Finite Element Analysis of a Cantilevered Beam

Computer Homework 3

MEMS1047, Finite Element Analysis

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October 22, 2018

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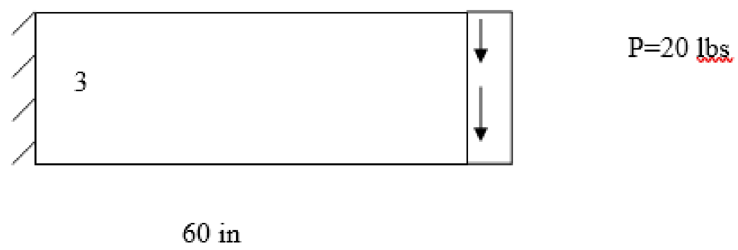
Introduction:

The aim of this homework was in analyzing two simple cantilevered beam models using two-dimensional finite element analysis. The beam models are shown in Figure 1 below.

A



B

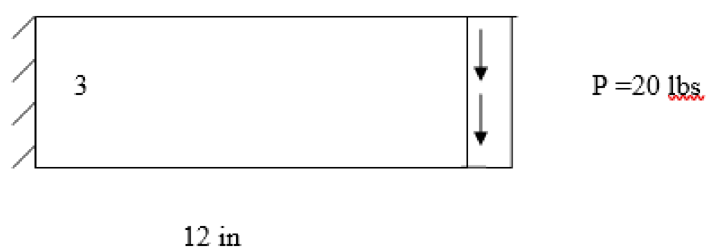


Figure 1: Loading Scenarios

We wish to solve these models in ANSYS APDL and analyze how their solutions compare to that predicted by beam theory. For this analysis, we will use unit thickness, E = 29e06 psi and Poisson's Ratio, υ, of 0. We would like to use 6 or more elements over the height (vertical direction) of the beam and elements that are close to being square. There will be a uniform load placed on the tip of the free-end of the beam. We would like to analyze how the tip displacement, bending stress distribution, and shear stress distribution compares to that as predicted by beam theory.

Problem Statement:

As stated above, we would like to analyze how the results for this beam analysis compare to those as predicted by beam theory. To do this, for each beam model, we will plot our desired results along three different paths: one, along the top edge of the beam, two, along the middle of the beam, and three, on the bottom edge of the beam. The paths will go from x = 0 in. to x = L, where L is the length of the respective beam. In this way, we can very easily tell how the stress distribution changes as a function of x. We plot three different paths to ensure that, at the top of the beam, we receive a tensile bending stress, in the middle, we receive approximately a zero-bending-stress condition, and at the bottom, we receive a compressive bending stress, as predicted by beam theory for the loading conditions shown in Figure 1.

<u>Analysis/Results:</u>

Let us first consider the loading scenario for Problem A as shown in Figure 2 below.

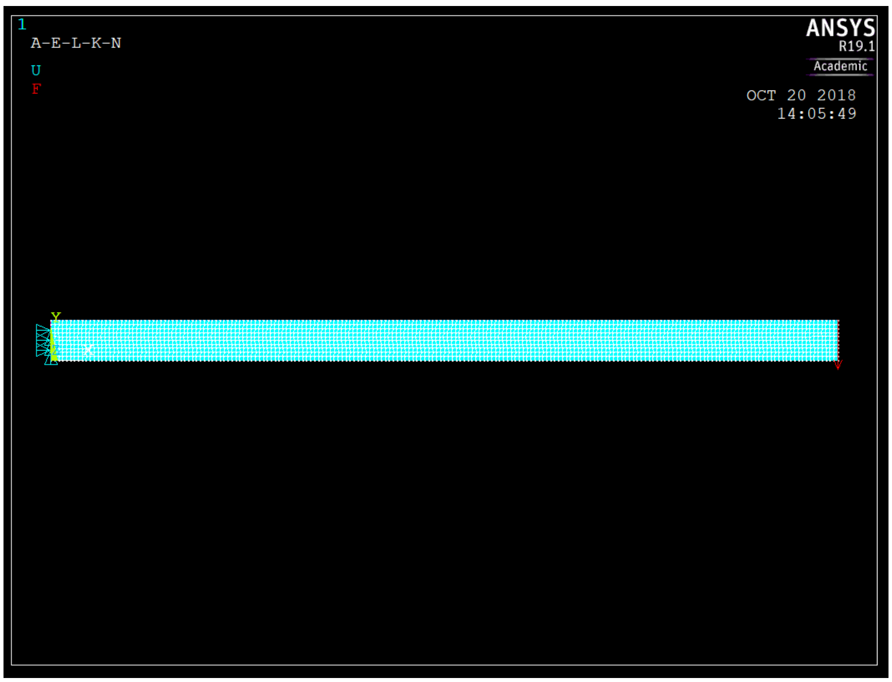


Figure 2: Loading Scenario for Problem A

This problem has dimensions given by height, h = 3 in., and length, L = 60 in. There is a zero-displacement condition placed along the edge at x = 0 in., i.e., the model acts as a cantilever beam. We have placed a load of P = 20 lbs on the top node at the free end of the beam. Note that we expect this point loading to cause a stress concentration at that node. Let us first solve for the anticipated displacement, bending stress, and shear stress as predicted by beam theory. First, for a simple point load at the free end of the beam, the displacement function is given by (Riley, Table A-19))

$$y(x) = \frac{-Px^2}{6EI}(3L - x) \tag{1}$$

For simplicity, let us consider the maximum deflection at x = L given by:

$$y(L) = y_{max} = \frac{-PL^3}{3EI} \tag{2}$$

Then for Problem A, we have P = 20 lbs, L = 60 in., E = 29e06 psi, and second moment of area, I, given by

$$I = \frac{1}{12}bh^3 = \frac{1}{12}(1)(3)^3 = 2.25\ in^4 \tag{3}$$

Then, from (2), we have

$$y(L) = y_{max} = \frac{-(20)(60)^3}{3(29 \cdot 10^6)(2.25)} = -0.0221\ in. \tag{4}$$

Equation 4 represents the expected deflection at the free-end of the beam for Problem A. A plot of the deflection curve as solved in APDL is shown in Figure 3 below.

