Blasius Similarity Solution

For Boundary Layer Flow Over a Flat Plate

MEMS1055, Computer Aided Analysis in Transport Phenomena
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Contents

1 Introduction

In this report we consider the Blasius Boundary Layer Solution for flow over a flat plate of length unity. The similarity variables are given by

$$\eta = y\sqrt{\frac{U}{x\nu}}, \quad f'(\eta) = \frac{u}{U}, \quad \frac{1}{2}\left[\sqrt{\frac{U\nu}{x}}(\eta f' - f)\right] \tag{1}$$

where U denotes the freestream velocity. We would like to analytically show that

$$ff'' + 2f''' = 0$$
$$f(0) = f'(0) = 0$$
$$f'(\infty) = 1$$

and use these results to calculate a numerical solution. Using the numerical solution, we would also like to:

- Determine a finite η such that the Blasius solution converges and discuss the optimum $\Delta\eta$
- Calculate the drag on the wall of the plate
- Determine a relationship for calculating the boundary layer thickness δ
- Determine the relationship between the shear stress and drag coefficients and Reynolds Number

2 Formulation

2.1 Analytical Derivation of the Blasius Solution

In this section, the Blasius third-order ODE and it's corresponding boundary conditions will be derived. From the definition of Navier-Stokes, we have that:

$$f_1(u, x, y, \nu, U) = 0 \tag{2}$$

$$f_2(v, x, y, \nu, U) = 0 \tag{3}$$

Using the Buckingham Pi Theorem, we can find nondimensionless parameters which accurately describe the system presented by Equations 2 and 3. Note that the derivation of these parameters is omitted. With regards to u,

$$\Pi_1 = \frac{u}{U}, \quad \Pi_2 = y\sqrt{\frac{U}{x\nu}} \tag{4}$$

such that:

$$\frac{u}{U} = f\left(y\sqrt{\frac{U}{x\nu}}\right) = F(\eta) \tag{5}$$

With regards to v,

$$\Pi_3 = v\sqrt{\frac{x}{U\nu}}, \quad \Pi_4 = y\sqrt{\frac{U}{x\nu}} \tag{6}$$

such that:

$$v\sqrt{\frac{x}{U\nu}} = f\left(y\sqrt{\frac{U}{x\nu}}\right) = G(\eta) \tag{7}$$

Based on these nondimensional parameters, we assume that the solution is of the form:

$$u = UF(\eta), \quad v = \sqrt{\frac{U\nu}{x}}G(\eta) \tag{8}$$

We now use the nondimensionalized quantities to go about solving the Navier-Stokes Equations. From continuity, we have,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{9}$$

such that:

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(UF(\eta)) = U\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial x} = U\frac{\partial F}{\partial \eta}\left(-\frac{1}{2}\frac{y\sqrt{U}}{x^{-3/2}\sqrt{\nu}}\right)$$

$$\frac{\partial u}{\partial x} = -\frac{U\eta}{2x}\frac{\partial F}{\partial \eta} \tag{10}$$

Likewise, for the y component:

$$\frac{\partial v}{\partial y} = \frac{U}{x}\frac{\partial G}{\partial \eta} \tag{11}$$

Then from Equations 9, 10, and 11:

$$-\frac{U\eta}{2x}\frac{\partial F}{\partial \eta} + \frac{U}{x}\frac{\partial G}{\partial \eta} = 0 \tag{12}$$

Upon factoring, we get:

$$\frac{U}{x}\left[-\frac{\eta}{2}\frac{\partial F}{\partial \eta} + \frac{\partial G}{\partial \eta}\right] = 0$$

$$\frac{\partial G}{\partial \eta} = \frac{\eta}{2}\frac{\partial F}{\partial \eta}$$

Integrating both sides with respect to η yields:

$$\int_0^{\eta} \frac{\partial G}{\partial \eta} d\eta = \frac{1}{2}\int_0^{\eta} \eta\frac{\partial F}{\partial \eta} d\eta$$

