beta(\mu)\big)=\lim_{i\to\infty}\uparrow J\big(t,x,\widetilde{\mu}^i,\beta(\widetilde{\mu}^i)\big),\quad P\text{-a.s.} \tag{4.28}$$

So $I(t,x,\beta)$ is $\mathcal{F}_t$-measurable.

For any $i\in\mathbb{N}$, we set $\widetilde{A}_i\stackrel{\triangle}{=}\big\{J\big(t,x,\widetilde{\mu}^i,\beta(\widetilde{\mu}^i)\big)\ge\big(I(t,x,\beta)-\varepsilon\big)\wedge\varepsilon^{-1}\big\}\in\mathcal{F}_t$ and $\widehat{A}_i\stackrel{\triangle}{=}\widetilde{A}_i\backslash\cup_{j<i}\widetilde{A}_j\in\mathcal{F}_t$. Fix $n\in\mathbb{N}$ and set $A_n\stackrel{\triangle}{=}\overset{n}{\underset{i=1}{\cup}}\widehat{A}_i\in\mathcal{F}_t$. Similar to $\widehat{\mu}$, $\mu^n\stackrel{\triangle}{=}\sum_{i=1}^n\mathbf{1}_{\widehat{A}_i}\widetilde{\mu}^i+\mathbf{1}_{A_n^c}\widetilde{\mu}^1$ also defines a $\mathcal{U}_t$-process. For $i=1,\dots,n$, as $\mu^n=\widetilde{\mu}^i$ on $[t,T]\times\widehat{A}_i$, taking $(\tau,A)=(t,\widehat{A}_i)$ in Definition 2.2 shows that $\beta(\mu^n)=\beta(\widetilde{\mu}^i)$, $ds\times dP$-a.s. on $[t,T]\times\widehat{A}_i$. Then (4.24) implies that $\mathbf{1}_{\widehat{A}_i}J\big(t,x,\mu^n,\beta(\mu^n)\big)=\mathbf{1}_{\widehat{A}_i}J\big(t,x,\widetilde{\mu}^i,\beta(\widetilde{\mu}^i)\big)$, P-a.s. Adding them up over $i\in\{1,\dots,n\}$ gives

$$\mathbf{1}_{A_n}J\big(t,x,\mu^n,\beta(\mu^n)\big)=\sum_{i=1}^n\mathbf{1}_{\widehat{A}_i}J\big(t,x,\widetilde{\mu}^i,\beta(\widetilde{\mu}^i)\big)\ge\mathbf{1}_{A_n}\big(\big(I(t,x,\beta)-\varepsilon\big)\wedge\varepsilon^{-1}\big),\quad P\text{-a.s.}$$

Let $\mathcal{N}$ be the P-null set such that (4.28) holds on $\mathcal{N}^c$. Clearly, $\{I(t,x,\beta)<\infty\}\cap\mathcal{N}^c\subset\cup_{i\in\mathbb{N}}\big\{J\big(t,x,\widetilde{\mu}^i,\beta(\widetilde{\mu}^i)\big)\ge I(t,x,\beta)-\varepsilon\big\}$ and $\{I(t,x,\beta)=\infty\}\cap\mathcal{N}^c\subset\cup_{i\in\mathbb{N}}\big\{J\big(t,x,\widetilde{\mu}^i,\beta(\widetilde{\mu}^i)\big)\ge\varepsilon^{-1}\big\}$. It follows that

$$\begin{aligned}\mathcal{N}^c&\subset\underset{i\in\mathbb{N}}{\cup}\Big(\big\{J\big(t,x,\widetilde{\mu}^i,\beta(\widetilde{\mu}^i)\big)\ge I(t,x,\beta)-\varepsilon\big\}\cup\big\{J\big(t,x,\widetilde{\mu}^i,\beta(\widetilde{\mu}^i)\big)\ge\varepsilon^{-1}\big\}\Big)\\&=\underset{i\in\mathbb{N}}{\cup}\widetilde{A}_i=\underset{i\in\mathbb{N}}{\cup}\widehat{A}_i=\underset{n\in\mathbb{N}}{\cup}A_n.\end{aligned}$$

Hence $\lim_{n\to\infty}\mathbf{1}_{A_n}=1$, P-a.s.

(ii) Let $\beta_1,\beta_2\in\mathfrak{B}_t$. We just showed that $I(t,x,\beta_1)$ and $I(t,x,\beta_2)$ are $\mathcal{F}_t$-measurable, so $\mathcal{A}_o\stackrel{\triangle}{=}\{I(t,x,\beta_1)\le I(t,x,\beta_2)\}$ belongs to $\mathcal{F}_t$. For any $\mu\in\mathcal{U}_t$, similar to $\widehat{\mu}$ above, $\beta_o(\mu)\stackrel{\triangle}{=}\mathbf{1}_{\mathcal{A}_o}\beta_1(\mu)+\mathbf{1}_{\mathcal{A}_o^c}\beta_2(\mu)$ defines a $\mathcal{V}_t$-process. For $i=1,2$, letting $C_i>0$ be the constant associated to β_i in Definition 2.2 (i), we see that

$$\begin{aligned}\big[(\beta_o(\mu))_s\big]_{\mathbb{V}}&=\mathbf{1}_{\mathcal{A}_o}\big[(\beta_1(\mu))_s\big]_{\mathbb{V}}+\mathbf{1}_{\mathcal{A}_o^c}\big[(\beta_2(\mu))_s\big]_{\mathbb{V}}\\&\le\kappa+(C_1\vee C_2)\,[\mu_s]_{\mathbb{U}},\quad ds\times dP\text{-}a.s.\end{aligned}$$

Let $\mu^1,\mu^2\in\mathcal{U}_t$ such that $\mu^1=\mu^2$, $ds\times dP$-a.s. on $[\![t,\tau[\![\cup[\![\tau,T]\!]_A$ for some $\tau\in\mathcal{S}_{t,T}$ and $A\in\mathcal{F}_\tau$. By Definition 2.2, $\beta_1(\mu^1)=\beta_1(\mu^2)$ and $\beta_2(\mu^1)=\beta_2(\mu^2)$, $ds\times dP$-a.s. on $[\![t,\tau[\![\cup[\![\tau,T]\!]_A$. Then for $ds\times dP$-a.s. $(s,\omega)\in[\![t,\tau[\![\cup[\![\tau,T]\!]_A$,

$$\begin{aligned}\big(\beta_o(\mu^1)\big)_s(\omega)&=\mathbf{1}_{\mathcal{A}_o}\big(\beta_1(\mu^1)\big)_s(\omega)+\mathbf{1}_{\mathcal{A}_o^c}\big(\beta_2(\mu^1)\big)_s(\omega)\\&=\mathbf{1}_{\mathcal{A}_o}\big(\beta_1(\mu^2)\big)_s(\omega)+\mathbf{1}_{\mathcal{A}_o^c}\big(\beta_2(\mu^2)\big)_s(\omega)=\big(\beta_o(\mu^2)\big)_s(\omega).\end{aligned}\tag{4.29}$$

Hence, $\beta_o\in\mathfrak{B}_t$.

For any $\mu\in\mathcal{U}_t$, (4.24) shows that $J\big(t,x,\mu,\beta_o(\mu)\big)=\mathbf{1}_{\mathcal{A}_o}J\big(t,x,\mu,\beta_1(\mu)\big)+\mathbf{1}_{\mathcal{A}_o^c}J\big(t,x,\mu,\beta_2(\mu)\big)$, P-a.s. Taking essential supremum over $\mu\in\mathcal{U}_t$ yields that $I(t,x,\beta_o)=\mathbf{1}_{\mathcal{A}_o}I(t,x,\beta_1)+\mathbf{1}_{\mathcal{A}_o^c}I(t,x,\beta_2)=I(t,x,\beta_1)\wedge I(t,x,\beta_2)$, P-a.s. Thus $\{I(t,x,\beta)\}_{\beta\in\mathfrak{B}_t}$ is directed downwards (see Theorem A.32 of [18]). By Proposition VI-**1**-1 of [27] or Theorem A.32 of [18], one can find $\big\{\widetilde{\beta}_i\big\}_{i\in\mathbb{N}}\subset\mathfrak{B}_t$ such that

$$w_1(t,x)=\underset{\beta\in\mathfrak{B}_t}{\operatorname{essinf}}\,I(t,x,\beta)=\lim_{i\to\infty}\downarrow I\big(t,x,\widetilde{\beta}_i\big),\quad P\text{-a.s.}\tag{4.30}$$

For any $i\in\mathbb{N}$, we set $\widetilde{\mathcal{A}}_i\stackrel{\triangle}{=}\big\{I\big(t,x,\widetilde{\beta}_i\big)\le w_1(t,x)+\varepsilon\big\}\in\mathcal{F}_t$ and $\widehat{\mathcal{A}}_i\stackrel{\triangle}{=}\widetilde{\mathcal{A}}_i\big\backslash\cup_{j<i}\widetilde{\mathcal{A}}_j\in\mathcal{F}_t$. Fix $n\in\mathbb{N}$ and set $\mathcal{A}_n\stackrel{\triangle}{=}\overset{n}{\underset{i=1}{\cup}}\widehat{\mathcal{A}}_i\in\mathcal{F}_t$. For any $\mu\in\mathcal{U}_t$, similar to $\widehat{\mu}$ above, $\beta_n(\mu)\stackrel{\triangle}{=}\sum_{i=1}^n\mathbf{1}_{\widehat{\mathcal{A}}_i}\widetilde{\beta}_i(\mu)+\mathbf{1}_{\mathcal{A}_n^c}\widetilde{\beta}_1(\mu)$ defines a $\mathcal{V}_t$-process. For $i=1,\dots,n$, let $\widetilde{C}_i>0$ be the constant associated to $\widetilde{\beta}_i$ in Definition 2.2 (i). Setting $C_n\stackrel{\triangle}{=}\max\{\widetilde{C}_i:i=1,\dots,n\}$, we can deduce that $\big[(\beta_n(\mu))_s\big]_{\mathbb{V}}=\sum_{i=1}^n\mathbf{1}_{\widehat{\mathcal{A}}_i}\big[(\widetilde{\beta}_i(\mu))_s\big]_{\mathbb{V}}+\mathbf{1}_{\mathcal{A}_n^c}\big[(\widetilde{\beta}_1(\mu))_s\big]_{\mathbb{V}}\le\kappa+C_n[\mu_s]_{\mathbb{U}}$, $ds\times dP$-a.s.

Let $\mu^1,\mu^2\in\mathcal{U}_t$ such that $\mu^1=\mu^2$, $ds\times dP$-a.s. on $[\![t,\tau[\![\cup[\![\tau,T]\!]_A$ for some $\tau\in\mathcal{S}_{t,T}$ and $A\in\mathcal{F}_\tau$. Similar to (4.29), it holds for $ds\times dP$-a.s. $(s,\omega)\in[\![t,\tau[\![\cup[\![\tau,T]\!]_A$ that

$$\begin{aligned}\big(\beta_n(\mu^1)\big)_s(\omega)&=\sum_{i=1}^n\mathbf{1}_{\widehat{\mathcal{A}}_i}\big(\widetilde{\beta}_i(\mu^1)\big)_s(\omega)+\mathbf{1}_{\mathcal{A}_n^c}\big(\widetilde{\beta}_1(\mu^1)\big)_s(\omega)\\&=\sum_{i=1}^n\mathbf{1}_{\widehat{\mathcal{A}}_i}\big(\widetilde{\beta}_i(\mu^2)\big)_s(\omega)+\mathbf{1}_{\mathcal{A}_n^c}\big(\widetilde{\beta}_1(\mu^2)\big)_s(\omega)=\big(\beta_n(\mu^2)\big)_s(\omega).\end{aligned}$$

So $\beta_n\in\mathfrak{B}_t$. For any $\mu\in\mathcal{U}_t$, using (4.24) again yields that $\mathbf{1}_{\mathcal{A}_n}J\big(t,x,\mu,\beta_n(\mu)\big)=\sum_{i=1}^n\mathbf{1}_{\widehat{\mathcal{A}}_i}J\big(t,x,\mu,\widetilde{\beta}_i(\mu)\big)$, P-a.s. Taking essential supremum over $\mu\in\mathcal{U}_t$ gives that $\mathbf{1}_{\mathcal{A}_n}I(t,x,\beta_n)=\sum_{i=1}^n\mathbf{1}_{\widehat{\mathcal{A}}_i}I\big(t,x,\widetilde{\beta}_i\big)\le\mathbf{1}_{\mathcal{A}_n}\big(w_1(t,x)+\varepsilon\big)$, P-a.s. Let $\widetilde{\mathcal{N}}$ be the P-null set such that (4.30) holds on $\widetilde{\mathcal{N}}^c$. As $|w_1(t,x)|<\infty$ by Propositions 2.1 and 2.2, we see that $\cup_{n\in\mathbb{N}}\mathcal{A}_n=\cup_{i\in\mathbb{N}}\widehat{\mathcal{A}}_i=\cup_{i\in\mathbb{N}}\widetilde{\mathcal{A}}_i=\widetilde{\mathcal{N}}^c$. □

In the proof of the weak dynamic programming principle below, we first use Lemma 4.2 to construct approximately optimal controls/strategies by pasting locally approximately optimal ones according to a finite partition of $\overline{O}_\delta(t,x)$ determined by the continuity of test functions ϕ and $\widetilde{\phi}$. After a series of estimates on state processes and payoff processes, we obtain the weak dynamic programming principle by using the stochastic backward semigroup property (2.11), the continuous dependence of payoff process on the initial state (see Lemma 2.3) as well as the control-neutralizer assumption and the growth condition on strategies.

Proof of Theorem 2.1. We prove only part (1) as (2) can be argued similarly or use the transformation (see [6]):

$$\begin{aligned}&g(x)\stackrel{\triangle}{=}-g(x)\ \text{ and }\ f(t,x,y,z,u,v)\stackrel{\triangle}{=}-f(t,x,-y,-z,u,v)\\&\qquad\forall\,(t,x,y,z,u,v)\in[0,T]\times\mathbb{R}^k\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{U}\times\mathbb{V}.\end{aligned}\tag{4.31}$$

For any $m \in \mathbb{N}$ and $(s,\mathfrak{x}) \in [t,T] \times \mathbb{R}^k$, the continuity of ϕ, $\widetilde{\phi}$ shows that there exists a $\delta^m_{s,\mathfrak{x}} \in (0,1/m)$ such that

$$\begin{aligned} &\big|\phi(s',\mathfrak{x}') - \phi(s,\mathfrak{x})\big| + \big|\widetilde{\phi}(s',\mathfrak{x}') - \widetilde{\phi}(s,\mathfrak{x})\big| \le 1/m \\ &\qquad \forall\, (s',\mathfrak{x}') \in \big[(s-\delta^m_{s,\mathfrak{x}}) \vee t, (s+\delta^m_{s,\mathfrak{x}}) \wedge T\big] \times \overline{O}_{\delta^m_{s,\mathfrak{x}}}(\mathfrak{x}). \end{aligned} \tag{4.32}$$

By classical covering theory, $\big\{\mathfrak{D}_m(s,\mathfrak{x}) \stackrel{\triangle}{=} \big(s-\delta^m_{s,\mathfrak{x}}, s+\delta^m_{s,\mathfrak{x}}\big) \times O_{\delta^m_{s,\mathfrak{x}}}(\mathfrak{x})\big\}_{(s,\mathfrak{x})\in[t,T]\times\mathbb{R}^k}$ has a finite subcollection $\{\mathfrak{D}_m(s_i,x_i)\}_{i=1}^{N_m}$ to cover $\overline{O}_\delta(t,x)$. For $i = 1,\dots,N_m$, we set $t_i \stackrel{\triangle}{=} (s_i + \delta^m_{s_i,x_i}) \wedge T$.

Fix $(\beta,\mu) \in \mathfrak{B}_t \times \mathcal{U}_t$ and simply denote $\tau_{\beta,\mu}$ by τ. By Lemma 2.1, $\widehat{\mu}_s \stackrel{\triangle}{=} \mathbf{1}_{\{s<\tau\}}\mu_s + \mathbf{1}_{\{s\ge\tau\}}u_0$, $s \in [t,T]$ defines a $\mathcal{U}_t$-control. We set $\Theta \stackrel{\triangle}{=} \big(t,x,\mu,\beta(\mu)\big)$ and $\widehat{\Theta} \stackrel{\triangle}{=} \big(t,x,\widehat{\mu},\beta(\widehat{\mu})\big)$.

(a) *Given $s \in [t,T)$, we first show that along $\widehat{\mu}|_{[t,s]}$, the restriction of β over $[s,T]$ is still an admissible strategy, which will be used in the next step to choose the locally approximately optimal controls; see* (4.34).

Let $\widetilde{\mu} \in \mathcal{U}_s$. The process $\big(\widehat{\mu} \oplus_s \widetilde{\mu}\big)_r \stackrel{\triangle}{=} \mathbf{1}_{\{r<s\}}\widehat{\mu}_r + \mathbf{1}_{\{r\ge s\}}\widetilde{\mu}_r$, $r \in [t,T]$ is clearly $\mathbf{F}$-progressively measurable. Suppose that $E\int_t^T [\mu_r]^q_{\mathbb{U}} dr + E\int_s^T [\widetilde{\mu}_r]^{\widetilde{q}}_{\mathbb{U}} dr < \infty$ for some $q,\widetilde{q} > 2$. It follows that $E\int_t^T \big[\big(\widehat{\mu} \oplus_s \widetilde{\mu}\big)_r\big]^{q\wedge\widetilde{q}}_{\mathbb{U}} dr \le E\int_t^T \big[\mu_r\big]^{q\wedge\widetilde{q}}_{\mathbb{U}} dr + E\int_s^T \big[\widetilde{\mu}_r\big]^{q\wedge\widetilde{q}}_{\mathbb{U}} dr < \infty$. Thus, $\widehat{\mu} \oplus_s \widetilde{\mu} \in \mathcal{U}_t$. Then we can define

$$\beta^s(\widetilde{\mu}) \stackrel{\triangle}{=} \big[\beta(\widehat{\mu} \oplus_s \widetilde{\mu})\big]^s \in \mathcal{V}_s. \tag{4.33}$$

For $dr \times dP$-a.s. $(r,\omega) \in [s,T] \times \Omega$, one has $\big[(\beta^s(\widetilde{\mu}))_r(\omega)\big]_{\mathbb{V}} = \big[(\beta(\widehat{\mu} \oplus_s \widetilde{\mu}))_r(\omega)\big]_{\mathbb{V}} \le \kappa + C_\beta\big[(\widehat{\mu} \oplus_s \widetilde{\mu})_r(\omega)\big]_{\mathbb{U}} = \kappa + C_\beta\big[\widetilde{\mu}_r(\omega)\big]_{\mathbb{U}}$. Let $\widetilde{\mu}^1, \widetilde{\mu}^2 \in \mathcal{U}_s$ such that $\widetilde{\mu}^1 = \widetilde{\mu}^2$, $dr \times dP$-a.s. on $[\![s,\zeta[\![\,\cup\, [\![\zeta,T]\!]_A$ for some $\zeta \in \mathcal{S}_{s,T}$ and $A \in \mathcal{F}_\zeta$. Then $\widehat{\mu} \oplus_s \widetilde{\mu}^1 = \widehat{\mu} \oplus_s \widetilde{\mu}^2$, $dr \times dP$-a.s. on $[\![t,\zeta[\![\,\cup\, [\![\zeta,T]\!]_A$. By Definition 2.2, $\beta(\widehat{\mu} \oplus_s \widetilde{\mu}^1) = \beta(\widehat{\mu} \oplus_s \widetilde{\mu}^2)$, $dr \times dP$-a.s. on $[\![t,\zeta[\![\,\cup\, [\![\zeta,T]\!]_A$. It follows that for $dr \times dP$-a.s. $(r,\omega) \in [\![s,\zeta[\![\,\cup\, [\![\zeta,T]\!]_A$, $\big(\beta^s(\widetilde{\mu}^1)\big)_r(\omega) = \big(\beta^s(\widetilde{\mu}^2)\big)_r(\omega)$. Hence, $\beta^s \in \mathfrak{B}_s$.