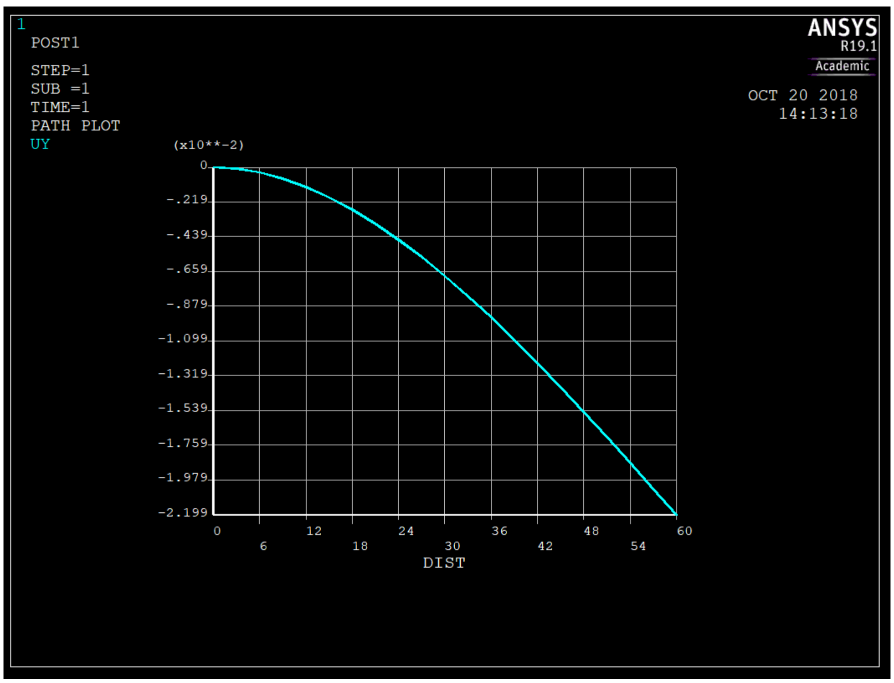


Figure 3: Deflection Curve for Problem A

This figure yields a maximum deflection of

$$y_{max} = -2.199 \cdot 10^{-2} in. = -0.02199\ in.$$

Which agrees with the result obtained from equation 4. Note that this plot was taken from the path along the top of the beam. We expect that the deflection is independent of the y-location we choose and each point in the y-direction will experience the same deflection for a given value of x. Thus, the deflection of the beam agrees with beam theory.

Next, let us consider the bending stress. From simple beam theory, the bending stress is given by

$$\sigma(x) = \frac{M(x)y}{I} = \frac{(Px)y}{I} \tag{5}$$

Where *M* is the applied moment at a given value of x, *y* is the distance from the neutral axis, and *I* is the second moment of area. Let us consider the *maximum* bending stress experienced along the top of the beam. The maximum bending stress will occur at x = 0 in., since the moment from the applied force is highest there. We have

$$\sigma_{max}^{top} = \frac{(20 \cdot 60)(1.5)}{\frac{1}{12}(1)(3)^3} = 800\ psi\ (T) \tag{6}$$

For this orientation, we expect that the bending stress will be in tension. A plot of the bending stress distribution experienced at the top of the beam is shown in Figure 4 below.

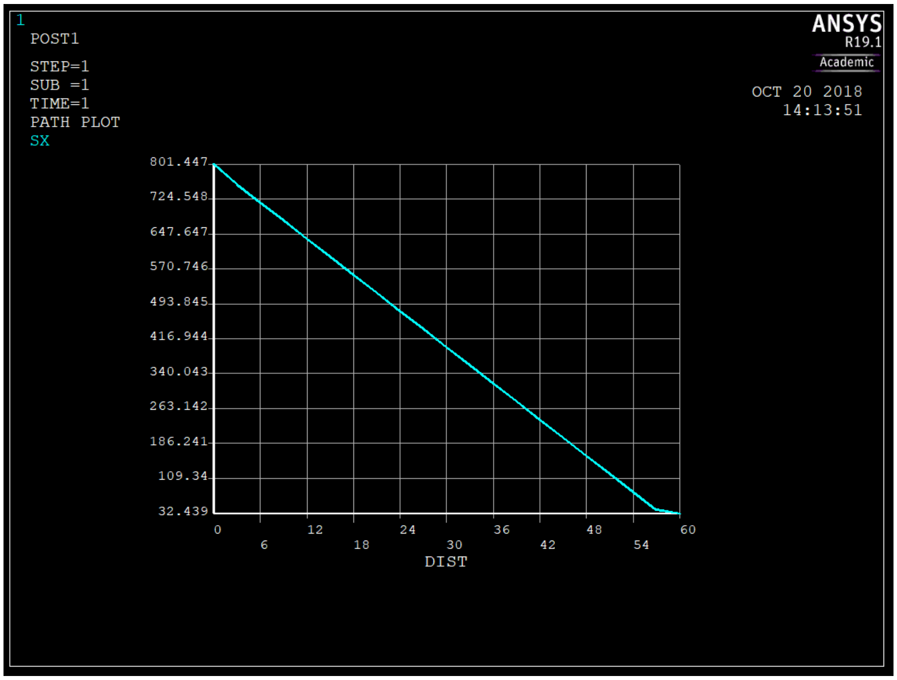


Figure 4: Bending Stress Distribution for Problem A along top of beam

The figure shows a maximum bending stress at the fixed end of the beam of

$$\sigma_{max,top} = 801.447\ psi$$

Which agrees with our result obtained in equation 6. We can check any other x-values and find that the results are verified for this setup. Let us verify the results along the neutral axis and bottom edge of the beam. From equation 5, the theoretical stress values are given by

$$\sigma_{ave}^{mid} = \frac{(20 \cdot 60)(0)}{\frac{1}{12}(1)(3)^3} = 0\ psi \tag{7}$$

$$\sigma_{max}^{bot} = \frac{(20 \cdot 60)(-1.5)}{\frac{1}{12}(1)(3)^3} = -800\ psi\ (C) \tag{8}$$

Plots of the bending stress distributions for the middle and bottom edge of the beam are shown in Figures 5 and 6, respectively.