For the left-hand side, $v(0) = 0$ due to the no-slip boundary condition implies $G(0) = 0$ from Equation 8. The right-hand side is integrated by parts. After some simple calculus, the function $G(\eta)$ is calculated as:

$$\boxed{G(\eta) = \frac{1}{2}\left[\eta f' - f\right]} \tag{13}$$

Next, we would like to solve the x-component of the momentum equation, presented below.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} \tag{14}$$

From Equation 10, we have:

$$\frac{\partial u}{\partial x} = -\frac{U\eta}{2x}\frac{\partial F}{\partial \eta} \tag{15}$$

Furthermore:

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\big(UF(\eta)\big) = U\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial y} = U\frac{\partial F}{\partial \eta}\sqrt{\frac{U}{x\nu}} \tag{16}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{U^2}{x\nu}\frac{\partial^2 F}{\partial \eta^2} \tag{17}$$

With u and v as prescribed in Equation 8, along with Equations 14, 15, 16, and 17, the momentum equation becomes:

$$UF(\eta)\left(-\frac{U\eta}{2x}\frac{\partial F}{\partial \eta}\right) + \sqrt{\frac{U\nu}{x}}G(\eta)\left(U\frac{\partial F}{\partial \eta}\sqrt{\frac{U}{x\nu}}\right) = \nu\left(\frac{U^2}{x\nu}\frac{\partial^2 F}{\partial \eta^2}\right)$$

Upon factoring and rearrangement:

$$\frac{U^2}{x}\left[G(\eta)\frac{\partial F}{\partial \eta} - \frac{\eta F(\eta)}{2}\frac{\partial F}{\partial \eta} = \frac{\partial^2 F}{\partial \eta^2}\right]$$

Since $F = F(\eta)$ and $G = G(\eta)$, we may write:

$$G(\eta)\frac{dF}{d\eta} - \frac{\eta F(\eta)}{2}\frac{dF}{d\eta} = \frac{d^2 F}{d\eta^2}$$

Letting $F = \frac{df}{d\eta} = f'$, and substituing in $G(\eta) = \frac{1}{2}\big(\eta f' - f\big)$ from Equation 13, we have:

$$\frac{1}{2}\big(\eta f' - f\big)f'' - \frac{\eta}{2}f'f'' = f'''$$

$$\frac{1}{2}\eta f'f'' - \frac{1}{2}ff'' - \frac{\eta}{2}f'f'' = f'''$$

$$-\frac{1}{2}ff'' = f'''$$

Finally, we obtain:

$$\boxed{ff'' + 2f''' = 0} \tag{18}$$

Let us now examine the boundary conditions. At $\eta = 0$, we have $u, v = 0$ due to the no-slip boundary condition. Thus, from Equations 8 and 13:

$$G(0) = 0 = -\frac{1}{2}f \Rightarrow \boxed{f(0) = 0} \tag{19}$$

From Equation 8, we have:

$$f'(0) = \frac{u(0)}{U} = 0 \Rightarrow \boxed{f'(0) = 0} \tag{20}$$

Finally, as $\eta \to \infty$, the velocity u should approach the freestream velocity, $u \to U$ such that:

$$f'(\eta \to \infty) = \frac{U}{U} = 1 \Rightarrow \boxed{f'(\infty) = 1} \tag{21}$$

3 Results

The MATLAB Script used to calculate the solution to Equation 18 is "Project_4.m". This script takes the third-order ODE and transforms it into three first-order ODE's. It then uses Euler's Method to solve the first-order ODE's, and also applies the shooting method to determine an initial value for the unprescribed boundary condition for $f''(0)$. Full code listings for this script are given in the Appendix, Section 5.1, Figures 3 and 4. The exact methodology behind solving a third-order ODE of this nature has been omitted from this report.

3.1 Solution Convergence and Boundary Layer Thickness

The numerical solution to Equation 18 for a step size of $\Delta\eta = 0.01$ is shown in Figure 1. We can see that the normalized velocity profile, $f' = u/U$, asymptotically approaches a value of one, indicating that the velocity approaches the freestream velocity as η is increased. Portions of the tabular data are presented in the Appendix, Table 1.