(b) *Fix $m \in \mathbb{N}$ with $m \ge C^\phi_{x,\delta} \stackrel{\triangle}{=} \sup\big\{|\phi(s,\mathfrak{x})| : (s,\mathfrak{x}) \in \overline{O}_{\delta+3}(t,x) \cap ([t,T] \times \mathbb{R}^k)\big\}$. According to the finite cover $\{\mathfrak{D}_m(s_i,x_i)\}_{i=1}^{N_m}$ of $\overline{O}_\delta(t,x)$, we use* (4.26) *to construct the $1/m$-optimal control μ^m for player* I *under strategy β by pasting together local $1/m$-optimal controls.*

Given $i = 1,\dots,N_m$, (4.26) shows that there exists $\big\{(A^{m,i}_n, \mu^{m,i}_n)\big\}_{n\in\mathbb{N}} \subset \mathcal{F}_{t_i} \times \mathcal{U}_{t_i}$ with $\lim_{n\to\infty} \mathbf{1}_{A^{m,i}_n} = 1$, P-a.s. such that for any $n \in \mathbb{N}$,

$$J\big(t_i, x_i, \mu^{m,i}_n, \beta^{t_i}(\mu^{m,i}_n)\big) \ge \big(I\big(t_i, x_i, \beta^{t_i}\big) - 1/m\big) \wedge m, \quad P\text{-a.s. on } A^{m,i}_n. \tag{4.34}$$

As $Y^{\widehat{\Theta}}(T, g(X^{\widehat{\Theta}}_T)) \in \mathbb{C}^p_{\mathbf{F}}([t,T])$, Monotone convergence theorem shows that $\lim_{n\to\infty} Eg[\mathbf{1}_{(A^{m,i}_n)^c}(\sup_{s\in[t,T]}|Y^{\widehat{\Theta}}_s(T, g(X^{\widehat{\Theta}}_T))|^p + (C^\phi_{x,\delta})^p)g] = 0$. So there is an $n(m,i) \in \mathbb{N}$ such that $Eg[\mathbf{1}_{(A^{m,i}_{n(m,i)})^c}(\sup_{s\in[t,T]}|Y^{\widehat{\Theta}}_s(T, g(X^{\widehat{\Theta}}_T))|^p + (C^\phi_{x,\delta})^p)g] \le m^{-(1+p)}N_m^{-1}$. Set $(A^m_i, \mu^m_i) \stackrel{\triangle}{=} (A^{m,i}_{n(m,i)}, \mu^{m,i}_{n(m,i)})$ and $\widetilde{A}^m_i \stackrel{\triangle}{=} \{(\tau, X^\Theta_\tau) \in \mathfrak{D}_m(s_i,x_i) \backslash \cup_{j<i} \mathfrak{D}_m(s_j,x_j)\} \in \mathcal{F}_\tau$. As $\widetilde{A}^m_i \subset \{(\tau, X^\Theta_\tau) \in \mathfrak{D}_m(s_i,x_i)\} \subset \{\tau \le t_i\}$, we see that $\widetilde{A}^m_i = \widetilde{A}^m_i \cap \{\tau \le t_i\} \in \mathcal{F}_{t_i}$.

By the continuity of process X^Θ, $(\tau, X^\Theta_\tau) \in \partial O_\delta(t,x)$, P-a.s. So $\{\widetilde{A}^m_i\}_{i=1}^{N_m}$ forms a partition of $\mathcal{N}^c$ for some P-null set $\mathcal{N}$. Then we can define an $\mathbf{F}$-stopping time $\tau_m \stackrel{\triangle}{=} \sum_{i=1}^{N_m} \mathbf{1}_{\widetilde{A}^m_i} t_i + \mathbf{1}_{\mathcal{N}} T \ge \tau$ as well as a process

$$\begin{aligned} \mu^m_s &\stackrel{\triangle}{=} \mathbf{1}_{\{s<\tau_m\}} \widehat{\mu}_s + \mathbf{1}_{\{s \ge \tau_m\}} \left(\sum_{i=1}^{N_m} \mathbf{1}_{\widetilde{A}^m_i \cap A^m_i} (\mu^m_i)_s + \mathbf{1}_{A_m} \widehat{\mu}_s \right) \\ &= \mathbf{1}_{A_m} \widehat{\mu}_s + \sum_{i=1}^{N_m} \mathbf{1}_{\widetilde{A}^m_i \cap A^m_i} \big(\mathbf{1}_{\{s<t_i\}} \widehat{\mu}_s + \mathbf{1}_{\{s \ge t_i\}} \big(\mu^m_i\big)_s \big) \ \ \forall\, s \in [t,T], \end{aligned}$$

where $A_m \stackrel{\triangle}{=} (\overset{N_m}{\cup} (\widetilde{A}^m_i \backslash A^m_i))_{i=1} \cup \mathcal{N}$. Let $s \in [t,T]$ and $U \in \mathscr{B}(\mathbb{U})$. As $[\![t,\tau[\![\in \mathscr{P}$, we see that $\mathcal{D} \stackrel{\triangle}{=} [\![t,\tau[\![\cap ([t,s] \times \Omega) \in \mathscr{B}([t,s]) \otimes \mathcal{F}_s$. The $\mathbf{F}$-progressive measurability of $\widehat{\mu}$ then implies that

$$\begin{aligned} \big\{(r,\omega) \in \mathcal{D} : \mu^m_r(\omega) \in U\big\} &= \big\{(r,\omega) \in \mathcal{D} : \widehat{\mu}_r(\omega) \in U\big\} \\ &= \mathcal{D} \cap \big\{(r,\omega) \in [t,s] \times \Omega : \widehat{\mu}_r(\omega) \in U\big\} \in \mathscr{B}\big([t,s]\big) \otimes \mathcal{F}_s. \end{aligned} \tag{4.35}$$

Given $i=1,\dots,N_m$, we set $\overline{A}^m_i \stackrel{\triangle}{=} (\widetilde{A}^m_i \backslash A^m_i) \cup \mathcal{N} \in \mathcal{F}_{t_i}$. If $s < t_i$, both $\mathcal{D}^m_i \stackrel{\triangle}{=} [\![\tau, T]\!]_{\widetilde{A}^m_i \cap A^m_i} \cap ([t,s] \times \Omega) = ([t_i,T] \cap [t,s]) \times (\widetilde{A}^m_i \cap A^m_i)$ and $\widehat{\mathcal{D}}^m_i \stackrel{\triangle}{=} [\![\tau,T]\!]_{\overline{A}^m_i} \cap ([t,s] \times \Omega) = ([t_i,T] \cap [t,s]) \times \overline{A}^m_i$ are empty. Otherwise, if $s \ge t_i$, both $\mathcal{D}^m_i = [t_i,s] \times (\widetilde{A}^m_i \cap A^m_i)$ and $\widehat{\mathcal{D}}^m_i = [t_i,s] \times \overline{A}^m_i$ belong to $\mathscr{B}([t_i,s]) \otimes \mathcal{F}_s$. Using a similar argument to (4.35) on the $\mathbf{F}$-progressive measurability of process μ^m_i yields that

$$\begin{aligned} &\big\{(r,\omega) \in \mathcal{D}^m_i : \mu^m_r(\omega) \in U\big\} \\ &\quad = \big\{(r,\omega) \in \mathcal{D}^m_i : \big(\mu^m_i\big)_r(\omega) \in U\big\} \in \mathscr{B}\big([t_i,s]\big) \otimes \mathcal{F}_s \subset \mathscr{B}\big([t,s]\big) \otimes \mathcal{F}_s \quad \text{and} \\ &\big\{(r,\omega) \in \widehat{\mathcal{D}}^m_i : \mu^m_r(\omega) \in U\big\} = \big\{(r,\omega) \in \widehat{\mathcal{D}}^m_i : \widehat{\mu}_r(\omega) \in U\big\} \in \mathscr{B}\big([t,s]\big) \otimes \mathcal{F}_s, \end{aligned}$$

both of which, together with (4.35), shows the $\mathbf{F}$-progressive measurability of μ^m. For $i=1,\dots,N_m$, suppose that $E\int_{t_i}^T \big[\big(\mu^m_i\big)_r\big]_{\mathbb{U}}^{q_i} dr < \infty$ for some $q_i > 2$. Setting $q_* \stackrel{\triangle}{=} q \wedge \min\{q_i : i=1,\dots,N_m\}$, we can deduce that $E\int_t^T \big[\mu^m_r\big]_{\mathbb{U}}^{q_*} dr \le E\int_t^T [\mu_r]_{\mathbb{U}}^{q_*} dr + \sum_{i=1}^{N_m} E\int_{t_i}^T \big[\big(\mu^m_i\big)_r\big]_{\mathbb{U}}^{q_*} dr < \infty$. Hence, $\mu^m \in \mathcal{U}_t$.

(c) *Next, set* $\Theta_m \stackrel{\triangle}{=} \big(t,x,\mu^m,\beta(\mu^m)\big)$. *We shall use a series of estimates on state processes* $X^{t,\xi,\mu,\nu}$*/payoff processes* $Y^{t,\xi,\mu,\nu}$*, a stochastic backward semigroup property* (2.11)*, as well as the continuous dependence of* $Y^{t,\xi,\mu,\nu}$ *on* ξ *to demonstrate how* $J\big(t,x,\mu^m,\beta(\mu^m)\big)$ *deviates from* $Y^\Theta_t\big(\tau, \phi\big(\tau, X^\Theta_\tau\big)\big)$*, which will eventually lead to*

$$w_1(t,x) \ge \underset{\beta \in \mathfrak{B}_t}{\operatorname{essinf}}\ \underset{\mu \in \mathcal{U}_t}{\operatorname{esssup}}\ Y^{t,x,\mu,\beta(\mu)}_t \Big(\tau_{\beta,\mu}, \phi\big(\tau_{\beta,\mu}, X^{t,x,\mu,\beta(\mu)}_{\tau_{\beta,\mu}}\big)\Big), \quad P\text{-a.s.} \tag{4.36}$$

As $\mu^m = \widehat{\mu} = \mu$ on $[\![t,\tau[\![$, taking $(\tau, A) = (\tau, \emptyset)$ in Definition 2.2 shows that $\beta(\mu^m) = \beta(\mu)$, $ds \times dP$-a.s. on $[\![t,\tau[\![$, and then applying (2.8) with $(\tau,A) = (\tau,\emptyset)$ yields that P-a.s.,

$$X^{\Theta_m}_s = X^\Theta_s \in \overline{O}_\delta(x) \quad \forall\, s \in [t,\tau]. \tag{4.37}$$

Thus, for any $\eta \in \mathbb{L}^p(\mathcal{F}_\tau)$, the BSDE$\big(t,\eta,f^{\Theta_m}_\tau\big)$ and the BSDE$\big(t,\eta,f^\Theta_\tau\big)$ are essentially the same. To wit,

$$\big(Y^{\Theta_m}(\tau,\eta), Z^{\Theta_m}(\tau,\eta)\big) = \big(Y^\Theta(\tau,\eta), Z^\Theta(\tau,\eta)\big). \tag{4.38}$$

Given $A \in \mathcal{F}_t$, we see from (4.37) that

$$\begin{aligned}\mathbf{1}_A X^{\Theta_m}_{\tau_m\wedge s} &= \mathbf{1}_A X^{\Theta_m}_{\tau\wedge s} + \mathbf{1}_A \int_{\tau\wedge s}^{\tau_m\wedge s} b\Big(r, X^{\Theta_m}_r, \mu^m_r, \big(\beta(\mu^m)\big)_r\Big) dr \\ &\quad + \mathbf{1}_A \int_{\tau\wedge s}^{\tau_m\wedge s} \sigma\Big(r, X^{\Theta_m}_r, \mu^m_r, \big(\beta(\mu^m)\big)_r\Big) dB_r, \\ &= \mathbf{1}_A X^{\Theta}_{\tau\wedge s} + \int_{\tau\wedge s}^{\tau_m\wedge s} \mathbf{1}_A b\big(r, X^{\Theta_m}_{\tau_m\wedge r}, u_0, \big(\beta(\mu^m)\big)_r\big) dr \\ &\quad + \int_{\tau\wedge s}^{\tau_m\wedge s} \mathbf{1}_A \sigma\big(r, X^{\Theta_m}_{\tau_m\wedge r}, u_0, \big(\beta(\mu^m)\big)_r\big) dB_r, \quad s \in [t,T].\end{aligned}$$

It follows that

$$\begin{aligned}\mathbf{1}_A \sup_{r\in[t,s]} \big|X^{\Theta_m}_{\tau_m\wedge r} - X^{\Theta}_{\tau\wedge r}\big| &\le \int_{\tau\wedge s}^{\tau_m\wedge s} \mathbf{1}_A \big|b\big(r, X^{\Theta_m}_{\tau_m\wedge r}, u_0, \big(\beta(\mu^m)\big)_r\big)\big| dr \\ &\quad + \sup_{r\in[t,s]} \left| \int_{\tau\wedge r}^{\tau_m\wedge r} \mathbf{1}_A \sigma\big(r', X^{\Theta_m}_{\tau_m\wedge r'}, u_0, \big(\beta(\mu^m)\big)_{r'}\big) dB_{r'} \right|, \quad s \in [t,T].\end{aligned} \tag{4.39}$$

Let $C(\kappa, x, \delta)$ denote a generic constant, depending on $\kappa + |x| + \delta$, $C^{\phi}_{x,\delta}$, T, γ, p, and $|g(0)|$, whose form may vary from line to line. Squaring both sides of (4.39) and taking expectation, we can deduce from Hölder's inequality, Doob's martingale inequality, (2.1), (2.2), (4.37), and Fubini's theorem that

$$\begin{aligned}& E\Big[\mathbf{1}_A \sup_{r\in[t,s]} \big|X^{\Theta_m}_{\tau_m\wedge r} - X^{\Theta}_{\tau\wedge r}\big|^2\Big] \\ &\le 4E \int_{\tau\wedge s}^{\tau_m\wedge s} \mathbf{1}_A \big|b\big(r, X^{\Theta_m}_{\tau_m\wedge r}, u_0, \big(\beta(\mu^m)\big)_r\big)\big|^2 dr \\ &\quad + 8E \int_{\tau\wedge s}^{\tau_m\wedge s} \mathbf{1}_A \big|\sigma\big(r, X^{\Theta_m}_{\tau_m\wedge r}, u_0, \big(\beta(\mu^m)\big)_r\big)\big|^2 dr \\ &\le 12\gamma^2 E \int_{\tau\wedge s}^{\tau_m\wedge s} \mathbf{1}_A \Big(\big|X^{\Theta_m}_{\tau_m\wedge r} - X^{\Theta}_{\tau\wedge r}\big| + \big|X^{\Theta}_{\tau\wedge r}\big| + 1 + \big[\big(\beta(\mu^m)\big)_r\big]_{\mathbb{V}}\Big)^2 dr \\ &\le 24\gamma^2 \int_t^s E\left[\mathbf{1}_A \sup_{r'\in[t,r]} \big|X^{\Theta_m}_{\tau_m\wedge r'} - X^{\Theta}_{\tau\wedge r'}\big|^2\right] dr + \frac{C(\kappa,x,\delta)}{m} P(A) \quad \forall\, s \in [t,T],\end{aligned} \tag{4.40}$$

where we used the facts that

$$\begin{aligned}&\tau_m - \tau \le \sum_{i=1}^{N_m} \mathbf{1}_{\widetilde{A}^m_i} 2\delta^m_{s_i,x_i} < \frac{2}{m}, \ P\text{-a.s.} \quad \text{and} \\ &\big[\big(\beta(\mu^m)\big)_r\big]_{\mathbb{V}} \le \kappa, \ dr \times dP\text{-}a.s. \text{ on } [\![\tau, \tau_m[\![.\end{aligned} \tag{4.41}$$

Then Gronwall's inequality implies $E[\mathbf{1}_A \sup_{r\in[t,T]} \big|X^{\Theta_m}_{\tau_m\wedge r} - X^{\Theta}_{\tau\wedge r}\big|^2] \le \frac{C(\kappa,x,\delta)}{m} P(A)$. Letting A vary in $\mathcal{F}_t$ yields that

$$E\left[\sup_{r\in[t,T]} \big|X^{\Theta_m}_{\tau_m\wedge r} - X^{\Theta}_{\tau\wedge r}\big|^2 \Big| \mathcal{F}_t\right] \le \frac{C(\kappa,x,\delta)}{m} \quad P\text{-a.s.} \tag{4.42}$$

Let $i=1,\ldots,N_m$ and $\Theta_m^{t_i} \stackrel{\triangle}{=} \big(t_i, X_{t_i}^{\Theta_m}, [\mu^m]^{t_i}, [\beta(\mu^m)]^{t_i}\big)$. By (2.7), $X_T^{\Theta_m} = X_T^{\Theta_m^{t_i}}$, P-a.s. Then (2.13) shows that

$$
\begin{aligned}
Y_{t_i}^{\Theta_m}\big(T, g\big(X_T^{\Theta_m}\big)\big) &= Y_{t_i}^{\Theta_m^{t_i}}\big(T, g\big(X_T^{\Theta_m}\big)\big) \\
&= Y_{t_i}^{\Theta_m^{t_i}}\big(T, g\big(X_T^{\Theta_m^{t_i}}\big)\big) = J(\Theta_m^{t_i}), \quad P\text{-a.s.}
\end{aligned} \tag{4.43}
$$

Similar to μ^m, $\big(\widehat{\mu}_i^m\big)_s \stackrel{\triangle}{=} \mathbf{1}_{\{s<\tau_m\}}\widehat{\mu}_s + \mathbf{1}_{\{s\ge\tau_m\}}\big(\mathbf{1}_{\widetilde{A}_i^m\cap A_i^m}\big(\mu_i^m\big)_s + \mathbf{1}_{(\widetilde{A}_i^m\cap A_i^m)^c}\widehat{\mu}_s\big)$, $s\in[t,T]$ also defines a $\mathcal{U}_t$-process. As $\mu^m = \widehat{\mu}_i^m$ on $[\![t,\tau_m[\![\cup [\![\tau_m, T]\!]_{\widetilde{A}_i^m\cap A_i^m}$ and $\widehat{\mu}_i^m = \widehat{\mu}\oplus_{t_i}\mu_i^m$ on $\big([t,t_i)\times\Omega\big)\cup\big([t_i,T]\times(\widetilde{A}_i^m\cap A_i^m)\big)$, Definition 2.2 shows that $\beta(\mu^m)=\beta\big(\widehat{\mu}_i^m\big)$, $ds\times dP$-a.s. on $[\![t,\tau_m[\![\cup[\![\tau_m,T]\!]_{\widetilde{A}_i^m\cap A_i^m}$, and $\beta\big(\widehat{\mu}_i^m\big)=\beta\big(\widehat{\mu}\oplus_{t_i}\mu_i^m\big)$, $ds\times dP$-a.s. on $\big([t,t_i)\times\Omega\big)\cup\big([t_i,T]\times(\widetilde{A}_i^m\cap A_i^m)\big)$. Thus $\big(\mu^m,\beta(\mu^m)\big)=\big(\widehat{\mu}\oplus_{t_i}\mu_i^m,\beta\big(\widehat{\mu}\oplus_{t_i}\mu_i^m\big)\big)$, $ds\times dP$-a.s. on $[\![\tau_m,T]\!]_{\widetilde{A}_i^m\cap A_i^m}=[t_i,T]\times(\widetilde{A}_i^m\cap A_i^m)$. From (4.33), one has $\big([\mu^m]^{t_i},[\beta(\mu^m)]^{t_i}\big)=\big(\mu_i^m,\beta^{t_i}(\mu_i^m)\big)$, $ds\times dP$-a.s. on $[t_i,T]\times(\widetilde{A}_i^m\cap A_i^m)$. Then by (4.43), (4.24), and (2.14), it holds P-a.s. on $\widetilde{A}_i^m\cap A_i^m\in\mathcal{F}_{t_i}$ that

$$
\begin{aligned}
Y_{\tau_m}^{\Theta_m}\big(T,g\big(X_T^{\Theta_m}\big)\big) &= Y_{t_i}^{\Theta_m}\big(T,g\big(X_T^{\Theta_m}\big)\big) = J\Big(t_i, X_{\tau_m}^{\Theta_m}, \mu_i^m, \beta^{t_i}\big(\mu_i^m\big)\Big) \\
&\ge J\Big(t_i, X_\tau^{\Theta}, \mu_i^m, \beta^{t_i}\big(\mu_i^m\big)\Big) - c_0|X_{\tau_m}^{\Theta_m}-X_\tau^{\Theta}|^{2/p}.
\end{aligned}
$$