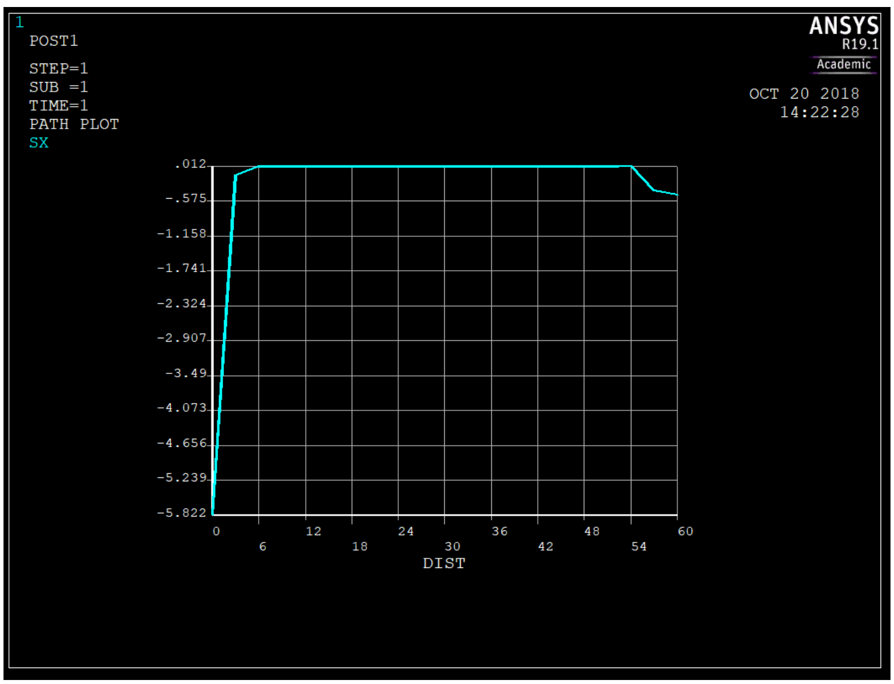


Figure 5: Bending Stress Distribution for Problem A along Neutral Axis of the Beam

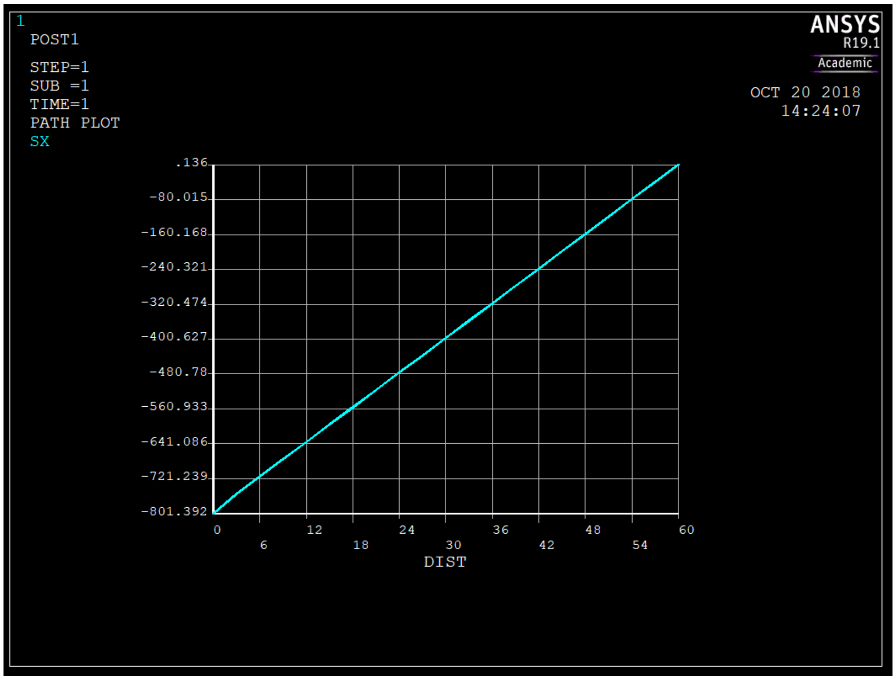


Figure 6: Bending Stress Distribution for Problem A along bottom of beam

Examining the figures, we see

$$\sigma_{ave}^{mid} \approx 0\, psi$$

$$\sigma_{max}^{bot} = -801.392\, psi$$

Which agrees with the results as obtained in equations 7 and 8. We note that there is a slight deviation in the stress distribution along the neutral axis of the beam. At x = 0 in., there is a stress concentration due to the boundary conditions that causes the stress to be non-zero, and, at x = L, there is a stress concentration likely due to the point loading that causes the stress to be non-zero. However, all the results still coincide with that as predicted by beam theory.

Finally, let us consider the shear stress distribution. The shear stress distribution is given by

$$\sigma_{xy} = \frac{V}{2I}(c^2 - y^2) \tag{9}$$

Where V is the shear force, I is the second moment of area, given by equation 3, c is the maximum distance from the neutral axis (namely, $c = 1.5$ in.), and y is the distance from the neutral axis. We note here that one, the distribution is quadratic in terms of y, and two, the shear stress on the top and bottom edge of the beam should be equal to zero, as in this case, $y = c$.