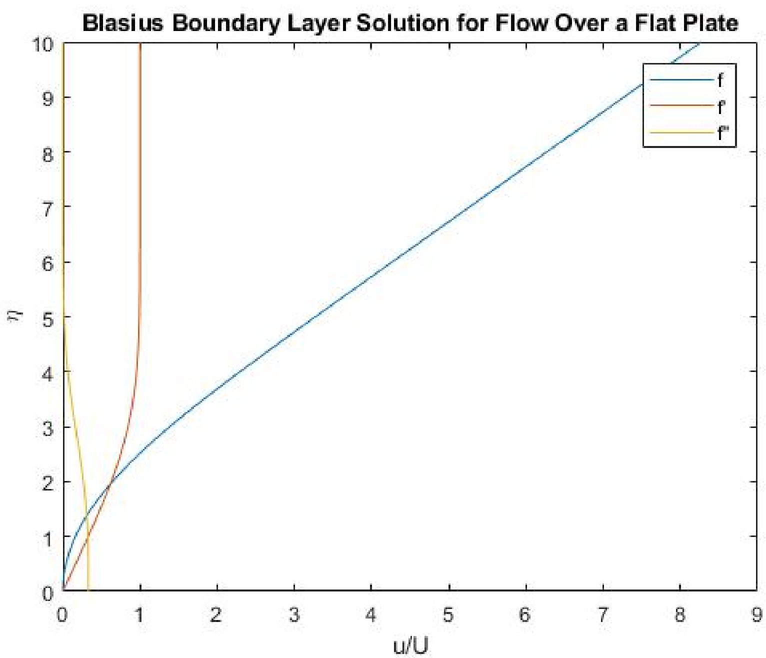

Figure 1: Blasius Solution for $\Delta\eta = 0.01$

Given the boundary condition presented by Equation 21, we would like to determine a finite value of η where the condition begins to hold. This value determines where convergence of the system occurs. Given the asymptotic nature of the solution, typically η is chosen such that the velocity is within 99% of the freestream velocity, $u = 0.99U$. By examining the tabular data presented in Table 1, we observe that at $\eta = 4.92$, $u = 0.99009$. This is the first value to come within 99% of the freestream velocity. Thus, replacing $\eta = \infty$ with $\eta = \eta^*$ (finite):

$$\boxed{\eta^* = 4.92} \tag{22}$$

Using this same concept, if we define the boundary layer thickness δ as the value of y for which $u = 0.99U$, along with the definition of η in Equation 1, we obtain:

$$\delta = y = \eta\sqrt{\frac{x\nu}{U}} = 4.92\sqrt{\frac{x\nu}{U}} \approx \boxed{5\sqrt{\frac{x\nu}{U}}} \tag{23}$$

3.2 Optimum $\Delta\eta$

In this section, the optimum step size $\Delta\eta$ is calculated. The term optimized in this report will refer to the last value before which the change in the initial guess $h(1)$ used in regards to the shooting method no longer exceeds the change in step size. Results for varying mesh sizes are given in Figure 2.

Grid Optimization			
Step Size	**Difference**	**h(1, N)**	**Difference**
0.01	N/A	0.33031	N/A
0.005	0.005	0.33118	0.00087
0.0025	0.0025	0.33162	0.00044
0.00125	0.00125	0.33184	0.00022
0.00063	0.00062	0.33195	0.00011
0.00031	0.00032	0.332	5E-05
0.00016	0.00015	0.33203	3E-05
0.00008	0.00008	0.33204	1E-05
0.00004	0.00004	0.33205	1E-05

Figure 2: Step Size Optimization

From this figure, we can see that the change in the initial guess $h(1)$ *never* exceeds the change in step size. Thus, decreasing the step size from the initial size of $\Delta\eta = 0.01$ will only result in more computational expense with nearly the same amount of computational accuracy. Thus, we conclude that the optimum step size for this problem is:

$$\boxed{\Delta\eta = 0.01} \tag{24}$$

3.3 Drag

In this section, the drag on the wall of the plate is calculated. The shear stress on the wall τ_w is given by:

$$\tau_w = \mu\frac{\partial u}{\partial y}\bigg|_{y=0} = \mu U\sqrt{\frac{U}{x\nu}}\frac{\partial F}{\partial \eta}\bigg|_{\eta=0} = \mu U\sqrt{\frac{U}{x\nu}}f''\bigg|_{\eta=0} \tag{25}$$