Since $\mathfrak{D}_m(s_i,x_i)\cap\overline{O}_\delta(t,x)\neq\emptyset$, it is easy to see that

$$
\begin{aligned}
\overline{\mathfrak{D}}_m(s_i,x_i) = \big[&s_i-\delta_{s_i,x_i}^m, s_i+\delta_{s_i,x_i}^m\big] \\
&\times\overline{O}_{\delta_{s_i,x_i}^m}(x_i)\subset\overline{O}_{\delta+2\sqrt{2}\delta_{s_i,x_i}^m}(t,x)\subset\overline{O}_{\delta+\frac{2\sqrt{2}}{m}}(t,x)\subset\overline{O}_{\delta+3}(t,x).
\end{aligned}
$$

So $\phi(t_i,x_i)\le C_{x,\delta}^{\phi}<m+1/m$. On the other hand, $\phi(t_i,x_i)\le w_1(t_i,x_i)\le I\big(t_i,x_i,\beta^{t_i}\big)$, P-a.s. Then (4.34) shows that $\phi(t_i,x_i)\le I\big(t_i,x_i,\beta^{t_i}\big)\wedge(m+1/m)\le J\big(t_i,x_i,\mu_i^m,\beta^{t_i}(\mu_i^m)\big)+1/m$, P-a.s. on A_i^m. As $\big|X_\tau^\Theta-x_i\big|^{2/p}<(\delta_{s_i,x_i}^m)^{2/p}<m^{-2/p}\le1/m$ on $\widetilde{A}_i^m$, (2.14), (4.32), and the continuity of ϕ imply that it holds P-a.s. on $\widetilde{A}_i^m\cap A_i^m$ that

$$
\begin{aligned}
J\Big(t_i,X_\tau^\Theta,\mu_i^m,\beta^{t_i}\big(\mu_i^m\big)\Big) &\ge J\big(t_i,x_i,\mu_i^m,\beta^{t_i}\big(\mu_i^m\big)\big)-\frac{c_0}{m}\ge\phi\big(t_i,x_i\big)-\frac{c_0}{m} \\
&\ge\phi\big(s_i,x_i\big)-\frac{c_0}{m}\ge\phi\big(\tau,X_\tau^\Theta\big)-\frac{c_0}{m}\stackrel{\triangle}{=}\eta_m\in\mathbb{L}^\infty(\mathcal{F}_\tau).
\end{aligned}
$$

Thus it holds P-a.s. on$\cup_{i=1}^{N_m}(\widetilde{A}_i^m\cap A_i^m)$ that

$$
Y_{\tau_m}^{\Theta_m}\big(T,g\big(X_T^{\Theta_m}\big)\big)\ge\eta_m-c_0|X_{\tau_m}^{\Theta_m}-X_\tau^\Theta|^{2/p}\stackrel{\triangle}{=}\widetilde{\eta}_m\in\mathbb{L}^p(\mathcal{F}_{\tau_m}). \tag{4.44}
$$

By (2.10), it holds P-a.s. that

$$
\big|Y_t^\Theta(\tau,\eta_m)-Y_t^\Theta\big(\tau,\phi\big(\tau,X_\tau^\Theta\big)\big)\big|^p\le c_0E\Big[\big|\eta_m-\phi\big(\tau,X_\tau^\Theta\big)\big|^p\Big|\mathcal{F}_t\Big]\le\frac{c_0}{m^p}. \tag{4.45}
$$

Let $(Y^m,Z^m)\in\mathbb{G}_{\mathbf{F}}^p([t,T])$ be the unique solution of the following BSDE with zero generator:

$$
Y_s^m = Y_\tau^{\Theta_m}(\tau_m,\eta_m)-\int_s^T Z_r^m dB_r, \quad s\in[t,T].
$$

For any $s \in [t,T]$, we have $Y^m_{\tau\wedge s} = E[Y^m_{\tau\wedge s}|\mathcal{F}_\tau] = E\Big[Y^{\Theta_m}_\tau(\tau_m,\eta_m) - \int_{\tau\wedge s}^T Z^m_r dB_r \Big| \mathcal{F}_\tau\Big] = Y^{\Theta_m}_\tau(\tau_m,\eta_m) - \int_{\tau\wedge s}^\tau Z^m_r dB_r$, P-a.s. By the continuity of process Y^m, it holds P-a.s. that

$$
\begin{aligned}
Y^m_{\tau\wedge s} &= Y^{\Theta_m}_\tau(\tau_m,\eta_m) - \int_{\tau\wedge s}^{\tau} Z^m_r dB_r \\
&= Y^{\Theta_m}_\tau(\tau_m,\eta_m) - \int_s^T \mathbf{1}_{\{r<\tau\}} Z^m_r dB_r, \quad s \in [t,T].
\end{aligned} \tag{4.46}
$$

Thus, we see that $(Y^m_s, Z^m_s) = \big(Y^m_{\tau\wedge s}, \mathbf{1}_{\{s<\tau\}} Z^m_s\big)$, $s \in [t,T]$. Also, taking $[\cdot|\mathcal{F}_{\tau\wedge s}]$ in (4.46) shows that $P\big(Y^m_s = Y^m_{\tau\wedge s} = E\big[Y^{\Theta_m}_\tau(\tau_m,\eta_m)\big|\mathcal{F}_{\tau\wedge s}\big] \ \forall\, s \in [t,T]\big) = 1$.

On the other hand, let $(\widetilde{Y}^m, \widetilde{Z}^m) \in \mathbb{G}^p_{\mathbf{F}}([t,T])$ be the unique solution of the following BSDE with zero generator:

$$
\widetilde{Y}^m_s = \eta_m - \int_s^T \widetilde{Z}^m_r dB_r, \quad s \in [t,T]. \tag{4.47}
$$

Similar to (Y^m, Z^m), it holds P-a.s. that

$$
\big(\widetilde{Y}^m_s, \widetilde{Z}^m_s\big) = \big(\widetilde{Y}^m_{\tau\wedge s}, \mathbf{1}_{\{s<\tau\}} \widetilde{Z}^m_s\big) \quad \text{and} \quad \widetilde{Y}^m_s = E[\eta_m|\mathcal{F}_{\tau\wedge s}] \quad \forall\, s \in [t,T]. \tag{4.48}
$$

The processes $(\mathcal{Y}^m, \mathcal{Z}^m) \stackrel{\triangle}{=} \big\{\big(\mathbf{1}_{\{s<\tau\}} Y^m_s + \mathbf{1}_{\{s\ge\tau\}} Y^{\Theta_m}_s(\tau_m,\eta_m), \mathbf{1}_{\{s<\tau\}} Z^m_s + \mathbf{1}_{\{s\ge\tau\}} Z^{\Theta_m}_s(\tau_m,\eta_m)\big)\big\}_{s\in[t,T]} \in \mathbb{G}^p_{\mathbf{F}}([t,T])$ solves the following BSDE:

$$
\begin{aligned}
\mathcal{Y}^m_s &= \mathbf{1}_{\{s\ge\tau\}} Y^{\Theta_m}_s(\tau_m,\eta_m) + \mathbf{1}_{\{s<\tau\}} Y^{\Theta_m}_\tau(\tau_m,\eta_m) - \mathbf{1}_{\{s<\tau\}} \int_s^T Z^m_r dB_r \\
&= Y^{\Theta_m}_{\tau\vee s}(\tau_m,\eta_m) - \mathbf{1}_{\{s<\tau\}} \int_s^T \mathbf{1}_{\{r<\tau\}} Z^m_r dB_r \\
&= \eta_m + \int_{\tau\vee s}^T f^{\Theta_m}_{\tau_m}\big(r, Y^{\Theta_m}_r(\tau_m,\eta_m), Z^{\Theta_m}_r(\tau_m,\eta_m)\big) dr \\
&\quad\ - \int_{\tau\vee s}^T Z^{\Theta_m}_r(\tau_m,\eta_m) dB_r - \int_s^T \mathbf{1}_{\{r<\tau\}} Z^m_r dB_r \\
&= \eta_m + \int_s^T \mathbf{1}_{\{r\ge\tau\}} f^{\Theta_m}_{\tau_m}\big(r, \mathcal{Y}^m_r, \mathcal{Z}^m_r\big) dr - \int_s^T \mathcal{Z}^m_r dB_r, \quad s \in [t,T].
\end{aligned} \tag{4.49}
$$

Since (2.4), Hölder's inequality and (2.9) imply that

$$
\begin{aligned}
&E\bigg[\int_t^T \mathbf{1}_{\{s\ge\tau\}} \big|f^{\Theta_m}_{\tau_m}\big(s, \widetilde{Y}^m_s, \widetilde{Z}^m_s\big)\big|^p ds\bigg] \\
&\le c_p E\left[\int_t^T \big|f^{\Theta_m}_{\tau_m}\big(s,0,0\big)\big|^p ds + \sup_{s\in[t,T]} \big|\widetilde{Y}^m_s\big|^p + \bigg(\int_t^T \big|\widetilde{Z}^m_s\big|^2 ds\bigg)^{p/2}\right] < \infty,
\end{aligned}
$$

applying (1.5) to $\mathcal{Y}^m - \widetilde{Y}^m$ and using (4.48) yield that

$$
\begin{aligned}
&E\Big[\big|Y^{\Theta_m}_\tau(\tau_m,\eta_m)-\eta_m\big|^p\Big|\mathcal{F}_t\Big]\\
&\quad= E\Big[\big|\mathcal{Y}^m_\tau-\widetilde{Y}^m_\tau\big|^p\Big|\mathcal{F}_t\Big]\\
&\quad\le E\bigg[\sup_{s\in[t,T]}\big|\mathcal{Y}^m_s-\widetilde{Y}^m_s\big|^p\Big|\mathcal{F}_t\bigg]\le c_0E\bigg[\int_\tau^T\big|f^{\Theta_m}_{\tau_m}\big(s,\widetilde{Y}^m_s,\widetilde{Z}^m_s\big)\big|^p ds\Big|\mathcal{F}_t\bigg]\\
&\quad= c_0E\bigg[\int_\tau^{\tau_m}\big|f\big(s,X^{\Theta_m}_{\tau_m\wedge s},\eta_m,0,u_0,(\beta(\mu^m))_s\big)\big|^p ds\Big|\mathcal{F}_t\bigg],\quad P\text{-a.s.}
\end{aligned}\tag{4.50}
$$

Then one can deduce from (2.10), (2.3), (2.4), (4.37), (4.41), and (4.42) that

$$
\begin{aligned}
&\big|Y^{\Theta_m}_t\big(\tau,Y^{\Theta_m}_\tau(\tau_m,\eta_m)\big)-Y^{\Theta_m}_t(\tau,\eta_m)\big|^p\le c_0E\Big[\big|Y^{\Theta_m}_\tau(\tau_m,\eta_m)-\eta_m\big|^p\Big|\mathcal{F}_t\Big]\\
&\quad\le c_0E\bigg[\int_\tau^{\tau_m}\Big(1+\big|X^{\Theta_m}_{\tau_m\wedge s}-X^{\Theta}_{\tau\wedge s}\big|^2+|X^{\Theta}_{\tau\wedge s}|^2+|\eta_m|^p+\big[(\beta(\mu^m))_s\big]^2_{\mathbb{V}}\Big)ds\Big|\mathcal{F}_t\bigg]\\
&\quad\le c_0E\bigg[(\tau_m-\tau)\cdot\sup_{s\in[t,T]}\big|X^{\Theta_m}_{\tau_m\wedge s}-X^{\Theta}_{\tau\wedge s}\big|^2\Big|\mathcal{F}_t\bigg]\\
&\qquad+\frac{c_0}{m}\Big\{1+(|x|+\delta)^2+\Big(C^\phi_{x,\delta}+\frac{c_0}{m}\Big)^p+\kappa^2\Big\}\\
&\quad\le\frac{C(\kappa,x,\delta)}{m^2}+\frac{C(\kappa,x,\delta)}{m}+\frac{c_0}{m^{p+1}}\le\frac{C(\kappa,x,\delta)}{m},\quad P\text{-a.s.}
\end{aligned}\tag{4.51}
$$

Applying (2.11) with $(\zeta,\tau,\eta)=(\tau,\tau_m,\eta_m)$, applying (4.38) with $\eta=\eta_m$ and using (4.45) yield that P-a.s.

$$
\begin{aligned}
Y^{\Theta_m}_t\big(\tau_m,\eta_m\big)&=Y^{\Theta_m}_t\big(\tau,Y^{\Theta_m}_\tau\big(\tau_m,\eta_m\big)\big)\ge Y^{\Theta_m}_t\big(\tau,\eta_m\big)-\frac{C(\kappa,x,\delta)}{m^{1/p}}\\
&=Y^{\Theta}_t\big(\tau,\eta_m\big)-\frac{C(\kappa,x,\delta)}{m^{1/p}}\ge Y^{\Theta}_t\big(\tau,\phi\big(\tau,X^{\Theta}_\tau\big)\big)-\frac{C(\kappa,x,\delta)}{m^{1/p}}.
\end{aligned}\tag{4.52}
$$

As $\mu^m=\widehat{\mu}$ on $[\![t,\tau_m[\![$, taking $(\tau,A)=(\tau_m,\emptyset)$ in Definition 2.2 shows that $\beta(\mu^m)=\beta(\widehat{\mu})$, $ds\times dP$-a.s. on $[\![t,\tau_m[\![$, and then applying (2.8) with $(\tau,A)=(\tau_m,\emptyset)$ yields that P-a.s.

$$
X^{\Theta_m}_s=X^{\widehat{\Theta}}_s\quad\forall\, s\in[t,\tau_m].\tag{4.53}
$$

Given $i=1,\dots,N_m$, (4.53) shows that $X^{\Theta_m}_{t_i}=X^{\widehat{\Theta}}_{t_i}$, P-a.s. on $\widetilde{A}^m_i\backslash A^m_i$. As $\mu^m=\widehat{\mu}$ on $[\![t,\tau_m[\![\cup[\![\tau_m,T]\!]_{\widetilde{A}^m_i\backslash A^m_i}$, Definition 2.2 shows that $\beta(\mu^m)=\beta\big(\widehat{\mu}\big)$, $ds\times dP$-a.s. on $[\![t,\tau_m[\![\cup[\![\tau_m,T]\!]_{\widetilde{A}^m_i\backslash A^m_i}$. So $\big([\mu^m]^{t_i},[\beta(\mu^m)]^{t_i}\big)=\big([\widehat{\mu}]^{t_i},[\beta(\widehat{\mu})]^{t_i}\big)$ holds $ds\times dP$-a.s. on $[\![\tau_m,T]\!]_{\widetilde{A}^m_i\backslash A^m_i}=[t_i,T]\times(\widetilde{A}^m_i\backslash A^m_i)$. Set $\widehat{\Theta}^{t_i}\stackrel{\triangle}{=}\big(t_i,X^{\widehat{\Theta}}_{t_i},[\widehat{\mu}]^{t_i},[\beta(\widehat{\mu})]^{t_i}\big)$. By (4.24) and a similar argument to (4.43), it holds P-a.s. on $\widetilde{A}^m_i\backslash A^m_i$ that

$$
\begin{aligned}
Y^{\Theta_m}_{\tau_m}\big(T,g\big(X^{\Theta_m}_T\big)\big)&=Y^{\Theta_m}_{t_i}\big(T,g\big(X^{\Theta_m}_T\big)\big)=J\big(\Theta^{t_i}_m\big)=J\big(\widehat{\Theta}^{t_i}\big)\\
&=Y^{\widehat{\Theta}}_{t_i}\big(T,g\big(X^{\widehat{\Theta}}_T\big)\big)=Y^{\widehat{\Theta}}_{\tau_m}\big(T,g\big(X^{\widehat{\Theta}}_T\big)\big).
\end{aligned}\tag{4.54}
$$

Let $\widehat{\eta}_m\stackrel{\triangle}{=}Y^{\Theta_m}_{\tau_m}\big(T,g\big(X^{\Theta_m}_T\big)\big)\wedge\widetilde{\eta}_m\in\mathbb{L}^p\big(\mathcal{F}_{\tau_m}\big)$ and set $\widetilde{A}_m\stackrel{\triangle}{=}\big\{Y^{\Theta_m}_{\tau_m}\big(T,g\big(X^{\Theta_m}_T\big)\big)<\widetilde{\eta}_m\big\}\in\mathcal{F}_{\tau_m}$. Clearly, $\mathbf{1}_{\widetilde{A}_m}\le\mathbf{1}_{A_m}$, P-a.s. Applying (2.10) again, we can deduce from

(4.42) and (4.54) that P-a.s.

$$\begin{aligned}
&\big|Y_t^{\Theta_m}(\tau_m,\widehat{\eta}_m)-Y_t^{\Theta_m}(\tau_m,\eta_m)\big|^p\\
&\quad\le c_0E\Big[\big|\widehat{\eta}_m-\eta_m\big|^p\Big|\mathcal{F}_t\Big]=c_0E\Big[\mathbf{1}_{\widetilde{A}_m^c}\big|\widetilde{\eta}_m-\eta_m\big|^p\\
&\qquad+\mathbf{1}_{\widetilde{A}_m}\big|Y_{\tau_m}^{\Theta_m}\big(T,g\big(X_T^{\Theta_m}\big)\big)-\eta_m\big|^p\Big|\mathcal{F}_t\Big]\\
&\quad\le c_0E\Big[\big|X_{\tau_m}^{\Theta_m}-X_\tau^{\Theta}\big|^2+\mathbf{1}_{A_m}\big|Y_{\tau_m}^{\Theta_m}\big(T,g\big(X_T^{\Theta_m}\big)\big)-\eta_m\big|^p\Big|\mathcal{F}_t\Big]\\
&\quad\le\frac{C(\kappa,x,\delta)}{m}+c_0E\Big[\mathbf{1}_{A_m}\big|Y_{\tau_m}^{\widehat{\Theta}}\big(T,g\big(X_T^{\widehat{\Theta}}\big)\big)-\phi(\tau,X_\tau^{\Theta})\big|^p\Big|\mathcal{F}_t\Big]+\frac{c_0}{m^p}\\
&\quad\le\frac{C(\kappa,x,\delta)}{m}+c_0E\Big[\mathbf{1}_{A_m}\Big(\sup_{s\in[t,T]}\Big|Y_s^{\widehat{\Theta}}\big(T,g\big(X_T^{\widehat{\Theta}}\big)\big)\Big|^p+\big(C_{x,\delta}^\phi\big)^p\Big)\Big|\mathcal{F}_t\Big].
\end{aligned}\tag{4.55}$$

Applying (2.11) with $(\zeta,\tau,\eta)=\big(\tau_m,T,g\big(X_T^{\Theta_m}\big)\big)$, we see from Proposition 1.2 (2), (4.55), and (4.52) that P-a.s.

$$\begin{aligned}
Y_t^{\Theta_m}\Big(T,g\big(X_T^{\Theta_m}\big)\Big)&=Y_t^{\Theta_m}\Big(\tau_m,Y_{\tau_m}^{\Theta_m}\big(T,g\big(X_T^{\Theta_m}\big)\big)\Big)\ge Y_t^{\Theta_m}(\tau_m,\widehat{\eta}_m)\\
&\ge Y_t^{\Theta}\big(\tau,\phi\big(\tau,X_\tau^{\Theta}\big)\big)-\frac{C(\kappa,x,\delta)}{m^{1/p}}\\
&\quad-c_0\bigg\{E\Big[\mathbf{1}_{A_m}\Big(\sup_{s\in[t,T]}\Big|Y_s^{\widehat{\Theta}}\big(T,g\big(X_T^{\widehat{\Theta}}\big)\big)\Big|^p+\big(C_{x,\delta}^\phi\big)^p\Big)\Big|\mathcal{F}_t\Big]\bigg\}^{\frac1p}.
\end{aligned}\tag{4.56}$$

Letting $\widehat{A}_m\stackrel{\triangle}{=}\{E[\mathbf{1}_{A_m}(\sup_{s\in[t,T]}|Y_s^{\widehat{\Theta}}(T,g(X_T^{\widehat{\Theta}}))|^p+(C_{x,\delta}^\phi)^p)|\mathcal{F}_t]>1/m\}$, one can deduce that

$$\begin{aligned}
P(\widehat{A}_m)&\le mE\left[\mathbf{1}_{A_m}\left(\sup_{s\in[t,T]}\Big|Y_s^{\widehat{\Theta}}\big(T,g\big(X_T^{\widehat{\Theta}}\big)\big)\Big|^p+\big(C_{x,\delta}^\phi\big)^p\right)\right]\\
&\le\sum_{i=1}^{N_m}mE\left[\mathbf{1}_{(A_i^m)^c}\left(\sup_{s\in[t,T]}\Big|Y_s^{\widehat{\Theta}}\big(T,g\big(X_T^{\widehat{\Theta}}\big)\big)\Big|^p+\big(C_{x,\delta}^\phi\big)^p\right)\right]\le m^{-p}.
\end{aligned}$$

Multiplying $\mathbf{1}_{\widehat{A}_m^c}$ to both sides of (4.56) yields that

$$\begin{aligned}
\mathbf{1}_{\widehat{A}_m^c}I\big(t,x,\beta\big)&\ge\mathbf{1}_{\widehat{A}_m^c}J\big(t,x,\mu^m,\beta(\mu^m)\big)\\
&\ge\mathbf{1}_{\widehat{A}_m^c}Y_t^{\Theta}\big(\tau,\phi\big(\tau,X_\tau^{\Theta}\big)\big)-\frac{C(\kappa,x,\delta)}{m^{1/p}},\quad P\text{-a.s.}
\end{aligned}\tag{4.57}$$

As $\sum_{m\in\mathbb{N}}P\big(\widehat{A}_m\big)\le\sum_{m\in\mathbb{N}}m^{-p}<\infty$, the Borel–Cantelli theorem shows that $P\big(\overline{\lim}_{m\to\infty}\mathbf{1}_{\widehat{A}_m}=1\big)=0$. It follows that

$$P\big(\overline{\lim_{m\to\infty}}\,\mathbf{1}_{\widehat{A}_m}=0\big)=1\ \text{ and thus }\ \lim_{m\to\infty}\mathbf{1}_{\widehat{A}_m}=0,\quad P\text{-a.s.}\tag{4.58}$$

So letting $m\to\infty$ in (4.57) yields that $I\big(t,x,\beta\big)\ge Y_t^{t,x,\mu,\beta(\mu)}\big(\tau_{\beta,\mu},\phi\big(\tau_{\beta,\mu},X_{\tau_{\beta,\mu}}^{t,x,\mu,\beta(\mu)}\big)\big)$, P-a.s. Taking essential supremum over $\mu\in\mathcal{U}_t$ and then taking essential infimum over $\beta\in\mathfrak{B}_t$, we obtain (4.36).