Since there is only the point load of P = 20 lbs acting at the free-end of the beam we expect that the shear force throughout the beam is constant and

$$V = 20\ lbs \tag{10}$$

Thus, from equation 9, the maximum shear stress occurs along the neutral axis of the beam and is given by

$$\sigma_{xy,max} = \frac{20}{2(2.25)}(1.5^2 - 0^2) = 10\ psi \tag{11}$$

The shear stress distributions for Problem A along the top edge, neutral axis, and bottom edge of the beam are shown in Figures 7, 8, and 9, respectively. We could also plot the shear stress distribution as a function of the distance y from the neutral axis to ensure the correct shear stress distribution, however we will not examine it in this report. Below are the shear stress distributions as a function of x.

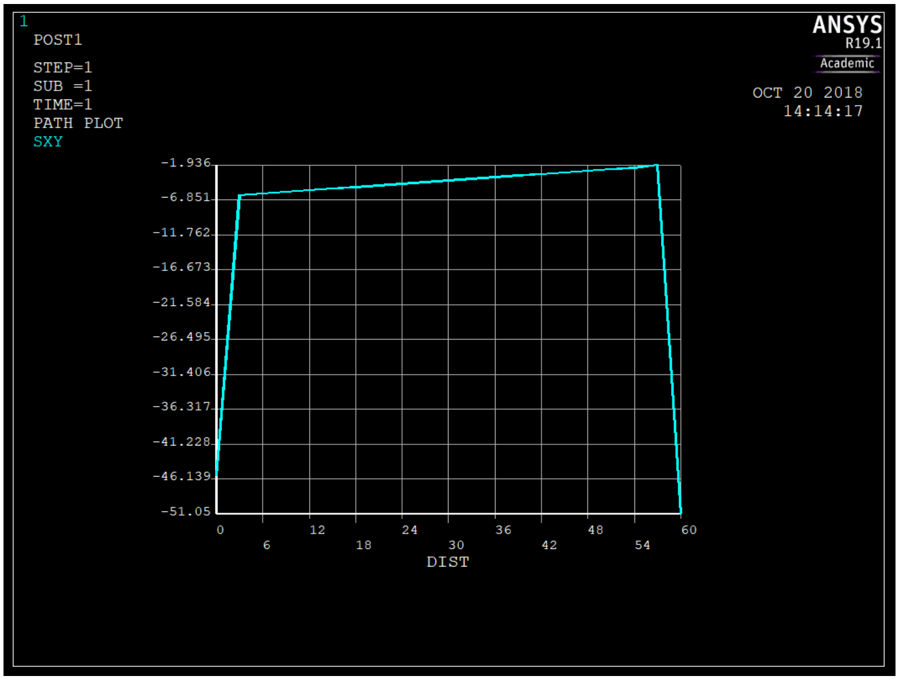


Figure 7: Shear Stress Distribution for Problem A along top of beam

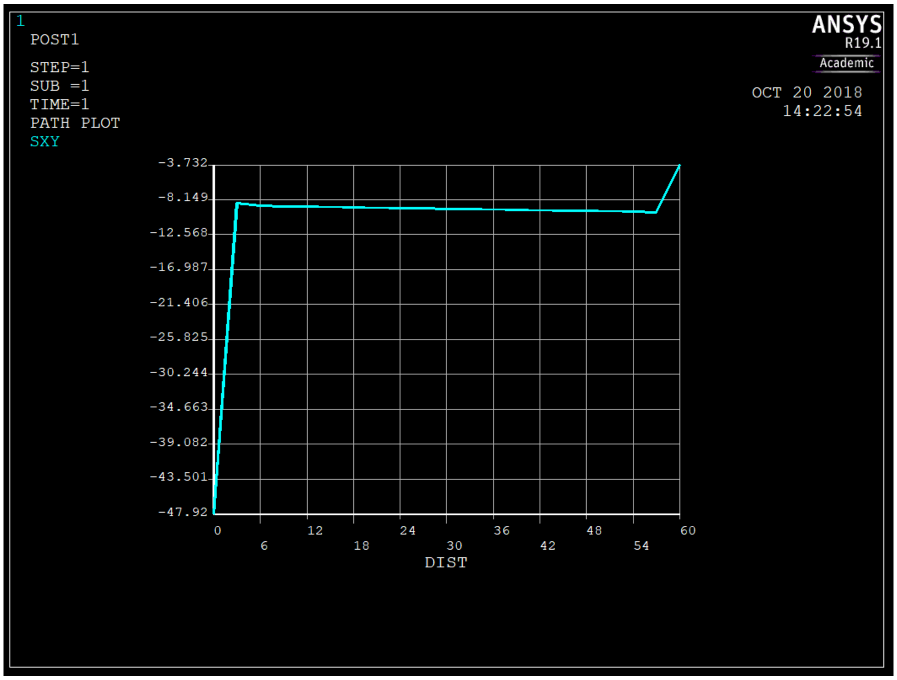


Figure 8: Shear Stress Distribution for Problem A along neutral axis of beam

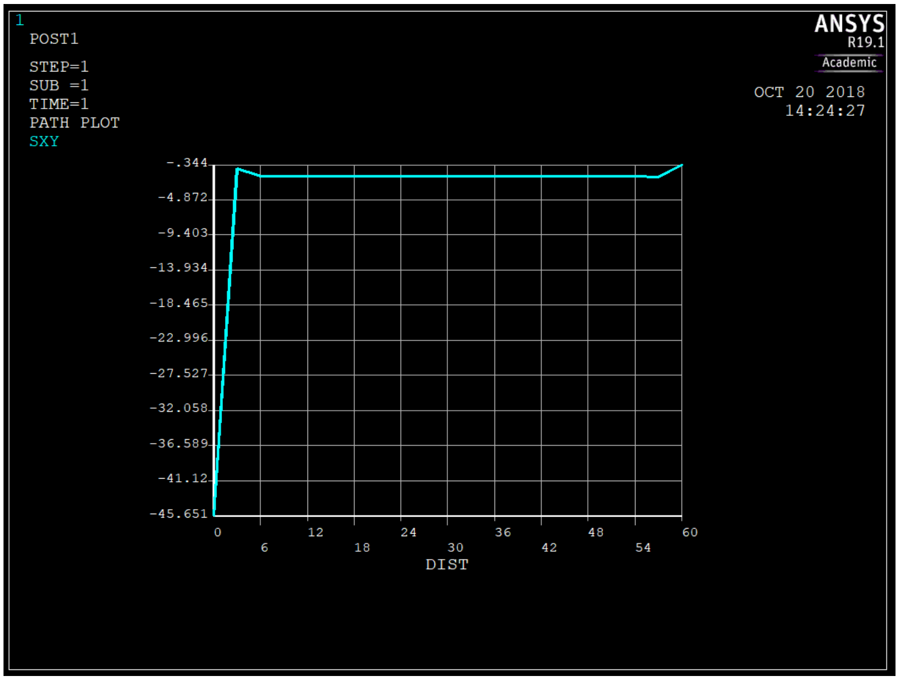


Figure 9: Shear Stress Distribution for Problem A along bottom of beam

These plots call for more investigation. We can see that in each case, there is a stress concentration due to boundary conditions at x = 0 in. We also see that there is a stress concentration due to the point loading at x = L, however note that it is less prevalent along the bottom edge of the beam. This is likely because the point load is located relatively far away from the point load, a phenomenon which can be described by St. Venant's Principle. Looking at the macroscopic effects, we see that the shear stress along the top and bottom edge of the beam is approximately 0, as predicted by beam theory. The average shear stress along the neutral axis of the beam is approximately