Then the drag force is given by:

$$D = \int_0^L \tau_w dx = \int_0^L \mu U\sqrt{\frac{U}{x\nu}}f''\bigg|_{\eta=0} dx = 2\mu U\sqrt{\frac{Ux}{\nu}}f''\bigg|_{\eta=0} \tag{26}$$

Evaluating for a unit length of $L = 1$, a freestream velocity equal to unity, dynamic viscosity $\mu = 1.01e-3$, and kinematic viscosity $\nu = 1.01e-6$, we obtain:

$$D = 2(1.01e-3)(1)\sqrt{\frac{1(1)}{1.01e-6}}f''\bigg|_{\eta=0} = 2.01f''|_{\eta=0} \tag{27}$$

From Table 1, $f''\big|_{\eta=0} = 0.33031$. Thus,

$$D = 2.01(0.33031) = \boxed{0.66391N} \tag{28}$$

3.4 Shear Stress and Drag Coefficient

In this section, we wish to find relations between the shear stress coefficient and the drag coefficient in terms of Reynolds Number. Consider the shear stress coefficient given by:

$$C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2} \tag{29}$$

From Equation 25, we can write:

$$C_f = \frac{\mu U\sqrt{\frac{U}{x\nu}} f''\big|_{\eta=0}}{\frac{1}{2}\rho U^2} = \frac{\nu\sqrt{\frac{U}{x\nu}} f''\big|_{\eta=0}}{\frac{1}{2}U}$$

$$C_f = 2\sqrt{\frac{\nu}{Ux}} f''\big|_{\eta=0} = \frac{2f''\big|_{\eta=0}}{\sqrt{Re_x}}$$

From Table 1, $f''\big|_{\eta=0} = 0.33031$ such that:

$$C_f = \frac{2(0.33031)}{\sqrt{Re_x}} = \frac{0.66}{\sqrt{Re_x}} \tag{30}$$

Thus, the shear stress coefficient can be approximated as:

$$\boxed{C_f \approx \frac{0.66}{\sqrt{Re_x}}} \tag{31}$$

Likewise, the drag coefficient is given by:

$$C_{fm} = \frac{D}{\frac{1}{2}\rho U^2 L} \tag{32}$$

From Equation 26, we can write:

$$C_{fm} = \frac{2\mu U\sqrt{\frac{Ux}{\nu}} f''\big|_{\eta=0}}{\frac{1}{2}\rho U^2 L} = \frac{4\nu\sqrt{\frac{Ux}{\nu}} f''\big|_{\eta=0}}{UL}$$

$$C_{fm} = \frac{4f''\big|_{\eta=0}}{\sqrt{Re_L}}$$

Again, from Table 1, $f''\big|_{\eta=0} = 0.33031$ such that:

$$C_{fm} = \frac{4(0.33031)}{\sqrt{Re_L}} = \frac{1.32}{\sqrt{Re_L}} \tag{33}$$

Thus, the drag coefficient can be approximated as:

$$\boxed{C_{fm} \approx \frac{1.32}{\sqrt{Re_L}}} \tag{34}$$

4 Conclusion

In this report we were able to calculate the Blasius Boundary Layer Solution for flow over a flat plate of length unity. In the formulation we derived the Blasius Equation and its corresponding boundary conditions given by

$$ff'' + 2f''' = 0$$
$$f(0) = f'(0) = 0$$
$$f'(\infty) = 1$$

The third order differential equation was split up into three first-order differential equations and the numerical solution was calculated via the use of Euler's Method and the Shooting Method in the MATLAB Script "Project_4.m". Using the numerical data presented in Table 1, the following results were calculated:

- Finite η: $\eta = \infty \Rightarrow \eta^* = 4.92$
- Boundary Layer Thickness: $\delta = 5\sqrt{\frac{x\nu}{U}}$
- Optimum step size: $\Delta\eta = 0.01$
- Drag on the plate: $D = 0.66391$ [N]
- Shear Stress Coefficient: $C_f \approx \frac{0.66}{\sqrt{Re_x}}$
- Drag Coefficient: $C_{fm} \approx \frac{1.32}{\sqrt{Re_L}}$