(d) *Now let us show the other inequality of Theorem* 2.1 (1). *Similar to* μ^m, *we shall first use* (4.27) *to construct the* $1/m$*-optimal strategy* β_m *by pasting together local* $1/m$*-optimal strategies with respect to the finite cover* $\{\mathfrak{D}_m(s_i,x_i)\}_{i=1}^{N_m}$ *of* $\overline{O}_\delta(t,x)$.

Fix $m\in\mathbb{N}$. For $i=1,\dots,N_m$, (4.27) shows that there exists $(\mathcal{A}_i^m,\beta_i^m)\in\mathcal{F}_{t_i}\times\mathfrak{B}_{t_i}$ with $P\big(\mathcal{A}_i^m\big)\geq 1-m^{\frac{1+p^2}{1-p}}N_m^{-1}$ such that

$$\widetilde{\phi}(t_i,x_i)\geq w_1(t_i,x_i)\geq I\big(t_i,x_i,\beta_i^m\big)-1/m,\quad P\text{-a.s. on }\mathcal{A}_i^m. \tag{4.59}$$

Let β_ψ be the $\mathfrak{B}_t$-strategy considered in (4.7) and fix $\beta\in\mathfrak{B}_t$. For any $\mu\in\mathcal{U}_t$, we denote $\tau_{\beta,\mu}$ by τ_μ and define

$$\big(\widehat{\beta}(\mu)\big)_s\stackrel{\triangle}{=}\mathbf{1}_{\{s<\tau_\mu\}}\big(\beta(\mu)\big)_s+\mathbf{1}_{\{s\geq\tau_\mu\}}\big(\beta_\psi(\mu)\big)_s\quad\forall\,s\in[t,T],$$

which is a $\mathcal{V}_t$-control by Lemma 2.1. By (A-u), it holds $ds\times dP$-a.s. that

$$\big[(\widehat{\beta}(\mu))_s\big]_{\mathbb{V}}=\mathbf{1}_{\{s<\tau_\mu\}}\big[\big(\beta(\mu)\big)_s\big]_{\mathbb{V}}+\mathbf{1}_{\{s\geq\tau_\mu\}}\big[\big(\beta_\psi(\mu)\big)_s\big]_{\mathbb{V}}\leq\kappa+(C_\beta\vee\kappa)[\mu_s]_{\mathbb{U}}. \tag{4.60}$$

To see $\widehat{\beta}\in\mathfrak{B}_t$, we let $\mu^1,\mu^2\in\mathcal{U}_t$ such that $\mu^1=\mu^2$, $ds\times dP$-a.s. on $[\![t,\tau[\![\,\cup\,[\![\tau,T]\!]_A$ for some $\tau\in\mathcal{S}_{t,T}$ and $A\in\mathcal{F}_\tau$. Since $\beta(\mu^1)=\beta(\mu^2)$, $ds\times dP$-a.s. on $[\![t,\tau[\![\,\cup\,[\![\tau,T]\!]_A$ by Definition 2.2, it holds $ds\times dP$-a.s. on $\big([\![t,\tau[\![\,\cup\,[\![t,T]\!]_A\big)\cap[\![t,\tau_{\mu^1}\wedge\tau_{\mu^2}[\![$ that

$$\big(\widehat{\beta}(\mu^1)\big)_s=\big(\beta(\mu^1)\big)_s=\big(\beta(\mu^2)\big)_s=\big(\widehat{\beta}(\mu^2)\big)_s. \tag{4.61}$$

And (2.8) shows that except on a P-null set $\mathcal{N}$,

$$\mathbf{1}_A X_s^{\Theta_{\mu^1}}+\mathbf{1}_{A^c}X_{\tau\wedge s}^{\Theta_{\mu^1}}=\mathbf{1}_A X_s^{\Theta_{\mu^2}}+\mathbf{1}_{A^c}X_{\tau\wedge s}^{\Theta_{\mu^2}}\quad\forall\,s\in[t,T]. \tag{4.62}$$

Then for any $\omega\in A\cap\mathcal{N}^c$, $\tau_{\mu^1}(\omega)=\inf\{s\in(t,T]:\big(s,X_s^{\Theta_{\mu^1}}(\omega)\big)\notin O_\delta(t,x)\}=\inf\{s\in(t,T]:\big(s,X_s^{\Theta_{\mu^2}}(\omega)\big)\notin O_\delta(t,x)\}=\tau_{\mu^2}(\omega)$. Let $A_o\stackrel{\triangle}{=}\{\tau\geq\tau_{\mu^1}\wedge\tau_{\mu^2}\}$. We can deduce from (4.62) that for any $\omega\in A_o\cap\{\tau_{\mu^1}\leq\tau_{\mu^2}\}\cap\mathcal{N}^c$,

$$\begin{aligned}\tau_{\mu^1}(\omega)&=\inf\Big\{s\in(t,T]:\big(s,X_s^{\Theta_{\mu^1}}(\omega)\big)\notin O_\delta(t,x)\Big\}\\&=\inf\Big\{s\in(t,\tau(\omega)]:\big(s,X_s^{\Theta_{\mu^1}}(\omega)\big)\notin O_\delta(t,x)\Big\}\\&=\inf\Big\{s\in(t,\tau(\omega)]:\big(s,X_s^{\Theta_{\mu^2}}(\omega)\big)\notin O_\delta(t,x)\Big\}\\&\geq\inf\Big\{s\in(t,T]:\big(s,X_s^{\Theta_{\mu^2}}(\omega)\big)\notin O_\delta(t,x)\Big\}=\tau_{\mu^2}(\omega)\geq\tau_{\mu^1}(\omega).\end{aligned}$$

Similarly, it holds on $A_o\cap\{\tau_{\mu^2}\leq\tau_{\mu^1}\}\cap\mathcal{N}^c$ that $\tau_{\mu^1}=\tau_{\mu^2}$. So

$$\tau_{\mu^1}=\tau_{\mu^2}\ \text{ on }\widetilde{A}\stackrel{\triangle}{=}(A\cup A_o)\cap\mathcal{N}^c. \tag{4.63}$$

As $[\![t,\tau[\![\,\cap\,[\![\tau_{\mu^1}\wedge\tau_{\mu^2},T]\!]=[\![\tau_{\mu^1}\wedge\tau_{\mu^2},\tau[\![_{A_o}$ and $[\![t,T]\!]_A\cap[\![\tau_{\mu^1}\wedge\tau_{\mu^2},T]\!]=[\![\tau_{\mu^1}\wedge\tau_{\mu^2},T]\!]_A$, (4.63) implies that $\big([\![t,\tau[\![\,\cup\,[\![t,T]\!]_A\big)\cap[\![\tau_{\mu^1}\wedge\tau_{\mu^2},T]\!]_{\mathcal{N}^c}\subset[\![\tau_{\mu^1}\wedge\tau_{\mu^2},T]\!]_{\widetilde{A}}=[\![\tau_{\mu^1},T]\!]_{\widetilde{A}}\cap[\![\tau_{\mu^2},T]\!]_{\widetilde{A}}$. Thus it holds $ds\times dP$-a.s. on $\big([\![t,\tau[\![\,\cup\,[\![t,T]\!]_A\big)\cap[\![\tau_{\mu^1}\wedge\tau_{\mu^2},T]\!]$ that $\big(\widehat{\beta}(\mu^1)\big)_s=\psi(s,\mu_s^1)=\psi(s,\mu_s^2)=\big(\widehat{\beta}(\mu^2)\big)_s$, which together with (4.61) shows that $\widehat{\beta}\in\mathfrak{B}_t$.

Given $\mu\in\mathcal{U}_t$, we set $\Theta_\mu\stackrel{\triangle}{=}\big(t,x,\mu,\beta(\mu)\big)$ and $\widehat{\Theta}_\mu\stackrel{\triangle}{=}\big(t,x,\mu,\widehat{\beta}(\mu)\big)$. For $i=1,\dots,N_m$, analogous to $\widetilde{A}_i^m$ of part (b), $\mathcal{A}_i^{\mu,m}\stackrel{\triangle}{=}\big\{\big(\tau_\mu,X_{\tau_\mu}^{\Theta_\mu}\big)\in\mathfrak{D}_m(s_i,x_i)\backslash\cup_{j<i}\mathfrak{D}_m(s_j,x_j)\big\}$ belongs to $\mathcal{F}_{\tau_\mu}\cap\mathcal{F}_{t_i}$. By the continuity of process X^{Θ_μ}, $(\tau_\mu,X_{\tau_\mu}^{\Theta_\mu})\in\partial O_\delta(t,x)$, P-a.s. So $\{\mathcal{A}_i^{\mu,m}\}_{i=1}^{N_m}$ forms a partition of $\mathcal{N}_\mu^c$ for some P-null set $\mathcal{N}_\mu$.

Then we can define an $\mathbf{F}$-stopping time $\tau^m_\mu \stackrel{\triangle}{=} \sum_{i=1}^{N_m} \mathbf{1}_{\mathcal{A}^{\mu,m}_i} t_i + \mathbf{1}_{\mathcal{N}_\mu} T \ge \tau_\mu$ as well as a process

$$\begin{aligned}\big(\beta_m(\mu)\big)_s &\stackrel{\triangle}{=} \mathbf{1}_{\{s<\tau^m_\mu\}}\big(\widehat{\beta}(\mu)\big)_s + \mathbf{1}_{\{s\ge\tau^m_\mu\}}\left(\sum_{i=1}^{N_m}\mathbf{1}_{\mathcal{A}^{\mu,m}_i\cap\mathcal{A}^m_i}\big(\beta^m_i([\mu]^{t_i})\big)_s + \mathbf{1}_{\mathcal{A}^m_\mu}\big(\widehat{\beta}(\mu)\big)_s\right)\\ &= \mathbf{1}_{\mathcal{A}^m_\mu}\big(\widehat{\beta}(\mu)\big)_s + \sum_{i=1}^{N_m}\mathbf{1}_{\mathcal{A}^{\mu,m}_i\cap\mathcal{A}^m_i}\Big(\mathbf{1}_{\{s<t_i\}}\big(\widehat{\beta}(\mu)\big)_s + \mathbf{1}_{\{s\ge t_i\}}\big(\beta^m_i([\mu]^{t_i})\big)_s\Big)\end{aligned}$$

$$\forall\, s\in[t,T], \tag{4.64}$$

where $\mathcal{A}^m_\mu = \big(\overset{N_m}{\cup}\big(\mathcal{A}^{\mu,m}_i\backslash\mathcal{A}^m_i\big)\big)_{i=1}\cup\mathcal{N}_\mu$. We claim that β_m is a $\mathfrak{B}_t$-strategy. Using a similar argument to that in part (b) for the measurability of the pasted control μ^m, one can deduce that the process $\beta_m(\mu)$ is $\mathbf{F}$-progressively measurable. For $i = 1,\dots,N_m$, let $C^m_i > 0$ be the constant associated to β^m_i in Definition 2.2 (i). Setting $C_m = C_\beta \vee \kappa \vee \max\{C^m_i : i = 1,\dots,N_m\}$, we can deduce from (4.60) and (A-u) that $ds \times dP$-a.s.

$$\begin{aligned}\big[(\beta_m(\mu))_s\big]_{\mathbb{V}} &= \mathbf{1}_{\{s<\tau^m_\mu\}}\big[\big(\widehat{\beta}(\mu)\big)_s\big]_{\mathbb{V}}\\ &\quad + \mathbf{1}_{\{s\ge\tau^m_\mu\}}\left(\sum_{i=1}^{N_m}\mathbf{1}_{\mathcal{A}^{\mu,m}_i\cap\mathcal{A}^m_i}\big[\big(\beta^m_i([\mu]^{t_i})\big)_s\big]_{\mathbb{V}} + \mathbf{1}_{\mathcal{A}^m_\mu}\big[\big(\widehat{\beta}(\mu)\big)_s\big]_{\mathbb{V}}\right)\\ &\le \big(\mathbf{1}_{\{s<\tau^m_\mu\}} + \mathbf{1}_{\{s\ge\tau^m_\mu\}}\mathbf{1}_{\mathcal{A}^m_\mu}\big)\big(\kappa + (C_\beta\vee\kappa)[\mu_s]_{\mathbb{U}}\big)\\ &\quad + \mathbf{1}_{\{s\ge\tau^m_\mu\}}\sum_{i=1}^{N_m}\mathbf{1}_{\mathcal{A}^{\mu,m}_i\cap\mathcal{A}^m_i}\big(\kappa + C^m_i\big[[\mu]^{t_i}_s\big]_{\mathbb{U}}\big) \le \kappa + C_m[\mu_s]_{\mathbb{U}}.\end{aligned} \tag{4.65}$$

Let $E\int_t^T\big[\mu_s\big]^q_{\mathbb{U}}ds<\infty$ for some $q>2$. It follows from (4.65) that $E\int_t^T\big[(\beta_m(\mu))_s\big]^q_{\mathbb{V}}ds\le 2^{q-1}\kappa^qT+2^{q-1}C^q_mE\int_t^T\big[\mu_s\big]^q_{\mathbb{U}}ds<\infty$. Hence $\beta_m(\mu)\in\mathcal{V}_t$.

Let $\mu^1,\mu^2\in\mathcal{U}_t$ such that $\mu^1=\mu^2$, $ds\times dP$-a.s. on $[\![t,\tau[\![\,\cup[\![\tau,T]\!]_A$ for some $\tau\in\mathcal{S}_{t,T}$ and $A\in\mathcal{F}_\tau$. As $\widehat{\beta}(\mu^1)=\widehat{\beta}(\mu^2)$, $ds\times dP$-a.s. on $[\![t,\tau[\![\,\cup[\![\tau,T]\!]_A$ by Definition 2.2, it holds $ds\times dP$-a.s. on $\big([\![t,\tau[\![\,\cup[\![t,T]\!]_A\big)\cap[\![t,\tau^m_{\mu^1}\wedge\tau^m_{\mu^2}[\![$ that

$$\big(\beta_m(\mu^1)\big)_s = \big(\widehat{\beta}(\mu^1)\big)_s = \big(\widehat{\beta}(\mu^2)\big)_s = \big(\beta_m(\mu^2)\big)_s. \tag{4.66}$$

Definition 2.2 also shows that $\big(\mu^1,\beta(\mu^1)\big)=\big(\mu^2,\beta(\mu^2)\big)$, $ds\times dP$-a.s. on $[\![t,\tau[\![\,\cup[\![\tau,T]\!]_A$. So we again have (4.62) except on a P-null set $\mathcal{N}$, and (4.63) still holds on $\widetilde{A}\stackrel{\triangle}{=}(A\cup A_o)\cap\mathcal{N}^c$ with $A_o=\{\tau\ge\tau_{\mu^1}\wedge\tau_{\mu^2}\}$. Plugging (4.63) into (4.62) yields that

$$X^{\Theta_{\mu^1}}_{\tau_{\mu^1}} = X^{\Theta_{\mu^2}}_{\tau_{\mu^2}} \text{ holds on } \widetilde{A}. \tag{4.67}$$

Given $i=1,\dots,N_m$, since it holds $ds\times dP$-a.s. on $\big([\![t,\tau[\![\,\cup[\![\tau,T]\!]_A\big)\cap\big([t_i,T]\times\Omega\big)=[\![t_i,\tau\vee t_i[\![\,\cup[\![\tau\vee t_i,T]\!]_A$ that $\big([\mu^1]^{t_i}\big)_s=\mu^1_s=\mu^2_s=\big([\mu^2]^{t_i}\big)_s$, taking $(\tau,A)=(\tau\vee t_i,A)$ in Definition 2.2 with respect to β^m_i yields that for $ds\times dP$-a.s. $(s,\omega)\in[\![t_i,\tau\vee t_i[\![\,\cup[\![\tau\vee t_i,T]\!]_A=\big([\![t,\tau[\![\,\cup[\![t,T]\!]_A\big)\cap\big([t_i,T]\times\Omega\big)$,

$$\big(\beta^m_i([\mu^1]^{t_i})\big)_s(\omega) = \big(\beta^m_i([\mu^2]^{t_i})\big)_s(\omega). \tag{4.68}$$

Given $\omega \in \mathcal{A}_i \stackrel{\triangle}{=} \widetilde{A} \cap \mathcal{A}_i^{\mu^1,m}$, (4.63) and (4.67) imply that

$$\Big(\tau_{\mu^2}(\omega), X^{\Theta_{\mu^2}}_{\tau_{\mu^2}(\omega)}(\omega)\Big) = \Big(\tau_{\mu^1}(\omega), X^{\Theta_{\mu^1}}_{\tau_{\mu^1}(\omega)}(\omega)\Big) \in \mathfrak{D}_m(s_i,x_i)\backslash \underset{j<i}{\cup} \mathfrak{D}_m(s_j,x_j), \quad \text{i.e., } \omega \in \mathcal{A}_i^{\mu^2,m}.$$