$$\sigma_{xy,ave}^{mid} = -8.149 \, psi$$

Which is close to our predicted value of shear stress. We conclude that the deflection and bending stress distributions largely agree with the predicted values as explained by beam theory, but due to our loading/boundary conditions, the shear stress distributions seem cause for concern.

Next, let us switch our focus to Problem B. This problem has dimensions given by height, h = 3 in., and length, L = 12 in. There is a zero-displacement condition placed along the edge at x = 0 in., i.e., the model acts as a cantilever beam. We have placed a load of P = 20 lbs on the top node at the free end of the beam. Note that we expect this point loading to cause a stress concentration at that node. The loading scenario for Problem B is shown in Figure 10, below.

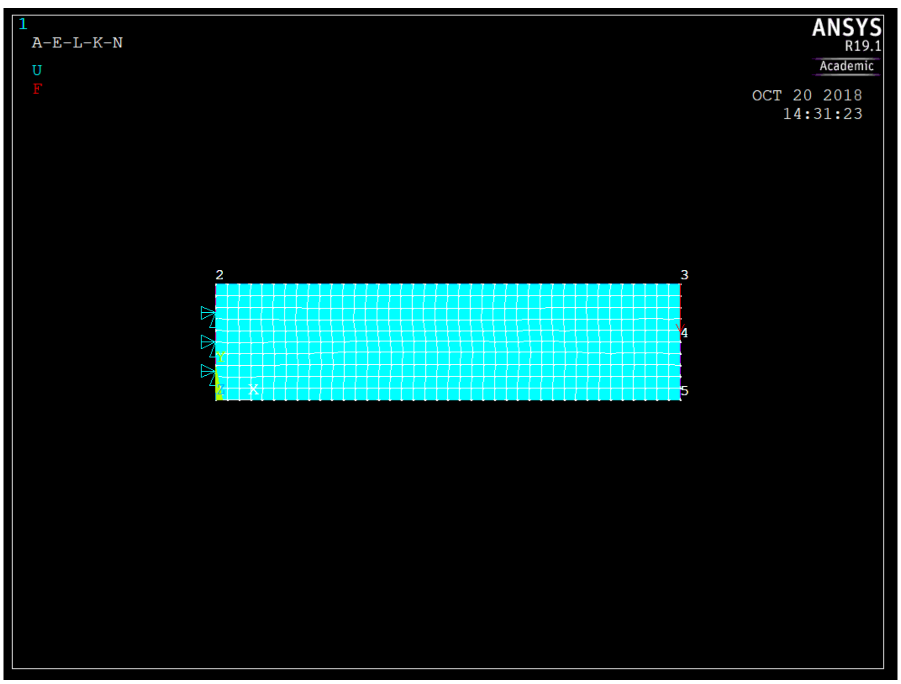


Figure 10: Loading Scenario for Problem B

Prior to performing any analyses on this beam, we first note that we expect issues with this beam scenario since the length of the beam is not large compared to the height of the beam. In practical beam theory, it is assumed that the radius of curvature is large as compared to the dimensions of the cross-section. Since the length of the beam is only four times that of the height of the beam, we should expect issues with our solutions that may lead to errors in our solutions, i.e., uneven stress distributions, etc.

The second moment of area, *I*, is the same as in Problem A (given by equation 3). The length of this beam is given by *L* = 12 in. Then from equation 4, we calculate the theoretical maximum deflection for this problem as

$$y(L) = y_{max} = \frac{-(20)(12)^3}{3(29 \cdot 10^6)(2.25)} = -1.77 \cdot 10^{-4}\, in. \tag{12}$$

A plot of the deflection curve as solved in APDL is shown in Figure 11, below.