5 Appendix

5.1 Additional Tables and Figures

η	f'	f''	f''
0	0	0	0.33031
0.5	0.04045	0.16503	0.32924
1	0.16309	0.32819	0.32161
1.5	0.36595	0.48471	0.30166
2	0.64403	0.62755	0.26643
2.5	0.98864	0.74927	0.21762
3	1.38778	0.84452	0.16187
3.5	1.8277	0.91205	0.10831
4	2.29516	0.95499	0.06463
4.5	2.77924	0.97929	0.03418
4.91	3.1831	0.98991	0.01849
4.92	3.193	0.99009	0.01819
5	3.27225	0.99147	0.01597
6	4.26854	0.99898	0.00239
7	5.26816	0.99993	0.00021
8	6.26814	1	0.00001

Table 1: Blasius Solution for $\Delta\eta = 0.01$

```matlab
% Seth Strayer
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% This script is used to calculate the solution to the Blasius Equation
% given by ff'' + 2f''' = 0.  The script uses the shooting method to
% determine the initial condition h(1, N) based on initial guesses h(1, 1)
% and h(1, 2).  Results are printed to file and the graph of f, f', and f''
% as a function of eta is produced.

clear;
clc;

% defining parameters
eta = 10;
dt = 0.01;
N = eta/dt + 1;
t = linspace(0, eta, N);
K = 1001;
tol = 1e-5;

% % creating output file used to write data
fid1 = fopen('blasius_eta_data.txt', 'wt');
fid2 = fopen('blasius_f_data.txt', 'wt');
fid3 = fopen('blasius_g_data.txt', 'wt');
fid4 = fopen('blasius_h_data.txt', 'wt');
fid5 = fopen('guess_data.txt', 'at');

% defining matrices
f = zeros(N, K);
g = zeros(N, K);
h = zeros(N, K);

% defining boundary conditions
f(1, :) = 0;
g(1, :) = 0;
% defining initial guesses for shooting method
h(1, 1) = 10;
h(1, 2) = 50;

% iteratively solving the Blasius Equation
for k = 1:K-1
    for i = 2:N
        f(i, k) = f(i-1, k) + g(i-1, k)*dt;
        g(i, k) = g(i-1, k) + h(i-1, k)*dt;
        h(i, k) = h(i-1, k) - (f(i-1, k)*h(i-1, k)*dt)/2;
```

Figure 3: MATLAB Script (1)

```matlab
        end

        % if tolerance is met, iter = k and break out of the loop
        if abs(g(N, k) - 1) < tol
            iter = k;
            break;
        end

        % want to skip this case
        if k == 1
            continue;
        else
            % applying shooting method to determine the next guess
            h(1, k+1) = h(1, k) + (1.0 - g(N, k))*(h(1, k) - h(1, k-1))/(g(N, k) - g(N, k-1));

            % condition on h(1, :) - cannot be a negative value
            if sign(h(1, k+1)) == -1
                h(1, k+1) = -h(1, k+1);
            end
        end

end

% transposing for data collection purposes
t_col = transpose(t);

% printing results to file
fprintf(fid1, '%.5f\n', t_col);
fprintf(fid2, '%.5f\n', f(:, iter));
fprintf(fid3, '%.5f\n', g(:, iter));
fprintf(fid4, '%.5f\n', h(:, iter));
fprintf(fid5, '%.5f   %.5f\n', [dt; h(1, iter)]);

% plotting the solution
plot(f(:, iter), t);
hold on;
plot(g(:, iter), t);
plot(h(:, iter), t);
legend('f', 'f''', 'f"');
title('Blasius Boundary Layer Solution for Flow Over a Flat Plate');
xlabel('u/U');
ylabel('\eta');

% end of script
```

Figure 4: MATLAB Script (2)

5.2 References

[1] Çengel, Yunus A., and John M. Cimbala. Fluid Mechanics: Fundamentals and Applications. McGraw-Hill Higher Education, 2006.

List of Figures