So $\mathcal{A}_i \subset \mathcal{A}_i^{\mu^1,m} \cap \mathcal{A}_i^{\mu^2,m}$, and it follows that $\mathbf{1}_{\mathcal{A}_i}\tau^m_{\mu^1} = \mathbf{1}_{\mathcal{A}_i} t_i = \mathbf{1}_{\mathcal{A}_i}\tau^m_{\mu^2}$. Then one can deduce that

$$\begin{aligned}&\big([\![t,\tau[\![\cup[\![t,T]\!]_A\big)\cap[\![\tau^m_{\mu^1}\wedge\tau^m_{\mu^2},T]\!]_{\mathcal{A}_i\cap\mathcal{A}_i^m}\\ &= \big([\![t,\tau[\![\cup[\![t,T]\!]_A\big)\cap\big([t_i,T]\times(\mathcal{A}_i\cap\mathcal{A}_i^m)\big)\subset[t_i,T]\times\big(\mathcal{A}_i^{\mu^1,m}\cap\mathcal{A}_i^{\mu^2,m}\cap\mathcal{A}_i^m\big),\end{aligned} \tag{4.69}$$

which together with (4.68) shows that for $ds\times dP$-a.s. $(s,\omega)\in\big([\![t,\tau[\![\cup[\![t,T]\!]_A\big)\cap [\![\tau^m_{\mu^1}\wedge\tau^m_{\mu^2},T]\!]_{\mathcal{A}_i\cap\mathcal{A}_i^m}$,

$$\big(\beta_m(\mu^1)\big)_s(\omega) = \big(\beta_i^m([\mu^1]^{t_i})\big)_s(\omega) = \big(\beta_i^m([\mu^2]^{t_i})\big)_s(\omega) = \big(\beta_m(\mu^2)\big)_s(\omega). \tag{4.70}$$

Analogous to (4.69), $\big([\![t,\tau[\![\cup[\![t,T]\!]_A\big)\cap[\![\tau^m_{\mu^1}\wedge\tau^m_{\mu^2},T]\!]_{\mathcal{A}_i\backslash\mathcal{A}_i^m}\subset[t_i,T]\times\big((\mathcal{A}_i^{\mu^1,m}\backslash\mathcal{A}_i^m)\cap(\mathcal{A}_i^{\mu^2,m}\backslash\mathcal{A}_i^m)\big)$. So (4.66) also holds $ds\times dP$-a.s. on $\big([\![t,\tau[\![\cup[\![t,T]\!]_A\big)\cap[\![\tau^m_{\mu^1}\wedge\tau^m_{\mu^2},T]\!]_{\mathcal{A}_i\backslash\mathcal{A}_i^m}$. Combining this with (4.70) and then letting i run over $\{1,\dots,N_m\}$ yield that

$$\big(\beta_m(\mu^1)\big)_s = \big(\beta_m(\mu^2)\big)_s, \quad ds\times dP\text{-a.s. on } \big([\![t,\tau[\![\cup[\![t,T]\!]_A\big) \cap[\![\tau^m_{\mu^1}\wedge\tau^m_{\mu^2},T]\!]_{A\cup A_o}. \tag{4.71}$$

As $[\![\tau^m_{\mu^1}\wedge\tau^m_{\mu^2},T]\!]_{A^c\cap A^c_o}\subset[\![\tau_{\mu^1}\wedge\tau_{\mu^2},T]\!]_{A^c\cap A^c_o}\subset[\![\tau,T]\!]_{A^c\cap A^c_o}\subset[\![\tau,T]\!]_{A^c}$, one can deduce that $\big([\![t,\tau[\![\cup[\![t,T]\!]_A\big)\cap[\![\tau^m_{\mu^1}\wedge\tau^m_{\mu^2},T]\!]_{A\cup A_o} = \big([\![t,\tau[\![\cup[\![t,T]\!]_A\big)\cap[\![\tau^m_{\mu^1}\wedge\tau^m_{\mu^2},T]\!]$. Therefore, (4.71) together with (4.66) implies that $\beta_m\in\mathfrak{B}_t$.

(e) *Next, let* $\mu\in\mathcal{U}_t$ *and* $\Theta^m_\mu \stackrel{\triangle}{=} \big(t,x,\mu,\beta_m(\mu)\big)$. *We shall do similar estimates to those in part* (c) *to conclude*

$$w_1(t,x)\le \underset{\beta\in\mathfrak{B}_t}{\operatorname{essinf}}\ \underset{\mu\in\mathcal{U}_t}{\operatorname{esssup}}\ Y_t^{t,x,\mu,\beta(\mu)}\Big(\tau_{\beta,\mu},\widetilde{\phi}\big(\tau_{\beta,\mu},X^{t,x,\mu,\beta(\mu)}_{\tau_{\beta,\mu}}\big)\Big), \quad P\text{-a.s.} \tag{4.72}$$

As $\beta_m(\mu)=\widehat{\beta}(\mu)=\beta(\mu)$ on $[\![t,\tau_\mu[\![$, taking $(\tau,A)=(\tau_\mu,\emptyset)$ in (2.8) shows

$$P\Big(X_s^{\Theta^m_\mu}=X_s^{\widehat{\Theta}_\mu}=X_s^{\Theta_\mu}\in\overline{O}_\delta(x)\ \ \forall\, s\in[t,\tau_\mu]\Big)=1. \tag{4.73}$$

Thus, for any $\eta\in\mathbb{L}^p(\mathcal{F}_{\tau_\mu})$, the BSDE$(t,\eta,f^{\Theta^m_\mu}_{\tau_\mu})$ and the BSDE$(t,\eta,f^{\Theta_\mu}_{\tau_\mu})$ are essentially the same. To wit,

$$\big(Y^{\Theta^m_\mu}(\tau_\mu,\eta),Z^{\Theta^m_\mu}(\tau_\mu,\eta)\big)=\big(Y^{\Theta_\mu}(\tau_\mu,\eta),Z^{\Theta_\mu}(\tau_\mu,\eta)\big). \tag{4.74}$$

Given $A\in\mathcal{F}_t$, similar to (4.39), we can deduce from (4.73) that

$$\begin{aligned}&\mathbf{1}_A\sup_{r\in[t,s]}\big|X^{\Theta^m_\mu}_{\tau^m_\mu\wedge r}-X^{\Theta_\mu}_{\tau_\mu\wedge r}\big|\\ &\le\int_{\tau_\mu\wedge s}^{\tau^m_\mu\wedge s}\mathbf{1}_A\big|b\big(r,X^{\Theta^m_\mu}_{\tau^m_\mu\wedge r},\mu_r,\psi(r,\mu_r)\big)\big|dr\\ &\quad+\sup_{r\in[t,s]}\left|\int_{\tau_\mu\wedge r}^{\tau^m_\mu\wedge r}\mathbf{1}_A\sigma\big(r',X^{\Theta^m_\mu}_{\tau^m_\mu\wedge r'},\mu_{r'},\psi(r',\mu_{r'})\big)dB_{r'}\right|,\quad s\in[t,T].\end{aligned}$$

where we used the fact that $\beta_m(\mu) = \widehat{\beta}(\mu) = \beta_\psi(\mu)$ on $[\![\tau_\mu, \tau_\mu^m[\![$. Let $\widetilde{C}(\kappa, x, \delta)$ denote a generic constant, depending on $\kappa + |x| + \delta$, $C^{\widetilde{\phi}}_{x,\delta} \stackrel{\triangle}{=} \sup\{|\widetilde{\phi}(s,\mathfrak{x})| : (s,\mathfrak{x}) \in \overline{O}_{\delta+3}(t,x) \cap ([t,T]\times\mathbb{R}^k)\}$, T, γ, p and $|g(0)|$, whose form may vary from line to line. Since $\tau_\mu^m - \tau_\mu \le \sum_{i=1}^{N_m} \mathbf{1}_{\mathcal{A}_i^{\mu,m}} 2\delta^m_{s_i,x_i} < \frac{2}{m}$, P-a.s., using similar arguments to those that lead to (4.40) and using an analogous decomposition and estimation to (4.8), we can deduce that

$$\begin{aligned}
&E\bigg[\mathbf{1}_A \sup_{r\in[t,s]} \big|X^{\Theta_\mu^m}_{\tau_\mu^m\wedge r} - X^{\Theta_\mu}_{\tau_\mu\wedge r}\big|^2\bigg]\\
&\quad\le 4E\int_{\tau_\mu\wedge s}^{\tau_\mu^m\wedge s} \mathbf{1}_A \Big|b\Big(r, X^{\Theta_\mu^m}_{\tau_\mu^m\wedge r}, \mu_r, \psi(r,\mu_r)\Big)\Big|^2 dr\\
&\qquad + 8E\int_{\tau_\mu\wedge s}^{\tau_\mu^m\wedge s} \mathbf{1}_A \Big|\sigma\Big(r, X^{\Theta_\mu^m}_{\tau_\mu^m\wedge r}, \mu_r, \psi(r,\mu_r)\Big)\Big|^2 dr\\
&\quad\le 24\gamma^2\int_t^s E\bigg[\mathbf{1}_A \sup_{r'\in[t,r]} \big|X^{\Theta_\mu^m}_{\tau_\mu^m\wedge r'} - X^{\Theta_\mu}_{\tau_\mu\wedge r'}\big|^2\bigg] dr + \frac{\widetilde{C}(\kappa,x,\delta)}{m} P(A) \quad \forall\, s\in[t,T].
\end{aligned}$$

Then, similar to (4.42), an application of Gronwall's inequality leads to

$$E\bigg[\sup_{r\in[t,T]} \big|X^{\Theta_\mu^m}_{\tau_\mu^m\wedge r} - X^{\Theta_\mu}_{\tau_\mu\wedge r}\big|^2\Big|\mathcal{F}_t\bigg] \le \frac{\widetilde{C}(\kappa,x,\delta)}{m}, \quad P\text{-a.s.} \tag{4.75}$$

Let $i = 1, \dots, N_m$ and set $\Theta_\mu^{m,t_i} \stackrel{\triangle}{=} (t_i, X^{\Theta_\mu^m}_{t_i}, [\mu]^{t_i}, [\beta_m(\mu)]^{t_i})$. Similar to (4.43), it holds P-a.s. that

$$Y^{\Theta_\mu^m}_{t_i}\Big(T, g\Big(X^{\Theta_\mu^m}_T\Big)\Big) = J\big(\Theta_\mu^{m,t_i}\big). \tag{4.76}$$

Since $\big[\beta_m(\mu)\big]^{t_i}_r(\omega) = \big(\beta_m(\mu)\big)_r(\omega) = \big(\beta_i^m\big([\mu]^{t_i}\big)\big)_r(\omega)$ for any $(r,\omega)\in[t_i,T]\times\big(\mathcal{A}_i^{\mu,m}\cap \mathcal{A}_i^m\big)$, one can deduce from (4.76), (4.24), and (2.14) that it holds P-a.s. on $\mathcal{A}_i^{\mu,m}\cap \mathcal{A}_i^m \in \mathcal{F}_{t_i}$ that

$$\begin{aligned}
Y^{\Theta_\mu^m}_{\tau_\mu^m}\Big(T, g\Big(X^{\Theta_\mu^m}_T\Big)\Big) &= Y^{\Theta_\mu^m}_{t_i}\Big(T, g\Big(X^{\Theta_\mu^m}_T\Big)\Big) = J\Big(t_i, X^{\Theta_\mu^m}_{\tau_\mu^m}, [\mu]^{t_i}, \beta_i^m\big([\mu]^{t_i}\big)\Big)\\
&\le J\Big(t_i, X^{\Theta_\mu}_{\tau_\mu}, [\mu]^{t_i}, \beta_i^m\big([\mu]^{t_i}\big)\Big) + c_0\Big|X^{\Theta_\mu^m}_{\tau_\mu^m} - X^{\Theta_\mu}_{\tau_\mu}\Big|^{2/p}.
\end{aligned}$$

As $\big|X^{\Theta_\mu}_{\tau_\mu} - x_i\big|^{2/p} < (\delta^m_{s_i,x_i})^{2/p} < m^{-2/p} \le 1/m$ on $\mathcal{A}_i^{\mu,m}$, we can also deduce from (2.14), (4.59), (4.32), and the continuity of $\widetilde{\phi}$ that it holds P-a.s. on $\mathcal{A}_i^{\mu,m}\cap\mathcal{A}_i^m$ that

$$\begin{aligned}
&J\Big(t_i, X^{\Theta_\mu}_{\tau_\mu}, [\mu]^{t_i}, \beta_i^m\big([\mu]^{t_i}\big)\Big)\\
&\quad\le J\Big(t_i, x_i, [\mu]^{t_i}, \beta_i^m\big([\mu]^{t_i}\big)\Big) + \frac{c_0}{m} \le I\big(t_i, x_i, \beta_i^m\big) + \frac{c_0}{m} \le \widetilde{\phi}\big(t_i, x_i\big) + \frac{c_0}{m}\\
&\quad\le \widetilde{\phi}\big(s_i, x_i\big) + \frac{c_0}{m} \le \widetilde{\phi}\big(\tau_\mu, X^{\Theta_\mu}_{\tau_\mu}\big) + \frac{c_0}{m} \stackrel{\triangle}{=} \eta_\mu^m \in \mathbb{L}^\infty(\mathcal{F}_{\tau_\mu}).
\end{aligned}$$

Thus it holds P-a.s. on $\bigcup_{i=1}^{N_m}\big(\mathcal{A}_i^{\mu,m}\cap\mathcal{A}_i^m\big)$ that $Y^{\Theta_\mu^m}_{\tau_\mu^m}(T, g(X^{\Theta_\mu^m}_T)) \le \eta_\mu^m + c_0|X^{\Theta_\mu^m}_{\tau_\mu^m} - X^{\Theta_\mu}_{\tau_\mu}|^{2/p} \stackrel{\triangle}{=} \widetilde{\eta}_\mu^m \in \mathbb{L}^p\big(\mathcal{F}_{\tau_\mu^m}\big)$. By (2.10)

$$\begin{aligned}
&\Big|Y^{\Theta_\mu}_t\big(\tau_\mu, \eta_\mu^m\big) - Y^{\Theta_\mu}_t\big(\tau_\mu, \widetilde{\phi}(\tau_\mu, X^{\Theta_\mu}_{\tau_\mu})\big)\Big|^p\\
&\quad\le c_0 E\Big[\big|\eta_\mu^m - \widetilde{\phi}\big(\tau_\mu, X^{\Theta_\mu}_{\tau_\mu}\big)\big|^p\Big|\mathcal{F}_t\Big] \le \frac{c_0}{m^p}, \quad P\text{-a.s.}
\end{aligned} \tag{4.77}$$

Similar to (4.50), one can deduce that P-a.s.

$$E\bigg[\Big|Y^{\Theta^m_\mu}_{\tau_\mu}(\tau^m_\mu,\eta^m_\mu)-\eta^m_\mu\Big|^p\Big|\mathcal{F}_t\bigg]\le c_0E\bigg[\int_{\tau_\mu}^T\big|f^{\Theta^m_\mu}_{\tau^m_\mu}\big(s,\widetilde{Y}^{m,\mu}_s,\widetilde{Z}^{m,\mu}_s\big)\big|^p ds\Big|\mathcal{F}_t\bigg]$$
$$=c_0E\bigg[\int_{\tau_\mu}^{\tau^m_\mu}\Big|f\Big(s,X^{\Theta^m_\mu}_{\tau^m_\mu\wedge s},\eta^m_\mu,0,\mu_s,\psi(s,\mu_s)\Big)\Big|^p ds\Big|\mathcal{F}_t\bigg].$$

Using an analogous decomposition and estimation to (4.8), similar to (4.51), we can deduce from (4.75) that

$$\Big|Y^{\Theta^m_\mu}_t\big(\tau_\mu,Y^{\Theta^m_\mu}_{\tau_\mu}(\tau^m_\mu,\eta^m_\mu)\big)-Y^{\Theta^m_\mu}_t\big(\tau_\mu,\eta^m_\mu\big)\Big|^p\le E\bigg[\Big|Y^{\Theta^m_\mu}_{\tau_\mu}(\tau^m_\mu,\eta^m_\mu)-\eta^m_\mu\Big|^p\Big|\mathcal{F}_t\bigg]$$
$$\le c_0E\bigg[\int_{\tau_\mu}^{\tau^m_\mu}\Big(\Big|X^{\Theta^m_\mu}_{\tau^m_\mu\wedge s}-X^{\Theta_\mu}_{\tau_\mu\wedge s}\Big|^2+\big|X^{\Theta_\mu}_{\tau_\mu\wedge s}\big|^2+|\eta^m_\mu|^p+c_\kappa\Big)ds\Big|\mathcal{F}_t\bigg]$$
$$\le\frac{\widetilde{C}(\kappa,x,\delta)}{m},\quad P\text{-a.s.}$$

Applying (2.11) with $(\zeta,\tau,\eta)=\big(\tau_\mu,\tau^m_\mu,\eta^m_\mu\big)$, applying (4.74) with $\eta=\eta^m_\mu$, and using (4.77) yields that P-a.s.

$$Y^{\Theta^m_\mu}_t\big(\tau^m_\mu,\eta^m_\mu\big)=Y^{\Theta^m_\mu}_t\Big(\tau_\mu,Y^{\Theta^m_\mu}_{\tau_\mu}\big(\tau^m_\mu,\eta^m_\mu\big)\Big)\le Y^{\Theta^m_\mu}_t\big(\tau_\mu,\eta^m_\mu\big)+\frac{\widetilde{C}(\kappa,x,\delta)}{m^{1/p}}$$
$$=Y^{\Theta_\mu}_t\big(\tau_\mu,\eta^m_\mu\big)+\frac{\widetilde{C}(\kappa,x,\delta)}{m^{1/p}}$$
$$\le Y^{\Theta_\mu}_t\big(\tau_\mu,\widetilde{\phi}\big(\tau_\mu,X^{\Theta_\mu}_{\tau_\mu}\big)\big)+\frac{\widetilde{C}(\kappa,x,\delta)}{m^{1/p}}$$
$$\le\operatorname*{esssup}_{\mu\in\mathcal{U}_t}Y^{t,x,\mu,\beta(\mu)}_t\Big(\tau_{\beta,\mu},\widetilde{\phi}\big(\tau_{\beta,\mu},X^{t,x,\mu,\beta(\mu)}_{\tau_{\beta,\mu}}\big)\Big)+\frac{\widetilde{C}(\kappa,x,\delta)}{m^{1/p}}.\tag{4.78}$$

As $\beta_m(\mu)=\widehat{\beta}(\mu)$, $ds\times dP$-a.s. on $[\![t,\tau^m_\mu[\![$, applying (2.8) with $(\tau,A)=(\tau^m_\mu,\emptyset)$ yields that P-a.s.

$$X^{\Theta^m_\mu}_s=X^{\widehat{\Theta}_\mu}_s\quad\forall\,s\in[t,\tau^m_\mu].\tag{4.79}$$

Given $i=1,\dots,N_m$, (4.79) shows that $X^{\Theta^m_\mu}_{t_i}=X^{\widehat{\Theta}_\mu}_{t_i}$, P-a.s. on $\mathcal{A}^{\mu,m}_i\backslash\mathcal{A}^m_i$. Since $\big[\beta_m(\mu)\big]^{t_i}_r(\omega)=\big(\beta_m(\mu)\big)_r(\omega)=\big(\widehat{\beta}(\mu)\big)_r(\omega)=\big[\widehat{\beta}(\mu)\big]^{t_i}_r(\omega)$ holds $ds\times dP$-a.s. on $[\![\tau^m_\mu,T]\!]_{\mathcal{A}^{\mu,m}_i\backslash\mathcal{A}^m_i}=[t_i,T]\times(\mathcal{A}^{\mu,m}_i\backslash\mathcal{A}^m_i)$. Then by (4.24) and a similar argument to (4.76), it holds P-a.s. on $\mathcal{A}^{\mu,m}_i\backslash\mathcal{A}^m_i$ that

$$Y^{\Theta^m_\mu}_{\tau^m_\mu}\Big(T,g\Big(X^{\Theta^m_\mu}_T\Big)\Big)=Y^{\Theta^m_\mu}_{t_i}\Big(T,g\Big(X^{\Theta^m_\mu}_T\Big)\Big)=J\big(\Theta^{m,t_i}_\mu\big)$$
$$=J\big(\widehat{\Theta}^{t_i}_\mu\big)=Y^{\widehat{\Theta}_\mu}_{t_i}\big(T,g\big(X^{\widehat{\Theta}_\mu}_T\big)\big)=Y^{\widehat{\Theta}_\mu}_{\tau^m_\mu}\big(T,g\big(X^{\widehat{\Theta}_\mu}_T\big)\big),\tag{4.80}$$

where $\widehat{\Theta}^{t_i}_\mu\stackrel{\triangle}{=}\big(t_i,X^{\widehat{\Theta}_\mu}_{t_i},[\mu]^{t_i},\big[\widehat{\beta}(\mu)\big]^{t_i}\big)$. Given $A\in\mathcal{F}_t$, one can deduce that

$$\mathbf{1}_AX^{\widehat{\Theta}_\mu}_{\tau_\mu\vee s}=\mathbf{1}_AX^{\widehat{\Theta}_\mu}_{\tau_\mu}+\mathbf{1}_A\int_{\tau_\mu}^{\tau_\mu\vee s}b\Big(r,X^{\widehat{\Theta}_\mu}_r,\mu_r,(\widehat{\beta}(\mu))_r\Big)dr$$
$$+\mathbf{1}_A\int_{\tau_\mu}^{\tau_\mu\vee s}\sigma\Big(r,X^{\widehat{\Theta}_\mu}_r,\mu_r,(\widehat{\beta}(\mu))_r\Big)dB_r$$

$$= \mathbf{1}_A X^{\widehat{\Theta}_\mu}_{\tau_\mu} + \int_t^s \mathbf{1}_{\{r\ge\tau_\mu\}}\mathbf{1}_A b\Big(r, X^{\widehat{\Theta}_\mu}_r, \mu_r, \psi(r,\mu_r)\Big)dr$$
$$+ \int_t^s \mathbf{1}_{\{r\ge\tau_\mu\}}\mathbf{1}_A \sigma\Big(r, X^{\widehat{\Theta}_\mu}_r, \mu_r, \psi(r,\mu_r)\Big)dB_r, \quad s\in[t,T].$$