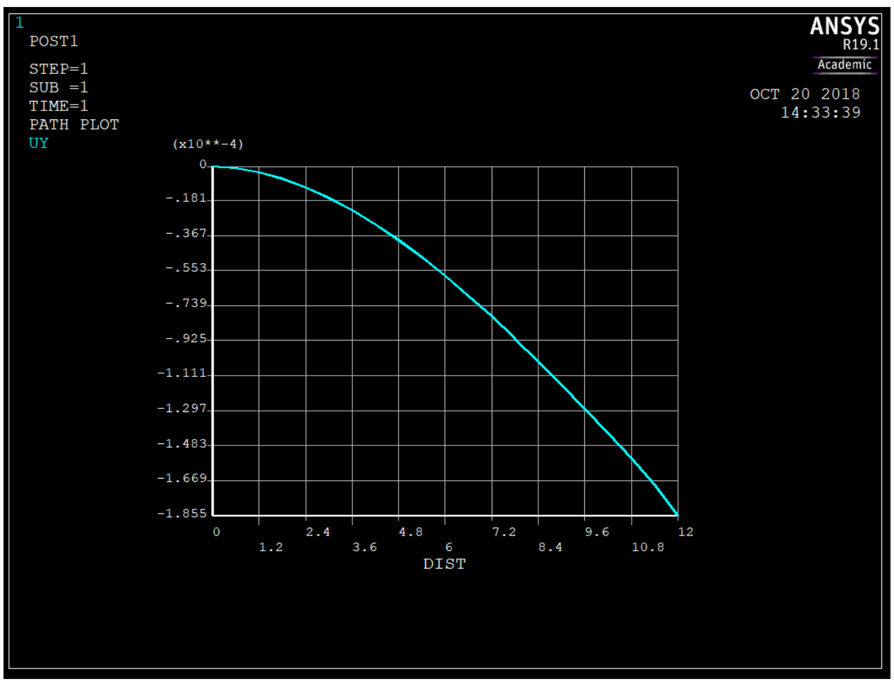


Figure 11: Deflection Curve for Problem B

The curve yields a maximum deflection of

$$y_{max} = -1.855 \cdot 10^{-4} in.$$

Which agrees with the result as obtained in equation 12. Again, we expect that the deflection is independent of the y-location we choose and each point in the y-direction will experience the same deflection for a given value of x. Thus, the deflection for Problem B agrees with that as predicted by beam theory.

Next, let us examine the bending stress. From equations 5 and 6, the maximum bending stress experienced at the top of the beam for this orientation is given by

$$\sigma_{max}^{top} = \frac{(20 \cdot 12)(1.5)}{\frac{1}{12}(1)(3)^3} = 160\, psi\ (T) \tag{13}$$

Likewise, from equations 7 and 8,

$$\sigma_{ave}^{mid} = \frac{(20 \cdot 12)(0)}{\frac{1}{12}(1)(3)^3} = 0\, psi \tag{14}$$

$$\sigma_{max}^{bot} = \frac{(20 \cdot 12)(-1.5)}{\frac{1}{12}(1)(3)^3} = -160\, psi\ (C) \tag{15}$$

Plots of the bending stress distributions for the top, middle and bottom edge of the beam are shown in Figures 12, 13, and 14, respectively.

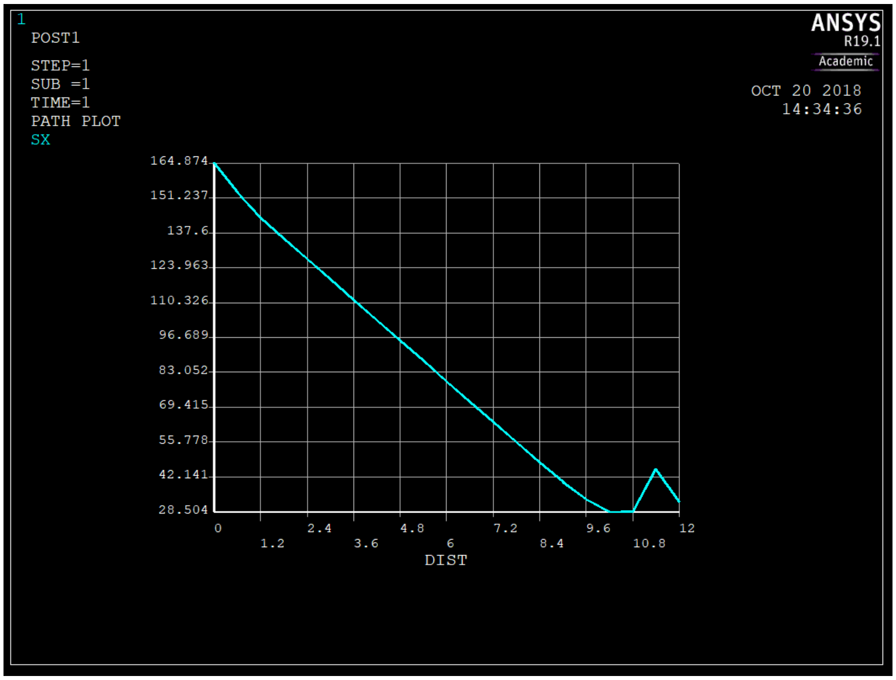


Figure 12: Bending Stress Distribution for Problem B along top of beam

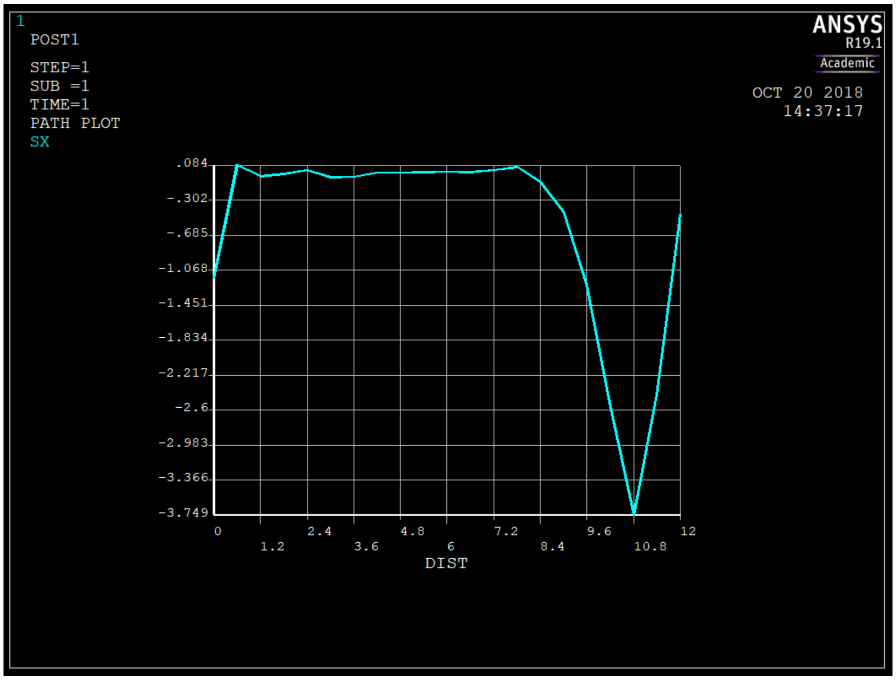


Figure 13: Bending Stress Distribution for Problem A along Neutral Axis of the Beam