It then follows from (4.73) that

$$\mathbf{1}_A \sup_{r\in[t,s]}\Big|X^{\widehat{\Theta}_\mu}_{\tau_\mu\vee r}\Big| \le \mathbf{1}_A(|x|+\delta) + \int_t^s \mathbf{1}_{\{r\ge\tau_\mu\}}\mathbf{1}_A\Big|b\Big(r, X^{\widehat{\Theta}_\mu}_r, \mu_r, \psi(r,\mu_r)\Big)\Big|dr$$
$$+ \sup_{r\in[t,s]}\bigg|\int_t^r \mathbf{1}_{\{r'\ge\tau_\mu\}}\mathbf{1}_A\sigma\Big(r', X^{\widehat{\Theta}_\mu}_{r'}, \mu_{r'}, \psi(r',\mu_{r'})\Big)dB_{r'}\bigg|, \quad s\in[t,T].$$

Using an analogous decomposition and estimation to (4.8), one can deduce from Hölder's inequality, Doob's martingale inequality, (2.1), (2.2), (4.73), and Fubini's theorem that

$$E\bigg[\mathbf{1}_A \sup_{r\in[t,s]}\Big|X^{\widehat{\Theta}_\mu}_{\tau_\mu\vee r}\Big|^2\bigg] \le \widetilde{C}(\kappa,x,\delta)P(A) + c_0 E\int_t^s \mathbf{1}_{\{r\ge\tau_\mu\}}\mathbf{1}_A\Big(\big|b\big(r, X^{\widehat{\Theta}_\mu}_r, \mu_r, \psi(r,\mu_r)\big)\big|^2$$
$$+ \big|\sigma\big(r, X^{\widehat{\Theta}_\mu}_r, \mu_r, \psi(r,\mu_r)\big)\big|^2\Big)dr$$
$$\le \widetilde{C}(\kappa,x,\delta)P(A) + c_0 E\int_t^s \mathbf{1}_{\{r\ge\tau_\mu\}}\mathbf{1}_A\Big|X^{\widehat{\Theta}_\mu}_{\tau_\mu\vee r}\Big|^2 dr$$
$$\le \widetilde{C}(\kappa,x,\delta)P(A) + c_0\int_t^s E\bigg[\mathbf{1}_A \sup_{r'\in[t,r]}\Big|X^{\widehat{\Theta}_\mu}_{\tau_\mu\vee r'}\Big|^2\bigg]dr \ \ \forall\, s\in[t,T].$$

Then, similar to (4.42), an application of Gronwall's inequality leads to

$$E\bigg[\sup_{r\in[\tau_\mu,T]}\Big|X^{\widehat{\Theta}_\mu}_r\Big|^2\bigg|\mathcal{F}_t\bigg] \le \widetilde{C}(\kappa,x,\delta), \quad P\text{-a.s.} \tag{4.81}$$

Let $(\widehat{Y}^\mu, \widehat{Z}^\mu) \in \mathbb{G}^p_{\mathbf{F}}([t,T])$ be the unique solution of the following BSDE with zero generator:

$$\widehat{Y}^\mu_s = Y^{\widehat{\Theta}_\mu}_{\tau_\mu}\Big(T, g\big(X^{\widehat{\Theta}_\mu}_T\big)\Big) - \int_s^T \widehat{Z}^\mu_r dB_r, \quad s\in[t,T].$$

Analogous to (4.49), $\big(\widehat{\mathcal{Y}}^\mu, \widehat{\mathcal{Z}}^\mu\big) \triangleq \big\{\big(\mathbf{1}_{\{s<\tau_\mu\}}\widehat{Y}^\mu_s + \mathbf{1}_{\{s\ge\tau_\mu\}}Y^{\widehat{\Theta}_\mu}_s\big(T, g\big(X^{\widehat{\Theta}_\mu}_T\big)\big), \mathbf{1}_{\{s<\tau_\mu\}}\widehat{Z}^\mu_s + \mathbf{1}_{\{s\ge\tau_\mu\}}Z^{\widehat{\Theta}_\mu}_s\big(T, g\big(X^{\widehat{\Theta}_\mu}_T\big)\big)\big)\big\}_{s\in[t,T]} \in \mathbb{G}^p_{\mathbf{F}}([t,T])$ solves the following BSDE:

$$\widehat{\mathcal{Y}}^\mu_s = g\Big(X^{\widehat{\Theta}_\mu}_T\Big) + \int_s^T \mathbf{1}_{\{r\ge\tau_\mu\}} f^{\widehat{\Theta}_\mu}_T\big(r, \widehat{\mathcal{Y}}^\mu_r, \widehat{\mathcal{Z}}^\mu_r\big)dr - \int_s^T \widehat{\mathcal{Z}}^\mu_r dB_r, \quad s\in[0,T].$$

Then (2.9), (1.4), and Hölder's inequality imply that P-a.s.

$$E\left[\sup_{s\in[\tau_\mu,T]}\Big|Y^{\widehat{\Theta}_\mu}_s\Big(T, g\Big(X^{\widehat{\Theta}_\mu}_T\Big)\Big)\Big|^p\Big|\mathcal{F}_t\right]$$
$$\le E\left[\sup_{s\in[t,T]}\big|\widehat{\mathcal{Y}}^\mu_s\big|^p\Big|\mathcal{F}_t\right] \le c_0 E\bigg[\Big|g\Big(X^{\widehat{\Theta}_\mu}_T\Big)\Big|^p + \int_{\tau_\mu}^T \Big|f^{\widehat{\Theta}_\mu}_T(s,0,0)\Big|^p ds\Big|\mathcal{F}_t\bigg]$$
$$= c_0 E\bigg[\Big|g\Big(X^{\widehat{\Theta}_\mu}_T\Big)\Big|^p + \int_{\tau_\mu}^T \Big|f\big(s, X^{\widehat{\Theta}_\mu}_s, 0, 0, \mu_s, \psi(s,\mu_s)\big)\Big|^p ds\Big|\mathcal{F}_t\bigg].$$

Using an analogous decomposition and estimation to (4.8), we can then deduce from (2.3), (2.4), and (4.81) that

$$E\left[\sup_{s\in[\tau_\mu,T]}\left|Y_s^{\widehat{\Theta}_\mu}\left(T,g\left(X_T^{\widehat{\Theta}_\mu}\right)\right)\right|^p\middle|\mathcal{F}_t\right]\le c_\kappa+c_0E\left[\sup_{s\in[\tau_\mu,T]}\left|X_s^{\widehat{\Theta}_\mu}\right|^2\middle|\mathcal{F}_t\right]$$
$$\le \widetilde{C}(\kappa,x,\delta),\quad P\text{-a.s.}$$

Let $\widehat{\eta}_\mu^m \stackrel{\triangle}{=} Y_{\tau_\mu^m}^{\Theta_\mu^m}(T,g(X_T^{\Theta_\mu^m}))\vee\widetilde{\eta}_\mu^m\in\mathbb{L}^p\big(\mathcal{F}_{\tau_\mu^m}\big)$ and set $\widetilde{\mathcal{A}}_\mu^m\stackrel{\triangle}{=}\{Y_{\tau_\mu^m}^{\Theta_\mu^m}(T,g\big(X_T^{\Theta_\mu^m}\big))>\widetilde{\eta}_\mu^m\}\in\mathcal{F}_{\tau_\mu^m}$. Clearly, $\mathbf{1}_{\widetilde{\mathcal{A}}_\mu^m}\le\mathbf{1}_{\mathcal{A}_\mu^m}$, P-a.s. Applying (2.10) with $\widetilde{p}=\frac{1+p}{2}$, we can deduce from Hölder's inequality, (4.75), and (4.80) that

(4.82)

$$\begin{aligned}
&\left|Y_t^{\Theta_\mu^m}(\tau_\mu^m,\widehat{\eta}_\mu^m)-Y_t^{\Theta_\mu^m}(\tau_\mu^m,\eta_\mu^m)\right|^{\widetilde{p}}\\
&\le c_0E\left[\left|\widehat{\eta}_\mu^m-\eta_\mu^m\right|^{\widetilde{p}}\middle|\mathcal{F}_t\right]=c_0E\left[\mathbf{1}_{(\widetilde{\mathcal{A}}_\mu^m)^c}\left|\widetilde{\eta}_\mu^m-\eta_\mu^m\right|^{\widetilde{p}}+\mathbf{1}_{\widetilde{\mathcal{A}}_\mu^m}\left|Y_{\tau_\mu^m}^{\Theta_\mu^m}\left(T,g\left(X_T^{\Theta_\mu^m}\right)\right)-\eta_\mu^m\right|^{\widetilde{p}}\middle|\mathcal{F}_t\right]\\
&\le c_0E\left[\left|X_{\tau_\mu^m}^{\Theta_\mu^m}-X_{\tau_\mu}^{\Theta_\mu}\right|^{\frac{2\widetilde{p}}{p}}\right]+c_0\left\{E\left[\mathbf{1}_{\widetilde{\mathcal{A}}_\mu^m}\middle|\mathcal{F}_t\right]\right\}^{\frac{p-\widetilde{p}}{p}}\left\{E\left[\left|Y_{\tau_\mu^m}^{\widehat{\Theta}_\mu}\left(T,g(X_T^{\widehat{\Theta}_\mu})\right)-\eta_\mu^m\right|^p\middle|\mathcal{F}_t\right]\right\}^{\frac{\widetilde{p}}{p}}\\
&\le c_0\left\{E\left[\left|X_{\tau_\mu^m}^{\Theta_\mu^m}-X_{\tau_\mu}^{\Theta_\mu}\right|^2\right]\right\}^{\frac{\widetilde{p}}{p}}\\
&\quad+c_0\left\{E\left[\mathbf{1}_{\mathcal{A}_\mu^m}\middle|\mathcal{F}_t\right]\right\}^{\frac{p-\widetilde{p}}{p}}\left\{E\left[\sup_{s\in[\tau_\mu,T]}\left|Y_s^{\widehat{\Theta}_\mu}\left(T,g(X_T^{\widehat{\Theta}_\mu})\right)\right|^p+\left(C_{x,\delta}^{\widetilde{\phi}}+\frac{c_0}{m}\right)^p\middle|\mathcal{F}_t\right]\right\}^{\frac{\widetilde{p}}{p}}\\
&\le\frac{\widetilde{C}(\kappa,x,\delta)}{m^{\widetilde{p}/p}}+\widetilde{C}(\kappa,x,\delta)\left\{E\left[\mathbf{1}_{\cup_{i=1}^{N_m}(\mathcal{A}_i^m)^c}\middle|\mathcal{F}_t\right]\right\}^{\frac{p-\widetilde{p}}{p}},\quad P\text{-a.s.}
\end{aligned}$$

Applying (2.11) with $(\zeta,\tau,\eta)=(\tau_\mu^m,T,g(X_T^{\Theta_\mu^m}))$, we see from Proposition 1.2 (2), (4.82), and (4.78) that P-a.s.

$$\begin{aligned}
Y_t^{\Theta_\mu^m}\left(T,g\left(X_T^{\Theta_\mu^m}\right)\right)&=Y_t^{\Theta_\mu^m}\left(\tau_\mu^m,Y_{\tau_\mu^m}^{\Theta_\mu^m}\left(T,g\left(X_T^{\Theta_\mu^m}\right)\right)\right)\le Y_t^{\Theta_\mu^m}(\tau_\mu^m,\widehat{\eta}_\mu^m)\\
&\le Y_t^{\Theta_\mu^m}(\tau_\mu^m,\eta_\mu^m)+\frac{\widetilde{C}(\kappa,x,\delta)}{m^{1/p}}+\widetilde{C}(\kappa,x,\delta)\left\{E\left[\mathbf{1}_{\cup_{i=1}^{N_m}(\mathcal{A}_i^m)^c}\middle|\mathcal{F}_t\right]\right\}^{\frac{p-\widetilde{p}}{p\widetilde{p}}}\\
&\le \operatorname*{esssup}_{\mu\in\mathcal{U}_t} Y_t^{t,x,\mu,\beta(\mu)}\left(\tau_{\beta,\mu},\widetilde{\phi}\big(\tau_{\beta,\mu},X_{\tau_{\beta,\mu}}^{t,x,\mu,\beta(\mu)}\big)\right)\\
&\quad+\frac{\widetilde{C}(\kappa,x,\delta)}{m^{1/p}}+\widetilde{C}(\kappa,x,\delta)\left\{E\left[\mathbf{1}_{\cup_{i=1}^{N_m}(\mathcal{A}_i^m)^c}\middle|\mathcal{F}_t\right]\right\}^{\frac{p-\widetilde{p}}{p\widetilde{p}}}.
\end{aligned}\tag{4.83}$$

For $\widehat{\mathcal{A}}_m\stackrel{\triangle}{=}\{E[\mathbf{1}_{\cup_{i=1}^{N_m}(\mathcal{A}_i^m)^c}|\mathcal{F}_t]>m^{\frac{1+p}{1-p}}\}$, we have $P(\widehat{\mathcal{A}}_m)\le m^{\frac{1+p}{p-1}}P\big(\cup_{i=1}^{N_m}(\mathcal{A}_i^m)^c\big)\le m^{\frac{1+p}{p-1}}\sum_{i=1}^{N_m}P\big((\mathcal{A}_i^m)^c\big)\le m^{-p}$. Multiplying $\mathbf{1}_{\widehat{\mathcal{A}}_m^c}$ to both sides of (4.83) yields

$$\begin{aligned}
&\mathbf{1}_{\widehat{\mathcal{A}}_m^c}J\big(t,x,\mu,\beta_m(\mu)\big)\\
&\quad\le\mathbf{1}_{\widehat{\mathcal{A}}_m^c}\operatorname*{esssup}_{\mu\in\mathcal{U}_t} Y_t^{t,x,\mu,\beta(\mu)}\left(\tau_{\beta,\mu},\widetilde{\phi}\big(\tau_{\beta,\mu},X_{\tau_{\beta,\mu}}^{t,x,\mu,\beta(\mu)}\big)\right)+\frac{\widetilde{C}(\kappa,x,\delta)}{m^{1/p}},\quad P\text{-a.s.}
\end{aligned}$$

Since $\widehat{\mathcal{A}}_m$ does not depend on μ nor on β, taking essential supremum over $\mu \in \mathcal{U}_t$ yields

$$\begin{aligned}\mathbf{1}_{\widehat{\mathcal{A}}_m^c} w_1(t,x) &\le \mathbf{1}_{\widehat{\mathcal{A}}_m^c} I\big(t,x,\beta_m\big)\\ &\le \mathbf{1}_{\widehat{\mathcal{A}}_m^c} \underset{\mu\in\mathcal{U}_t}{\text{esssup}}\; Y_t^{t,x,\mu,\beta(\mu)}\Big(\tau_{\beta,\mu}, \widetilde{\phi}\big(\tau_{\beta,\mu}, X^{t,x,\mu,\beta(\mu)}_{\tau_{\beta,\mu}}\big)\Big) + \frac{\widetilde{C}(\kappa,x,\delta)}{m^{1/p}}, \quad P\text{-a.s.}\end{aligned}$$

Then taking essential infimum over $\beta \in \mathfrak{B}_t$, we obtain

$$\begin{aligned}\mathbf{1}_{\widehat{\mathcal{A}}_m^c} w_1(t,x) \le \mathbf{1}_{\widehat{\mathcal{A}}_m^c} \underset{\beta\in\mathfrak{B}_t}{\text{essinf}}\; \underset{\mu\in\mathcal{U}_t}{\text{esssup}}\; Y_t^{t,x,\mu,\beta(\mu)}\Big(\tau_{\beta,\mu}, \widetilde{\phi}\big(\tau_{\beta,\mu}, X^{t,x,\mu,\beta(\mu)}_{\tau_{\beta,\mu}}\big)\Big)\\ + \frac{\widetilde{C}(\kappa,x,\delta)}{m^{1/p}}, \quad P\text{-a.s.}\end{aligned} \tag{4.84}$$

As $\sum_{m\in\mathbb{N}} P\big(\widehat{\mathcal{A}}_m\big) \le \sum_{m\in\mathbb{N}} m^{-p} < \infty$, similar to (4.58), the Borel–Cantelli theorem implies that $\lim_{m\to\infty} \mathbf{1}_{\widehat{\mathcal{A}}_m} = 0$, P-a.s. Thus, letting $m \to \infty$ in (4.84) yields (4.72). ∎

4.3. Proofs of section 3. We will prove that $\underline{w}_i$ and $\overline{w}_i$, $i = 1, 2$, are viscosity solutions of (3.1) using contraposition: Assume the contrary that the corresponding inequality of (3.1) does not hold for some test function φ. We decompose $\underline{H}_i$ or $\overline{H}_i$ with φ in the reverse inequality until we reach a similar reverse inequality satisfied by a control $\widehat{\mu}$ or a strategy $\widehat{\beta}$. Then applying the comparison result of BSDEs, i.e., Proposition 1.2 (2), to such an inequality leads to a contradiction to the weak dynamic programming principle.

Proof of Theorem 3.1. We prove only for $\underline{w}_1$ and $\overline{w}_1$ as the results of $\overline{w}_2$ and $\underline{w}_2$ can be argued similarly.