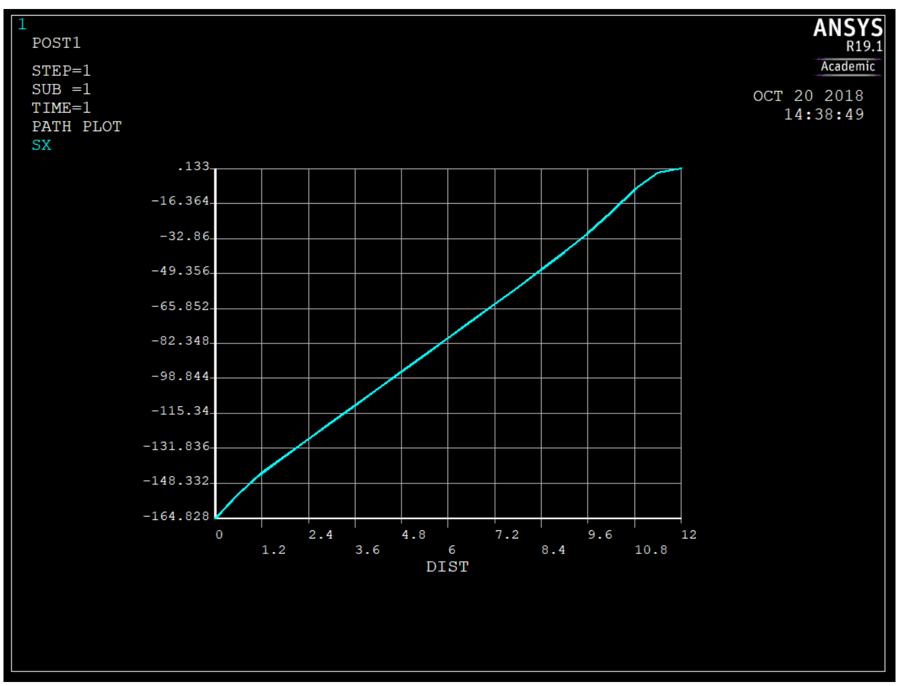


Figure 14: Bending Stress Distribution for Problem B along bottom of beam

Examining the figures, we see that

$$\sigma_{max}^{top} = 164.874\, psi$$

$$\sigma_{ave}^{mid} \cong 0\, psi$$

$$\sigma_{max}^{bot} = -164.828\, psi$$

These values all agree with those obtained in equations 13, 14, and 15. We note that there are stress concentrations due to the point loading along the top edge and neutral axis of the beam. The uneven stress distributions do not discredit the fact that the analytical values agree with the values as predicted by beam theory. There is no apparent stress concentration along the bottom edge of the beam, and as discussed prior, this phenomenon is explained by St. Venant's Principle. Thus, the bending stress distributions for Problem B agree with those predicted by beam theory.

Finally, let us examine the shear stress distribution. We expect that these distributions will be skewed based on our results from Problem A and the geometry of the problem. From equations 9 and 11, the maximum shear stress occurs along the neutral axis of the beam and is given by

$$\sigma_{xy,max} = \frac{20}{2(2.25)}(1.5^2 - 0^2) = 10\, psi$$

Again, we expect the shear stress along the top and bottom edge of the beam to be zero. The shear stress distributions for Problem B along the top edge, neutral axis, and bottom edge of the beam are shown in Figures 15, 16, and 17, respectively.

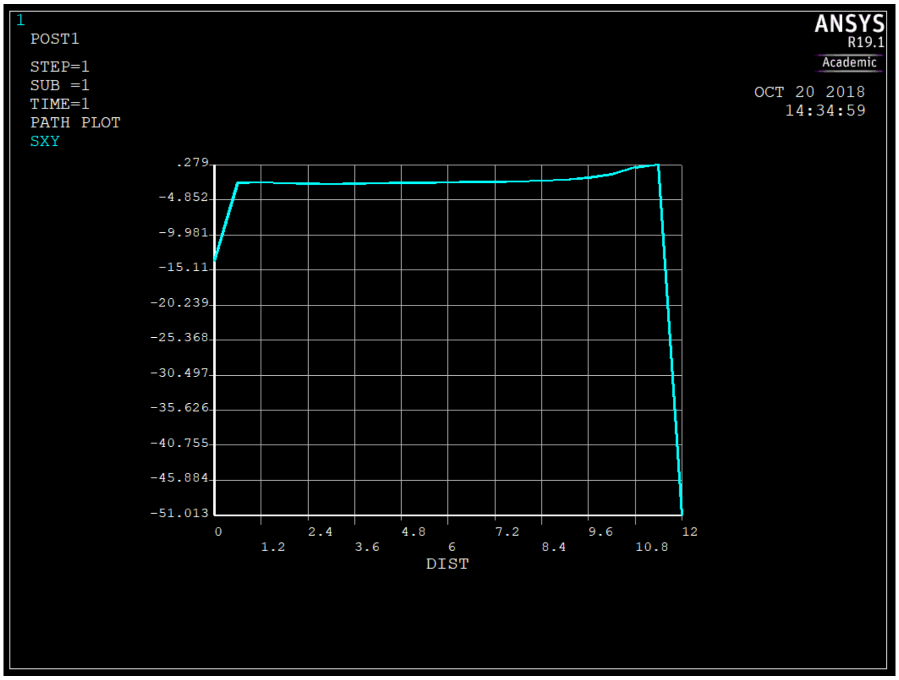


Figure 15: Shear Stress Distribution for Problem B along top of beam

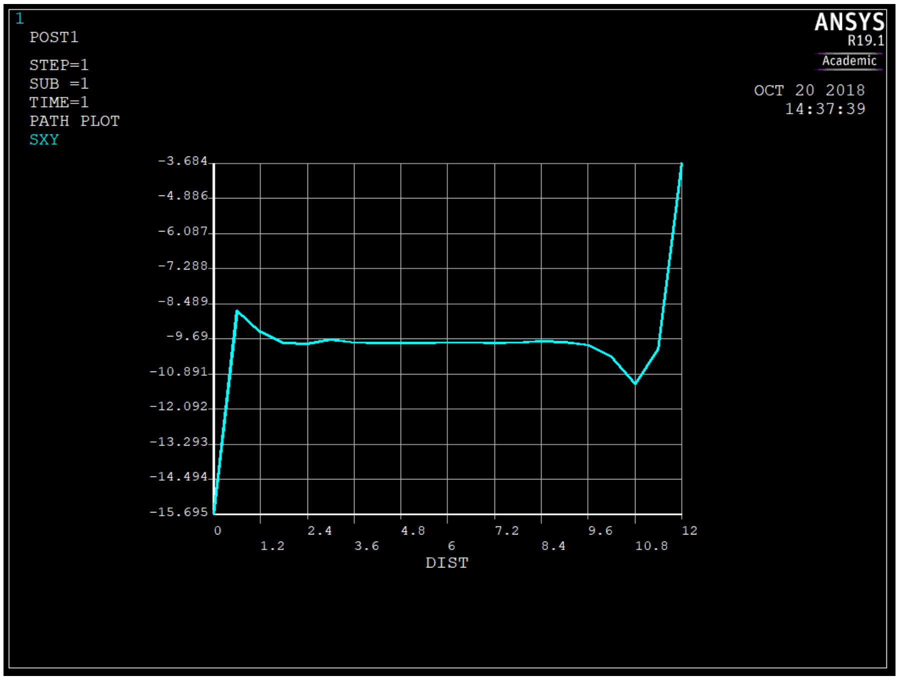


Figure 16: Shear Stress Distribution for Problem B along neutral axis of beam

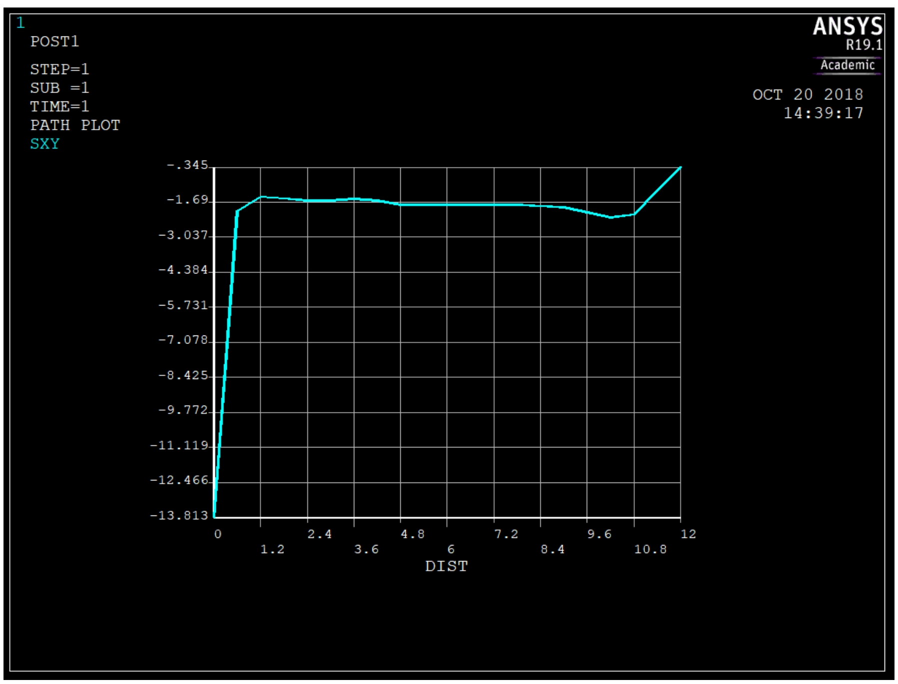


Figure 17: Shear Stress Distribution for Problem B along bottom of beam

In each case, there are stress concentrations experienced at the boundary conditions and the location of point loading. Along the top and bottom edge of the beam, we can say that the average shear stress is

$$\sigma_{xy,ave}^{top,bot} \cong 0\,psi$$

And, along the neutral axis of the beam,

$$\sigma_{xy,ave}^{mid} \cong -9.69\,psi$$

This value is very close to the theoretical value as predicted by beam theory. However, there are many deviations in the distributions that are explained by the loading/boundary conditions and the fact that the geometry of the beam poses natural problems for the solution. I.e., the length of the beam is not large enough compared to the height of the beam, which invokes issues in the shear stress distributions.

Discussion:

Overall, we are satisfied with the results and say that the analytical values agree with the values as predicted by beam theory. Typically we are only concerned with the maximum values associated with a particular solution, and in the case of this problem, we note that the maximum analytical values do in fact coincide with the theoretical values. As discussed, we see certain stress concentrations/uneven stress distributions that are explained by the boundary/loading conditions, and, in the case of Problem B, the geometry of the problem. We can conclude that APDL is a valuable resource for analyzing a simple cantilever beam.

References

Riley, William F., et al. Mechanics of Materials, 6th Edition. Wiley, 2007.