(a) We first show that $\underline{w}_1$ is a viscosity supersolution of (3.1) with Hamiltonian $\underline{H}_1$. Let $(t_0, x_0, \varphi) \in (0,T) \times \mathbb{R}^k \times \mathbb{C}^{1,2}\big([0,T] \times \mathbb{R}^k\big)$ be such that $\underline{w}_1 - \varphi$ attains a strict local minimum 0 at (t_0, x_0), i.e., for some $\delta_0 \in \big(0, t_0 \wedge (T - t_0)\big)$,

$$(\underline{w}_1 - \varphi)(t,x) > (\underline{w}_1 - \varphi)(t_0,x_0) = 0, \quad \forall\, (t,x) \in O_{\delta_0}(t_0,x_0)\big\backslash\big\{(t_0,x_0)\big\}. \tag{4.85}$$

We denote $\big(\varphi(t_0,x_0), D_x\varphi(t_0,x_0), D_x^2\varphi(t_0,x_0)\big)$ by (y_0, z_0, Γ_0). If $\underline{H}_1\big(t_0,x_0,y_0, z_0,\Gamma_0\big) = -\infty$, then $-\frac{\partial}{\partial t}\varphi(t_0,x_0) - \underline{H}_1\big(t_0,x_0,y_0,z_0,\Gamma_0\big) \ge 0$ clearly holds. To make a contradiction, we assume that when $\underline{H}_1\big(t_0,x_0,y_0,z_0,\Gamma_0\big) > -\infty$,

$$\varrho \overset{\triangle}{=} \frac{\partial}{\partial t}\varphi(t_0,x_0) + \underline{H}_1\big(t_0,x_0,y_0,z_0,\Gamma_0\big) > 0. \tag{4.86}$$

For any $(t,x,y,z,\Gamma,u,v) \in [0,T] \times \mathbb{R}^k \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}_k \times \mathbb{U} \times \mathbb{V}$, one can deduce from (2.1)–(2.4) that

$$\begin{aligned}\big|H(t,x,y,z,\Gamma,u,v)\big| &\le \frac{1}{4}|\sigma\sigma^T(t,x,u,v)|^2 + \frac{1}{4}|\Gamma|^2 + \gamma|z||b(t,x,u,v)|\\ &\quad + \gamma\Big(1 + |x|^{2/p} + |y| + \gamma|z||\sigma(t,x,u,v)| + [u]_{\mathbb{U}}^{2/p} + [v]_{\mathbb{V}}^{2/p}\Big)\\ &\le \frac{1}{4}\gamma^2\big(1 + |x| + [u]_{\mathbb{U}} + [v]_{\mathbb{V}}\big)^2 + \frac{1}{4}|\Gamma|^2\\ &\quad + (\gamma+\gamma^2)|z|\big(1 + |x| + [u]_{\mathbb{U}} + [v]_{\mathbb{V}}\big)\\ &\quad + \gamma\Big(1 + |x|^{2/p} + |y| + [u]_{\mathbb{U}}^{2/p} + [v]_{\mathbb{V}}^{2/p}\Big).\end{aligned} \tag{4.87}$$

Set $C^0_\varphi \stackrel{\triangle}{=} |y_0|+|z_0|+|\Gamma_0| = \big|\varphi(t_0,x_0)\big| + \big|D_x\varphi(t_0,x_0)\big| + \big|D^2_x\varphi(t_0,x_0)\big|$, and fix a $u_\sharp \in \partial O_\kappa(u_0)$. For any $u \notin O_\kappa(u_0)$, we see from (A-u) that $\psi(t_0,u)\in\mathscr{O}_u$, and it follows from (4.87) that

$$\inf_{v\in\mathscr{O}_u} H(t_0,x_0,y_0,z_0,\Gamma_0,u,v) \le \big|H(t_0,x_0,y_0,z_0,\Gamma_0,u,\psi(t_0,u))\big|$$
$$= \big|H(t_0,x_0,y_0,z_0,\Gamma_0,u_\sharp,\psi(t_0,u_\sharp))\big| \le \frac{1}{4}(C^0_\varphi)^2 + C^0_\varphi C(\kappa,x_0) + C(\kappa,x_0). \tag{4.88}$$

Here $C(\kappa,x_0)$ is a generic constant, depending on $\kappa, |x_0|, T, \gamma, p$, and $|g(0)|$, whose form may vary from line to line. Similarly, for $u\in O_\kappa(u_0)$, $\inf_{v\in\mathscr{O}_u} H(t_0,x_0,y_0,z_0,\Gamma_0,u,v) \le \big|H(t_0,x_0,y_0,z_0,\Gamma_0,u,v_0)\big| \le \frac{1}{4}(C^0_\varphi)^2 + C^0_\varphi C(\kappa,x_0)+C(\kappa,x_0)$, which together with (4.88) implies that $\underline{H}_1\big(t_0,x_0,y_0,z_0,\Gamma_0\big) \le \sup_{u\in\mathbb{U}} \inf_{v\in\mathscr{O}_u} H(t_0,x_0,y_0,z_0,\Gamma_0,u,v) < \infty$. Thus $\varrho<\infty$. As $\varphi\in\mathbb{C}^{1,2}\big([0,T]\times\mathbb{R}^k\big)$, (4.86) shows that for some $\widehat{u}\in\mathbb{U}$, $\underline{\lim}_{(t,x)\to(t_0,x_0)} \inf_{v\in\mathscr{O}_{\widehat{u}}} H(t,x,\varphi(t,x),D_x\varphi(t,x),D^2_x\varphi(t,x),\widehat{u},v) \ge \frac{3}{4}\varrho - \frac{\partial}{\partial t}\varphi(t_0,x_0)$. Moreover, there exists a $\delta\in(0,\delta_0)$ such that

$$\inf_{v\in\mathscr{O}_{\widehat{u}}} H(t,x,\varphi(t,x),D_x\varphi(t,x),D^2_x\varphi(t,x),\widehat{u},v) \ge \frac{1}{2}\varrho - \frac{\partial}{\partial t}\varphi(t,x) \quad \forall\,(t,x)\in\overline{O}_\delta(t_0,x_0). \tag{4.89}$$

Let $\wp \stackrel{\triangle}{=} \inf\big\{(\underline{w}_1-\varphi)(t,x) : (t,x)\in\overline{O}_\delta(t_0,x_0)\big\backslash O_{\frac{\delta}{3}}(t_0,x_0)\big\}$. Since $\overline{O}_\delta(t_0,x_0)\big\backslash O_{\frac{\delta}{3}}(t_0,x_0)$ is compact, there exists a sequence $\{(t_n,x_n)\}_{n\in\mathbb{N}}$ on $\overline{O}_\delta(t_0,x_0)\big\backslash O_{\frac{\delta}{3}}(t_0,x_0)$ that converges to some $(t_*,x_*)\in\overline{O}_\delta(t_0,x_0)\big\backslash O_{\frac{\delta}{3}}(t_0,x_0)$ and satisfies $\wp=\lim_{n\to\infty}(\underline{w}_1-\varphi)(t_n,x_n)$. The lower semicontinuity of $\underline{w}_1$ and the continuity of φ imply that $\underline{w}_1-\varphi$ is also lower semicontinuous. It follows that $\wp\le(\underline{w}_1-\varphi)(t_*,x_*)\le\lim_{n\to\infty}(\underline{w}_1-\varphi)(t_n,x_n)=\wp$, which together with (4.85) shows

$$\wp=\min\big\{(\underline{w}_1-\varphi)(t,x):(t,x)\in\overline{O}_\delta(t_0,x_0)\big\backslash O_{\frac{\delta}{3}}(t_0,x_0)\big\} = (\underline{w}_1-\varphi)(t_*,x_*)>0. \tag{4.90}$$

Then we set $\widetilde{\wp}\stackrel{\triangle}{=}\frac{\wp\wedge\varrho}{2(1\vee\gamma)T}>0$ and let $\big\{(t_j,x_j)\big\}_{j\in\mathbb{N}}$ be a sequence of $O_{\frac{\delta}{6}}(t_0,x_0)$ such that $\lim_{j\to\infty}(t_j,x_j)=(t_0,x_0)$ and that $\lim_{j\to\infty} w_1(t_j,x_j)=\underline{w}_1(t_0,x_0)=\varphi(t_0,x_0)=\lim_{j\to\infty}\varphi(t_j,x_j)$. So there exists a $j\in\mathbb{N}$ such that

$$\big|w_1(t_j,x_j)-\varphi(t_j,x_j)\big| < \frac{5}{6}\widetilde{\wp}t_0. \tag{4.91}$$

Clearly, $\widehat{\mu}_s \stackrel{\triangle}{=} \widehat{u}$, $s\in[t_j,T]$ is a constant $\mathcal{U}_{t_j}$-process. Fix $\beta\in\mathfrak{B}_{t_j}$. We set $\Theta\stackrel{\triangle}{=}\big(t_j,x_j,\widehat{\mu},\beta(\widehat{\mu})\big)$ and define

$$\tau=\tau_{\beta,\widehat{\mu}}\stackrel{\triangle}{=}\inf\Big\{s\in(t_j,T]:\big(s,X^\Theta_s\big)\notin O_{\frac{2}{3}\delta}(t_j,x_j)\Big\}\in\mathcal{S}_{t_j,T}.$$

Since $\big|\big(T,X^\Theta_T\big)-(t_j,x_j)\big|\ge T-t_j\ge T-t_0-|t_j-t_0|>\delta_0-\frac{\delta}{6}>\frac{5}{6}\delta>\frac{2}{3}\delta$, the continuity of X^Θ implies that P-a.s.

$$\tau<T \ \text{ and } \ \big(\tau\wedge s,X^\Theta_{\tau\wedge s}\big)\in\overline{O}_{\frac{2}{3}\delta}(t_j,x_j)\subset\overline{O}_{\frac{5}{6}\delta}(t_0,x_0) \quad \forall\,s\in[t_j,T]; \tag{4.92}$$

in particular,

$$\big(\tau,X^\Theta_\tau\big)\in\partial O_{\frac{2}{3}\delta}(t_j,x_j)\subset\overline{O}_{\frac{5}{6}\delta}(t_0,x_0)\big\backslash O_{\frac{\delta}{2}}(t_0,x_0). \tag{4.93}$$

The continuity of φ, X^Θ, and (4.92) show that $\mathcal{Y}_s \stackrel{\triangle}{=} \varphi\big(\tau\wedge s, X^\Theta_{\tau\wedge s}\big) + \widetilde{\wp}(\tau\wedge s)$, $s\in[t_j,T]$ defines a bounded $\mathbf{F}$-adapted continuous process. By Itô's formula,

$$\mathcal{Y}_s = \mathcal{Y}_T + \int_s^T \mathfrak{f}_r dr - \int_s^T \mathcal{Z}_r dB_r, \quad s\in[t_j,T],$$

where $\mathcal{Z}_r = \mathbf{1}_{\{r<\tau\}} D_x\varphi\big(r,X^\Theta_r\big)\cdot\sigma\big(r,X^\Theta_r,\widehat{u},\big(\beta(\widehat{\mu})\big)_r\big)$ and

$$\mathfrak{f}_r = -\mathbf{1}_{\{r<\tau\}}\bigg\{\widetilde{\wp} + \frac{\partial\varphi}{\partial t}\big(r,X^\Theta_r\big) + D_x\varphi\big(r,X^\Theta_r\big)\cdot b\big(r,X^\Theta_r,\widehat{u},(\beta(\widehat{\mu}))_r\big) + \frac{1}{2}\mathrm{trace}\Big(\sigma\sigma^T\big(r,X^\Theta_r,\widehat{u},(\beta(\widehat{\mu}))_r\big)\cdot D_x^2\varphi\big(r,X^\Theta_r\big)\Big)\bigg\}.$$

As $\varphi\in\mathbb{C}^{1,2}\big([t,T]\times\mathbb{R}^k\big)$, the measurability of b, σ, X^Θ, $\widehat{u}$, and $\beta(\widehat{\mu})$ implies that both $\mathcal{Z}$ and $\mathfrak{f}$ are $\mathbf{F}$-progressively measurable. And one can deduce from (2.1), (2.2), (4.92), and Hölder's inequality that

$$\begin{aligned} E\Bigg[\bigg(\int_{t_j}^T |\mathcal{Z}_s|^2\,ds\bigg)^{p/2}\Bigg] &\le (\gamma\widetilde{C}_\varphi)^p\, E\Bigg[\bigg(\int_{t_j}^\tau \Big(1+\big|X^\Theta_s\big| + [\widehat{u}]_{\mathbb{U}} + \big[(\beta(\widehat{\mu}))_s\big]_{\mathbb{V}}\Big)^2 ds\bigg)^{p/2}\Bigg] \\ &\le c_0\widetilde{C}^p_\varphi\bigg(\big(1+|x_0|+\delta+[\widehat{u}]_{\mathbb{U}}\big)^p + \bigg\{E\int_{t_j}^T \big[(\beta(\widehat{\mu}))_s\big]^2_{\mathbb{V}}\,ds\bigg\}^{p/2}\bigg) \\ &< \infty, \quad \text{i.e., } \mathcal{Z}\in\mathbb{H}^{2,p}_{\mathbf{F}}\big([t_j,T],\mathbb{R}^d\big), \end{aligned} \tag{4.94}$$

where $\widetilde{C}_\varphi \stackrel{\triangle}{=} \sup_{(t,x)\in\overline{O}_{\frac{5}{6}\delta}(t_0,x_0)}\big|D_x\varphi(t,x)\big| < \infty$. Hence, $\big\{\big(\mathcal{Y}_s,\mathcal{Z}_s\big)\big\}_{s\in[t_j,T]}$ solves the BSDE$\big(t_j,\mathcal{Y}_T,\mathfrak{f}\big)$.

Let $\ell(x) = c_\kappa + c_0|x|^{2/p}$, $x\in\mathbb{R}^k$, be the function appeared in Proposition 2.1. Let $\theta_1: [0,T]\times\mathbb{R}^k\to[0,1]$ be a continuous function such that $\theta_1\equiv 0$ on $\overline{O}_{\frac{5}{6}\delta}(t_0,x_0)$ and $\theta_1\equiv 1$ on $([0,T]\times\mathbb{R}^k)\backslash O_\delta(t_0,x_0)$. Also, let $\theta_2: [0,T]\times\mathbb{R}^k\to[0,1]$ be another continuous function such that $\theta_2\equiv 0$ on $\overline{O}_{\frac{\delta}{3}}(t_0,x_0)$ and $\theta_2\equiv 1$ on $([0,T]\times\mathbb{R}^k)\backslash O_{\frac{\delta}{2}}(t_0,x_0)$. Define

$$\phi(t,x) \stackrel{\triangle}{=} -\theta_1(t,x)\ell(x) + \big(1-\theta_1(t,x)\big)\big(\varphi(t,x) + \wp\theta_2(t,x)\big) \quad \forall\,(t,x)\in[t_j,T]\times\mathbb{R}^k, \tag{4.95}$$

which is a continuous function satisfying $\phi\le w_1$: given $(t,x)\in[t_j,T]\times\mathbb{R}^k$,

- if $(t,x)\in\overline{O}_{\frac{\delta}{3}}(t_0,x_0)$, (4.85) shows that $\phi(t,x)=\varphi(t,x)\le\underline{w}_1(t,x)\le w_1(t,x)$;
- if $(t,x)\in O_\delta(t_0,x_0)\backslash\overline{O}_{\frac{\delta}{3}}(t_0,x_0)$, since $\varphi(t,x)+\wp\theta_2(t,x)\le\varphi(t,x)+\wp\le\underline{w}_1(t,x)\le w_1(t,x)$ by (4.90), one can deduce from Proposition 2.1 that $\phi(t,x)\le w_1(t,x)$;
- if $(t,x)\notin O_\delta(t_0,x_0)$, $\phi(t,x) = -\ell(x)\le w_1(t,x)$.

Then we can deduce from (4.93) that

$$\mathcal{Y}_T = \varphi\big(\tau,X^\Theta_\tau\big) + \widetilde{\wp}T < \varphi\big(\tau,X^\Theta_\tau\big) + \wp = \phi\big(\tau,X^\Theta_\tau\big), \quad P\text{-a.s.} \tag{4.96}$$

Since it holds $ds\times dP$-a.s. on $[t_j,T]\times\Omega$ that $\big[\big(\beta(\widehat{\mu})\big)_s\big]_{\mathbb{V}}\le\kappa+C_\beta[\widehat{\mu}_s]_{\mathbb{U}} = \kappa+C_\beta[\widehat{u}]_{\mathbb{U}}\in\mathscr{O}_{\widehat{u}}$, (4.92), (4.89), and (2.4) imply that for $ds\times dP$-a.s. $(s,\omega)\in[t_j,T]\times\Omega$,

(4.97)

$$\begin{aligned}\mathfrak{f}_s(\omega) &\le \mathbf{1}_{\{s<\tau(\omega)\}}\Big\{-\widetilde{\wp}-\frac{1}{2}\varrho+f\left(s,\omega,X_s^\Theta(\omega),\mathcal{Y}_s(\omega)-\widetilde{\wp}s,\mathcal{Z}_s(\omega),\widehat{u},(\beta(\widehat{\mu}))_s(\omega)\right)\Big\}\\ &\le \mathbf{1}_{\{s<\tau(\omega)\}}\Big\{-\widetilde{\wp}-\frac{1}{2}\varrho+\gamma\widetilde{\wp}T+f\left(s,\omega,X_s^\Theta(\omega),\mathcal{Y}_s(\omega),\mathcal{Z}_s(\omega),\widehat{u},(\beta(\widehat{\mu}))_s(\omega)\right)\Big\}\\ &\le f_\tau^\Theta\big(s,\omega,\mathcal{Y}_s(\omega),\mathcal{Z}_s(\omega)\big).\end{aligned}$$

As f_τ^Θ is Lipschitz continuous in (y,z), Proposition 1.2 (2) implies that $P(\mathcal{Y}_s \le Y_s^\Theta(\tau,\phi(\tau,X_\tau^\Theta))\ \forall\, s\in[t_j,T])=1$. Letting $s=t_j$ and using the fact that $t_j > t_0-\frac{1}{6}\delta > t_0-\frac{1}{6}\delta_0>\frac{5}{6}t_0$, we obtain P-a.s.

$$\begin{aligned}\varphi(t_j,x_j)+\frac{5}{6}\widetilde{\wp}t_0<\mathcal{Y}_{t_j}&\le Y_{t_j}^{t_j,x_j,\widehat{\mu},\beta(\widehat{\mu})}\Big(\tau,\phi\Big(\tau,X_\tau^{t_j,x_j,\widehat{\mu},\beta(\widehat{\mu})}\Big)\Big)\\ &\le \underset{\mu\in\mathcal{U}_{t_j}}{\operatorname{esssup}} Y_{t_j}^{t_j,x_j,\mu,\beta(\mu)}\Big(\tau_{\beta,\mu},\phi\big(\tau_{\beta,\mu},X_{\tau_{\beta,\mu}}^{t_j,x_j,\mu,\beta(\mu)}\big)\Big),\end{aligned}$$

where $\tau_{\beta,\mu}\overset{\triangle}{=}\inf\{s\in(t_j,T]:\big(s,X_s^{t_j,x_j,\mu,\beta(\mu)}\big)\notin O_{\frac{2}{3}\delta}(t_j,x_j)\}\ \forall\,\mu\in\mathcal{U}_{t_j}$. Taking essential infimum over $\beta\in\mathfrak{B}_{t_j}$ and applying Theorem 2.1 with $(t,x,\delta)=(t_j,x_j,\frac{2}{3}\delta)$, we see from (4.91) that P-a.s.

$$\begin{aligned}\varphi\big(t_j,x_j\big)+\frac{5}{6}\widetilde{\wp}t_0 &\le \underset{\beta\in\mathfrak{B}_{t_j}}{\operatorname{essinf}}\ \underset{\mu\in\mathcal{U}_{t_j}}{\operatorname{esssup}} Y_{t_j}^{t_j,x_j,\mu,\beta(\mu)}\Big(\tau_{\beta,\mu},\phi\Big(\tau_{\beta,\mu},X_{\tau_{\beta,\mu}}^{t_j,x_j,\mu,\beta(\mu)}\Big)\Big)\\ &\le w_1(t_j,x_j)<\varphi(t_j,x_j)+\frac{5}{6}\widetilde{\wp}t_0\,.\end{aligned}$$

A contradiction appears. Therefore, $\underline{w}_1$ is a viscosity supersolution of (3.1) with Hamiltonian $\underline{H}_1$.

(b) Next, we show that $\overline{w}_1$ is a viscosity subsolution of (3.1) with Hamiltonian $\overline{H}_1$. Let $(t_0,x_0,\varphi)\in(0,T)\times\mathbb{R}^k\times\mathbb{C}^{1,2}\big([0,T]\times\mathbb{R}^k\big)$ be such that $\overline{w}_1-\varphi$ attains a strict local maximum 0 at (t_0,x_0), i.e., for some $\delta_0\in\big(0,t_0\wedge(T-t_0)\big)$,

$$(\overline{w}_1-\varphi)(t,x)<(\overline{w}_1-\varphi)(t_0,x_0)=0 \quad \forall\,(t,x)\in O_{\delta_0}(t_0,x_0)\big\backslash\big\{(t_0,x_0)\big\}.$$

We still denote $\big(\varphi(t_0,x_0),D_x\varphi(t_0,x_0),D_x^2\varphi(t_0,x_0)\big)$ by (y_0,z_0,Γ_0). If $\overline{H}_1\big(t_0,x_0,y_0,z_0,\Gamma_0\big)=\infty$, then $-\frac{\partial}{\partial t}\varphi(t_0,x_0)-\overline{H}_1\big(t_0,x_0,y_0,z_0,\Gamma_0\big)\le 0$ clearly holds. To make a contradiction, we assume that when $\overline{H}_1\big(t_0,x_0,y_0,z_0,\Gamma_0\big)<\infty$,

$$\varrho\overset{\triangle}{=}-\frac{\partial}{\partial t}\varphi(t_0,x_0)-\overline{H}_1\big(t_0,x_0,y_0,z_0,\Gamma_0\big)>0. \tag{4.98}$$

For any $v\in\overline{O}_\kappa(v_0)$, one can deduce from (4.87) that $|H(t_0,x_0,y_0,z_0,\Gamma_0,u_0,v)|\le \frac{1}{4}(C_\varphi^0)^2+C_\varphi^0C(\kappa,x_0)+C(\kappa,x_0)$, where $C_\varphi^0=\big|\varphi(t_0,x_0)\big|+\big|D_x\varphi(t_0,x_0)\big|+\big|D_x^2\varphi(t_0,x_0)\big|$ as set in part (a). It then follows that

$$\begin{aligned}\overline{H}_1\big(t_0,x_0,y_0,z_0,\Gamma_0\big)&\ge \lim_{n\to\infty}\downarrow\ \sup_{u\in\mathbb{U}}\ \inf_{v\in\mathscr{O}_u^n} H(t_0,x_0,y_0,z_0,\Gamma_0,u,v)\\ &\ge \lim_{n\to\infty}\downarrow\ \inf_{v\in\mathscr{O}_{u_0}^n} H(t_0,x_0,y_0,z_0,\Gamma_0,u_0,v)\\ &=\inf_{v\in\overline{O}_\kappa(v_0)} H(t_0,x_0,y_0,z_0,\Gamma_0,u_0,v)\\ &\ge -\frac{1}{4}(C_\varphi^0)^2-C_\varphi^0C(\kappa,x_0)-C(\kappa,x_0)>-\infty.\end{aligned} \tag{4.99}$$

Thus $\varrho < \infty$. Then one can find an $m \in \mathbb{N}$ such that

$$\begin{aligned} & -\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{7}{8}\,\varrho \\ & \quad \ge \sup_{u\in\mathbb{U}} \inf_{v\in\mathscr{O}^m_u} \varlimsup_{u'\to u} \sup_{(t,x,y,z,\Gamma)\in O_{1/m}(t_0,x_0,y_0,z_0,\Gamma_0)} H(t,x,y,z,\Gamma,u',v). \end{aligned} \tag{4.100}$$

As $\varphi \in \mathbb{C}^{1,2}\big([0,T]\times\mathbb{R}^k\big)$, there exists a $\delta < \frac{1}{2m}\wedge\delta_0$ such that for any $(t,x) \in \overline{O}_\delta(t_0,x_0)$,

$$\begin{aligned} \Big|\frac{\partial \varphi}{\partial t}(t,x) - \frac{\partial \varphi}{\partial t}(t_0,x_0)\Big| \le \frac{1}{8}\,\varrho \quad \text{and} \quad & \big|\varpi(t,x) - \varpi(t_0,x_0)\big| \\ & \le \frac{1}{2m} \text{ for } \varpi = \varphi, D_x\varphi, D_x^2\varphi, \end{aligned} \tag{4.101}$$

the latter of which together with (4.100) implies that

$$\begin{aligned} & -\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{7}{8}\,\varrho \\ & \quad \ge \sup_{u\in\mathbb{U}} \inf_{v\in\mathscr{O}^m_u} \varlimsup_{u'\to u} \sup_{(t,x)\in\overline{O}_\delta(t_0,x_0)} H(t,x,\varphi(t,x),D_x\varphi(t,x),D_x^2\varphi(t,x),u',v). \end{aligned}$$

Then for any $u \in \mathbb{U}$, there exists a $\mathfrak{P}_o(u) \in \mathscr{O}^m_u$ such that

$$\begin{aligned} & -\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{3}{4}\,\varrho \\ & \quad \ge \varlimsup_{u'\to u} \sup_{(t,x)\in\overline{O}_\delta(t_0,x_0)} H\big(t,x,\varphi(t,x),D_x\varphi(t,x),D_x^2\varphi(t,x),u',\mathfrak{P}_o(u)\big), \end{aligned}$$

and we can find a $\lambda(u) \in (0,1)$ such that for any $u' \in O_{\lambda(u)}(u)$,

$$\begin{aligned} & -\frac{\partial \varphi}{\partial t}(t_0, x_0) - \frac{5}{8}\,\varrho \\ & \quad \ge \sup_{(t,x)\in\overline{O}_\delta(t_0,x_0)} H\big(t,x,\varphi(t,x),D_x\varphi(t,x),D_x^2\varphi(t,x),u',\mathfrak{P}_o(u)\big). \end{aligned} \tag{4.102}$$

Set $\widetilde{\lambda}(u_0) = \lambda(u_0)$ and $\widetilde{\lambda}(u) = \lambda(u) \wedge \big(\frac{1}{2}[u]_{\mathbb{U}}\big)$ for any $u \in \mathbb{U}\backslash\{u_0\}$. Since the separable metric space $\mathbb{U}$ is Lindelöf, $\big\{\mathfrak{O}(u) \stackrel{\triangle}{=} O_{\widetilde{\lambda}(u)}(u)\big\}_{u\in\mathbb{U}}$ has a countable subset $\{\mathfrak{O}(u_i)\}_{i\in\mathbb{N}}$ to cover $\mathbb{U}$. Clearly,

$$\mathfrak{P}(u) \stackrel{\triangle}{=} \sum_{i\in\mathbb{N}} \mathbf{1}_{\big\{u\in\mathfrak{O}(u_i)\backslash \underset{j<i}{\cup}\mathfrak{O}(u_j)\big\}} \mathfrak{P}_o(u_i) \in \mathbb{V} \quad \forall\, u \in \mathbb{U}$$

defines a $\mathscr{B}(\mathbb{U})/\mathscr{B}(\mathbb{V})$-measurable function. Given $u\in\mathbb{U}$, there is an $i\in\mathbb{N}$ such that $u\in\mathfrak{O}(u_i)\backslash\cup_{j<i}\mathfrak{O}(u_j)$. If $u_i = u_0$,

$$\big[\mathfrak{P}(u)\big]_{\mathbb{V}} = \big[\mathfrak{P}_0(u_i)\big]_{\mathbb{V}} \le \kappa + m[u_i]_{\mathbb{U}} = \kappa \le \kappa + m[u]_{\mathbb{U}}. \tag{4.103}$$

On the other hand, if $u_i \ne u_0$, then $[u_i]_{\mathbb{U}} \le [u]_{\mathbb{U}} + \rho_{\mathbb{U}}(u,u_i) \le [u]_{\mathbb{U}} + \widetilde{\lambda}(u_i) \le [u]_{\mathbb{U}} + \frac{1}{2}[u_i]_{\mathbb{U}}$, and it follows that

$$\big[\mathfrak{P}(u)\big]_{\mathbb{V}} = \big[\mathfrak{P}_0(u_i)\big]_{\mathbb{V}} \le \kappa + m[u_i]_{\mathbb{U}} \le \kappa + 2m[u]_{\mathbb{U}}. \tag{4.104}$$

Also, we see from (4.102) that $-\frac{\partial \varphi}{\partial t}(t_0,x_0) - \frac{5}{8}\,\varrho \ge \sup_{(t,x)\in\omega(t_0,x_0)} H\big(t,x,\varphi(t,x), D_x\varphi(t,x), D^2_x\varphi(t,x), u, \mathfrak{P}_0(u_i)\big) = \sup_{(t,x)\in\omega(t_0,x_0)} H(t,x,\varphi(t,x), D_x\varphi(t,x), D^2_x\varphi(t,x), u, \mathfrak{P}(u))$, which together with (4.101) implies that

$$-\frac{\partial \varphi}{\partial t}(t,x) - \frac{1}{2}\,\varrho \ge H(t,x,\varphi(t,x), D_x\varphi(t,x), D^2_x\varphi(t,x), u, \mathfrak{P}(u)) \quad \forall\,(t,x)\in \overline{O}_\delta(t_0,x_0),\ \ \forall\, u\in\mathbb{U}. \tag{4.105}$$

Similar to (4.90), we set $\wp \stackrel{\triangle}{=} \min\big\{(\varphi - \overline{w}_1)(t,x)\colon (t,x)\in \overline{O}_\delta(t_0,x_0)\backslash O_{\frac{\delta}{3}}(t_0,x_0)\big\} > 0$ and $\widetilde{\wp} \stackrel{\triangle}{=} \frac{\wp\wedge\varrho}{2(1\vee\gamma)T} > 0$. Let $\big\{(t_j,x_j)\big\}_{j\in\mathbb{N}}$ be a sequence of $O_{\frac{\delta}{6}}(t_0,x_0)$ such that $\lim_{j\to\infty}(t_j,x_j) = (t_0,x_0)$ and $\lim_{j\to\infty} w_1(t_j,x_j) = \overline{w}_1(t_0,x_0) = \varphi(t_0,x_0) = \lim_{j\to\infty}\varphi(t_j,x_j)$. So one can find a $j\in\mathbb{N}$ such that (4.91) still holds.

For any $\mu\in\mathcal{U}_{t_j}$, the measurability of function $\mathfrak{P}$ shows that $\big(\widehat{\beta}(\mu)\big)_s \stackrel{\triangle}{=} \mathfrak{P}(\mu_s)$, $s\in[t_j,T]$ is a $\mathbb{V}$-valued, $\mathbf{F}$-progressively measurable process. By (4.103) and (4.104), $\big[\big(\widehat{\beta}(\mu)\big)_s\big]_{\mathbb{V}} = \big[\mathfrak{P}(\mu_s)\big]_{\mathbb{V}} \le \kappa + 2m[\mu_s]_{\mathbb{U}}\ \ \forall\, s\in[t_j,T]$. Let $E\int_{t_j}^T[\mu_s]^q_{\mathbb{U}}\,ds < \infty$ for some $q>2$. It then follows that $E\int_{t_j}^T\big[\big(\widehat{\beta}(\mu)\big)_s\big]^q_{\mathbb{V}}ds \le 2^{q-1}\kappa^qT + 2^{2q-1}m^qE\int_{t_j}^T[\mu_s]^q_{\mathbb{U}}ds < \infty$. So $\widehat{\beta}(\mu)\in\mathcal{V}_{t_j}$. Let $\mu^1,\mu^2\in\mathcal{U}_{t_j}$ such that $\mu^1=\mu^2$, $ds\times dP$-a.s. on $[\![t_j,\tau[\![\,\cup\,[\![\tau,T]\!]_A$ for some $\tau\in\mathcal{S}_{t_j,T}$ and $A\in\mathcal{F}_\tau$. Then it directly follows that $\big(\widehat{\beta}(\mu^1)\big)_s = \mathfrak{P}(\mu^1_s) = \mathfrak{P}(\mu^2_s) = \big(\widehat{\beta}(\mu^2)\big)_s$, $ds\times dP$-a.s. on $[\![t_j,\tau[\![\,\cup\,[\![\tau,T]\!]_A$. Hence, $\widehat{\beta}\in\mathfrak{B}_{t_j}$.

Let $\mu\in\mathcal{U}_{t_j}$. We set $\Theta_\mu \stackrel{\triangle}{=} \big(t_j,x_j,\mu,\widehat{\beta}(\mu)\big)$ and define $\tau_\mu = \tau_{\widehat{\beta},\mu} \stackrel{\triangle}{=} \inf\{s\in(t_j,T]\colon \big(s,X^{\Theta_\mu}_s\big)\notin O_{\frac{2}{3}\delta}(t_j,x_j)\}\in\mathcal{S}_{t_j,T}$. As $\big|\big(T,X^{\Theta_\mu}_T\big)-(t_j,x_j)\big| \ge T-t_j \ge T-t_0-|t_j-t_0| > \delta_0 - \frac{\delta}{6} > \frac{2}{3}\delta$, the continuity of X^{Θ_μ} implies that P-a.s.

$$\tau_\mu < T \quad\text{and}\quad \big(\tau_\mu\wedge s, X^{\Theta_\mu}_{\tau_\mu\wedge s}\big)\in\overline{O}_{\frac{2}{3}\delta}(t_j,x_j)\subset\overline{O}_{\frac{5}{6}\delta}(t_0,x_0)\quad\forall\, s\in[t_j,T]. \tag{4.106}$$

In particular,

$$\big(\tau_\mu, X^{\Theta_\mu}_{\tau_\mu}\big)\in\partial O_{\frac{2}{3}\delta}(t_j,x_j)\subset\overline{O}_{\frac{5}{6}\delta}(t_0,x_0)\backslash O_{\frac{\delta}{2}}(t_0,x_0). \tag{4.107}$$

The continuity of φ, X^{Θ_μ} and (4.106) show that $\mathcal{Y}^\mu_s \stackrel{\triangle}{=} \varphi\big(\tau_\mu\wedge s, X^{\Theta_\mu}_{\tau_\mu\wedge s}\big) - \widetilde{\wp}(\tau_\mu\wedge s)$, $s\in[t_j,T]$ defines a bounded $\mathbf{F}$-adapted continuous process. Applying Itô's formula yields that

$$\mathcal{Y}^\mu_s = \mathcal{Y}^\mu_T + \int_s^T \mathfrak{f}^\mu_r dr - \int_s^T \mathcal{Z}^\mu_r dB_r, \quad s\in[t_j,T],$$

where $\mathcal{Z}^\mu_r \stackrel{\triangle}{=} \mathbf{1}_{\{r<\tau_\mu\}} D_x\varphi\big(r,X^{\Theta_\mu}_r\big)\cdot\sigma\big(r,X^{\Theta_\mu}_r,\mu_r,\big(\widehat{\beta}(\mu)\big)_r\big)$ and

$$\mathfrak{f}^\mu_r \stackrel{\triangle}{=} \mathbf{1}_{\{r<\tau_\mu\}}\bigg\{\widetilde{\wp} - \frac{\partial\varphi}{\partial t}\big(r,X^{\Theta_\mu}_r\big) - D_x\varphi\big(r,X^{\Theta_\mu}_r\big)\cdot b\big(r,X^{\Theta_\mu}_r,\mu_r,(\widehat{\beta}(\mu))_r\big) - \frac{1}{2}\mathrm{trace}\Big(\sigma\sigma^T\big(r,X^{\Theta_\mu}_r,\mu_r,(\widehat{\beta}(\mu))_r\big)\cdot D^2_x\varphi\big(r,X^{\Theta_\mu}_r\big)\Big)\bigg\}.$$

As $\varphi\in\mathbb{C}^{1,2}\big([t,T]\times\mathbb{R}^k\big)$, the measurability of b, σ, X^{Θ_μ}, μ, and $\widehat{\beta}(\mu)$ implies that both $\mathcal{Z}^\mu$ and $\mathfrak{f}^\mu$ are $\mathbf{F}$-progressively measurable. Let $\widetilde{C}_\varphi \stackrel{\triangle}{=} \sup_{(t,x)\in\overline{O}_{\frac{5}{6}\delta}(t_0,x_0)}\big|D_x\varphi(t,x)\big| < \infty$. Similar to (4.94), we see from (2.1), (2.2), and (4.106) that

$$E\left[\left(\int_{t_j}^{T}|\mathcal{Z}_s^\mu|^2\,ds\right)^{p/2}\right]$$
$$\le c_0\widetilde{C}_\varphi^p\left(\big(1+|x_0|+\delta\big)^p+\left\{E\int_{t_j}^T[\mu_s]_{\mathbb{U}}^2\,ds\right\}^{p/2}+\left\{E\int_{t_j}^T\big[(\widehat{\beta}(\mu))_s\big]_{\mathbb{V}}^2\,ds\right\}^{p/2}\right)<\infty,$$

i.e., $\mathcal{Z}^\mu\in\mathbb{H}_{\mathbf{F}}^{2,p}\big([t_j,T],\mathbb{R}^d\big)$. Hence, $\big\{\big(\mathcal{Y}_s^\mu,\mathcal{Z}_s^\mu\big)\big\}_{s\in[t_j,T]}$ solves the BSDE$\big(t_j,\mathcal{Y}_T^\mu,\mathfrak{f}^\mu\big)$.

Let ℓ, θ_1, and θ_2 still be the continuous functions considered in part (a). Like ϕ in (4.95),

$$\widetilde{\phi}(t,x)\stackrel{\triangle}{=}\theta_1(t,x)\ell(x)+\big(1-\theta_1(t,x)\big)(\varphi(t,x)-\wp\theta_2(t,x))\quad\forall\,(t,x)\in[t_j,T]\times\mathbb{R}^k$$

define a continuous function with $\widetilde{\phi}\ge w_1$. Similar to (4.96) and (4.97), we can deduce from (4.107), (4.106), (4.105), and (2.4) that $\mathcal{Y}_T^\mu\ge\widetilde{\phi}\big(\tau_\mu,X_{\tau_\mu}^{\Theta_\mu}\big)$, P-a.s. and that $\mathfrak{f}_s^\mu(\omega)\ge f_{\tau_\mu}^{\Theta_\mu}\big(s,\omega,\mathcal{Y}_s^\mu(\omega),\mathcal{Z}_s^\mu(\omega)\big)$ for $ds\times dP$-a.s. $(s,\omega)\in[t_j,T]\times\Omega$. As $f_{\tau_\mu}^{\Theta_\mu}$ is Lipschitz continuous in (y,z), we know from Proposition 1.2 (2) that $P(\mathcal{Y}_s^\mu\ge Y_s^{\Theta_\mu}\big(\tau_\mu,\widetilde{\phi}\big(\tau_\mu,X_{\tau_\mu}^{\Theta_\mu}\big)\big)\,\forall\,s\in[t_j,T])=1$. Letting $s=t_j$ and using the fact that $t_j>t_0-\frac{1}{6}\delta>t_0-\frac{1}{6}\delta_0>\frac{5}{6}t_0$ yields that $\varphi(t_j,x_j)-\frac{5}{6}\widetilde{\wp}t_0>\mathcal{Y}_{t_j}^\mu\ge Y_{t_j}^{t_j,x_j,\mu,\widehat{\beta}(\mu)}(\tau_\mu,\widetilde{\phi}(\tau_\mu,X_{\tau_\mu}^{t_j,x_j,\mu,\widehat{\beta}(\mu)}))$, P-a.s. Taking essential supremum over $\mu\in\mathcal{U}_{t_j}$ and applying Theorem 2.1 with $(t,x,\delta)=(t_j,x_j,\frac{2}{3}\delta)$ yields that

$$\begin{aligned}\varphi(t_j,x_j)-\frac{5}{6}\widetilde{\wp}t_0&\ge\operatorname*{esssup}_{\mu\in\mathcal{U}_{t_j}}Y_{t_j}^{t_j,x_j,\mu,\widehat{\beta}(\mu)}\Big(\tau_\mu,\widetilde{\phi}\Big(\tau_\mu,X_{\tau_\mu}^{t_j,x_j,\mu,\widehat{\beta}(\mu)}\Big)\Big)\\&\ge\operatorname*{essinf}_{\beta\in\mathfrak{B}_{t_j}}\operatorname*{esssup}_{\mu\in\mathcal{U}_{t_j}}Y_{t_j}^{t_j,x_j,\mu,\beta(\mu)}\Big(\tau_{\beta,\mu},\widetilde{\phi}\Big(\tau_{\beta,\mu},X_{\tau_{\beta,\mu}}^{t_j,x_j,\mu,\beta(\mu)}\Big)\Big)\\&\ge w_1(t_j,x_j)>\varphi(t_j,x_j)-\frac{5}{6}\widetilde{\wp}t_0\,,\quad P\text{-a.s.},\end{aligned}$$

where $\tau_{\beta,\mu}\stackrel{\triangle}{=}\inf\{s\in(t_j,T]:\big(s,X_s^{t_j,x_j,\mu,\beta(\mu)}\big)\notin O_{\frac{2}{3}\delta}(t_j,x_j)\}$. A contradiction appears. Therefore, $\overline{w}_1$ is a viscosity supersolution of (3.1) with Hamiltonian $\overline{H}_1$. ∎

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